Robust Control for Fuzzy Nonlinear Uncertain Systems with Discrete and Distributed Time Delays

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This paper addresses the problem of stability and stabilization issue for a class of fuzzy nonlinear uncertain systems with discrete and distributed time delays. By utilizing a new Lyapunov–Krasovskii functional together with free weighting matrix approach, a new set of delay-dependent sufficient conditions are derived which makes the closed loop system robustly asymptotically stable. In particular, the parameter uncertainties are assumed to be norm bounded. Further, a state feedback controller is proposed to guarantee the robust stabilization for uncertain systems and subsequently the controller is constructed in terms of the solution to a set of linear matrix inequalities (LMI). The derived conditions are expressed in the form of linear matrix inequalities which can be efficiently solved via standard LMI toolbox. Further, two numerical examples are provided to demonstrate the effectiveness and less conservatism of the obtained results.

Key words: Fuzzy Nonlinear Systems; Robust Control; Delay Fractioning Technique; Linear Matrix Inequality; Lyapunov–Krasovskii Functional.

1. Introduction

Fuzzy systems in the form of the Takagi–Sugeno (TS) model [1] have recently attracted great interest due to its applications in various fields of science and engineering. The TS fuzzy model gives an effective method to represent complex nonlinear systems by fuzzy sets and fuzzy reasoning. One of the most-popular application areas of fuzzy-logic systems is modelling and control of complex dynamical systems. Many issues related to the stability analysis and control synthesis of TS fuzzy systems due to its simpler form in the defuzzification have been reported in [2 – 7] and references therein. It is well known that the stability issue is one of the most important problems in analysis and design of control systems. In particular, asymptotical stability is one of the main properties of dynamical systems, which is a crucial feature in the design of dynamical systems. More precisely, the linear matrix inequality (LMI) approach has been widely applied to discuss stability analysis for TS fuzzy systems [8 – 11].

It is well known that time delays are unavoidable in hardware implementations. The existence of time delays may lead to oscillation, divergence, and even instability in dynamical systems. Therefore, the stability of time delayed systems has become a topic of theoretical and practical importance [12 – 15]. Moreover, uncertainties are inevitable in dynamical systems because of the existence of modelling errors and external disturbance. Also, the parameter uncertainties are significant while modelling the system and should be taken into account. Therefore, many interesting results are presented on the stability analysis in the presence of parameter uncertainties [16 – 20].

In recent years, there has been increasing interest in studying stabilization of dynamical system by choosing proper controller design [21 – 23]. In particular, the main issue in stabilization of dynamical system is how

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to reduce the possible conservatism by applying so many new ideas and techniques. More precisely, an important index for checking the conservatism of stability criteria is to find the maximum delay bound. Moreover the idea of delay-fractioning or partitioning becomes an increasing interest of many researchers due to yield of its less conservative results while studying the dynamical behaviours via LMI approach when the partitioning number goes thinner [10]. Kwon et al. [24] derived a novel delay-dependent stability criteria in terms of LMIs for uncertain dynamic systems with time-varying delays by using a new augmented Lyapunov functional with convex optimization algorithms. Senthilkumar and Balasubramaniam [9] discussed robust H_∞ control for nonlinear uncertain stochastic TS fuzzy systems with time delays. Very recently, Feng and Lam [25] studied the problem of robust delay-dependent stability analysis and stabilization for distributed delay systems with linear fractional uncertainties by introducing an integral partitioning technique.

To the best of authors' knowledge, the stability and stabilization of fuzzy nonlinear systems with discrete and distributed time delays is not yet fully investigated and still there is room for further research, and this motivates our research. By constructing a new type of Lyapunov–Krasovskii functional (LKF) based on some advanced ideas like delay fractioning technique and using the matrix inequality technique, a new set of sufficient conditions are derived to ensure the robust stability and stabilization of the considered fuzzy systems. The obtained conditions are formulated in terms of LMIs, which can be easily solved numerically by using the MATLAB LMI toolbox. Finally, we provide two numerical examples to demonstrate the effectiveness and less conservatism of the proposed results.

2. Problem Formulation and Preliminaries

In this section, we start by introducing some notations, definitions, and basic results that will be used throughout the paper. The superscripts 'T' and '(-1)' stand for matrix transposition and matrix inverse, respectively. $\mathbb{R}^n$ denotes the n-dimensional Euclidean space. $P > 0$ means that P is real symmetric and positive definite. I_n and 0_n represent n-dimensional identity matrix and zero matrix, respectively. In symmetric block matrices or long matrix expressions, we use an asterisk $(*)$ to represent a term that is induced by symmetry, and sym(A) is defined as $A + A^{\mathrm{T}}$; diag$\{.\}$ stands for a block-diagonal matrix. $\|\cdot\|$ refers to the Euclidean vector norm.

Consider the following uncertain TS fuzzy systems with discrete and distributed time delays and nonlinear perturbations together with the ith rule as follows:
Plant rule i: IF $\{\theta_1(t) \text{ is } M_1^i\}$ and ... and $\{\theta_p(t) \text{ is } M_p^i\}$ THEN

$$\begin{aligned}\dot{x}(t) &= A_i(t)x(t) + A_{di}(t)x(t-\tau) + A_{hi}(t) \\ &\quad \cdot \int_{t-h}^{t} x(s)\,\mathrm{d}s + B_i(t)u(t) + A_{fi}f_i(x(t), x(t-\tau), t), \\ x(t) &= \phi(t), \quad t \in [-\bar{\tau}, 0], \quad \bar{\tau} = \max(\tau, h), \end{aligned} \tag{1}$$

where $\theta_1(t), \ldots, \theta_p(t)$ is the premise variable vector; $M_q^i (i = 1, 2, \ldots, r,\ q = 1, 2, \ldots, p)$ are fuzzy sets, and r is the number of IF-THEN rules; $x(t)$ is the state vector, and $u(t)$ is the control input vector; τ and h represent the discrete and the distributed time delay, respectively; $\phi(t)$ is the initial condition defined on $[-\bar{\tau}, 0]$; $A_i(t) = A_i + \Delta A_i(t)$, $A_{di}(t) = A_{di} + \Delta A_{di}(t)$, $A_{hi}(t) = A_{hi} + \Delta A_{hi}(t)$, and $B_i(t) = B_i + \Delta B_i(t)$ in which A_i, A_{di}, A_{hi}, B_i are known constant matrices with appropriate dimensions and $\Delta A_i(t), \Delta A_{di}(t), \Delta A_{hi}(t), \Delta B_i(t)$ denote the parameter uncertainties in the system and are assumed to satisfy the following assumption [9, 26]:

Assumption 1. The parameter uncertainties are defined as

$$\begin{aligned}&\left[\Delta A_i\ \Delta A_{di}\ \Delta A_{hi}\ \Delta B_i\right] = \\ &\left[H_i F_i(t) E_{1i}\ \ H_i F_i(t) E_{2i}\ \ H_i F_i(t) E_{3i}\ \ H_i F_i(t) E_{4i}\right],\end{aligned}$$

where H_i, E_{1i}, E_{2i}, E_{3i}, and E_{4i} are known real constant matrices, and $F_i(t)$ denotes an unknown time-varying matrix function satisfying $F_i^{\mathrm{T}}(t)F_i(t) \le I$; here I is the identity matrix.

Assumption 2. There exist known real constant matrices G_{1i}, $G_{2i} \in \mathbb{R}^{n\times n}$ such that the nonlinear function $f_i(\cdot,\cdot,\cdot) \in \mathbb{R}^n$ satisfies the following bounded condition:

$$\begin{aligned}&|f_i(x(t), x(t-\tau), t)| \le G_{1i}|x(t)| + G_{2i}|x(t-\tau)|, \\ &i = 1, 2 \ldots, r.\end{aligned}$$

By inferring from the fuzzy models, the final output of fuzzy system is inferred as

$$\dot{x}(t) = \sum_{i=1}^{r} \eta_i(\theta(t)) \Big[A_i(t)x(t) + A_{di}(t)x(t-\tau) + A_{hi}(t)$$

$$\cdot \int_{t-h}^{t} x(s)\,\mathrm{d}s + B_i(t)u(t) + A_{f_i} f_i(x(t), x(t-\tau), t) \Bigg], \quad (2)$$

where $\eta_i(\theta(t))$ is the normalized membership function of the inferred fuzzy set $\omega_i(\theta(t))$. ie., $\eta_i(\theta(t)) = \frac{\omega_i(\theta(t))}{\sum_{i=1}^{r}\omega_i(\theta(t))}$, where $\omega_i(\theta(t)) = \prod_{q=1}^{p} M_q^i(\theta_q(t))$, here $M_q^i(\theta_q(t))$ is the grade of the membership function of $\theta_q(t)$ in M_q^i. According to the theory of fuzzy sets, it is obvious that $\omega_i(\theta(t)) \geq 0,\ i = 1,2,\ldots,r$, $\sum_{i=1}^{r}\omega_i(\theta(t)) > 0$ for any $\theta(t)$. Therefore, it implies that $\eta_i(\theta(t)) \geq 0, \sum_{i=1}^{r}\eta_i(\theta(t)) = 1$ for any $\theta(t)$.

Based on the parallel distributed compensation strategy, the fuzzy state-feedback controller obeys the following rule:

Control rule i: IF $\{\theta_1(t) \text{ is } M_1^i\}$ and ... and $\{\theta_p(t) \text{ is } M_p^i\}$ THEN

$$u(t) = K_i x(t),\ \ i = 1,2\ldots,r,$$

where $x(t) \in \mathbb{R}^n$ is the input of the controller, $u(t) \in \mathbb{R}^m$ is the output of the controller, and K_i is the gain matrix of the state feedback controller. Thus, the overall fuzzy state feedback controller can be represented by the input-output form

$$u(t) = \sum_{i=1}^{r} \eta_i(\theta(t)) K_i x(t). \quad (3)$$

Then under the control law (3), the overall closed-loop system can be written as

$$\begin{aligned}\dot{x}(t) = \sum_{i=1}^{r}\eta_i(\theta(t))\sum_{k=1}^{r}\eta_k(\theta(t))\Bigg[&(A_i(t) \\ &+ B_i(t)K_k)x(t) + A_{di}(t)x(t-\tau) \\ &+ A_{hi}(t)\int_{t-h}^{t} x(s)\,\mathrm{d}s + A_{f_i} f_i(x(t), x(t-\tau), t)\Bigg].\end{aligned} \quad (4)$$

The following lemmas will be used in the main proof:

Lemma 1. *[7] Given constant matrices Ω_1, Ω_2, and Ω_3 with appropriate dimensions, where $\Omega_1^{\mathrm{T}} = \Omega_1$ and $\Omega_2^{\mathrm{T}} = \Omega_2 > 0$, then $\Omega_1 + \Omega_3^{\mathrm{T}}\Omega_2^{-1}\Omega_3 < 0$ if and only if*

$$\begin{bmatrix} \Omega_1 & \Omega_3^{\mathrm{T}} \\ * & -\Omega_2 \end{bmatrix} < 0 \quad or \quad \begin{bmatrix} -\Omega_2 & \Omega_3 \\ * & \Omega_1 \end{bmatrix} < 0.$$

Lemma 2. *[25] For any constant matrix $Z > 0$ and a scalar $\tau > 0$, if there exists a Lebesgue measurable function $x(\cdot) : [t-\tau,\ t] \to \mathbb{R}^n$, then the following inequalities hold:*

$$\begin{aligned}&-\int_{t-\tau}^{t} x^{\mathrm{T}}(s)Zx(s)\,\mathrm{d}s \leq -\frac{1}{\tau}\left(\int_{t-\tau}^{t} x^{\mathrm{T}}(s)\,\mathrm{d}s\right) \\ &\cdot Z\left(\int_{t-\tau}^{t} x(s)\,\mathrm{d}s\right), \\ &-\int_{-\tau}^{0}\int_{t+\theta}^{t} x^{\mathrm{T}}(s)Zx(s)\,\mathrm{d}s\,\mathrm{d}\theta \leq \\ &-\frac{2}{\tau^2}\left(\int_{-\tau}^{0}\int_{t+\theta}^{t} x(s)\,\mathrm{d}s\,\mathrm{d}\theta\right)^{\mathrm{T}} Z\left(\int_{-\tau}^{0}\int_{t+\theta}^{t} x(s)\,\mathrm{d}s\,\mathrm{d}\theta\right).\end{aligned}$$

Lemma 3. *[7] Assume that Ω, M_i, and E are real matrices with appropriate dimensions, and $F_i(t)$ is a matrix function satisfying $F_i^{\mathrm{T}}(t)F_i(t) \leq I$. Then, $\Omega + M_iF_i(t)E_i + [M_iF_i(t)E_i]^{\mathrm{T}} < 0$ holds if and only if there exist a scalar $\varepsilon > 0$ satisfying $\Omega + \varepsilon^{-1}M_iM_i^{\mathrm{T}} + \varepsilon E_i^{\mathrm{T}}E_i < 0$.*

3. Main Results

In this section, we present our main results. First, we derive the sufficient condition for robust stability of TS fuzzy system of the form

$$\begin{aligned}\dot{x}(t) = \sum_{i=1}^{r}\eta_i(\theta(t))\Bigg[&A_i(t)x(t) + Ad_i(t)x(t-\tau) \\ &+ Ah_i(t)\int_{t-h}^{t} x(s)\,\mathrm{d}s\Bigg].\end{aligned} \quad (5)$$

More precisely, by constructing a novel LKF together with free-weighting matrix technique, a delay dependent condition is derived to make system (5) robustly asymptotically stable. Then the results obtained for system (5) can be extended to study robust stabilization for the nonlinear uncertain TS fuzzy system (4). Further, an LMI based algorithm will be developed to design a suitable feedback controller for the uncertain TS fuzzy system (4) such that the resulting closed loop system is asymptotically stable.

Theorem 1. *Assume that Assumption 1 holds. For given integers $m, l \geq 1$, the closed loop system given in (5) is robustly asymptotically stable, if there exist $n \times n$ matrices $R_{1j} > 0,\ R_{2j} > 0,\ Z_j > 0,\ j = 1,2,\ 3n \times 3n$ matrix $\bar{P} > 0,\ ln \times ln$ matrices $Q_1 > 0,\ Q_2 > 0$, any appropriately dimensioned matrices $N_1 \in \mathbb{R}^{ln\times ln}$, $N_2 \in \mathbb{R}^{n\times n}$, $N_3 \in \mathbb{R}^{mn\times mn}$, $N_4 \in \mathbb{R}^{n\times n}$, and $L_1,\ L_2,\ M_1,\ M_2$, and scalar $\varepsilon_{1i} > 0$, such that the following LMIs hold*

for $i = 1,2,\ldots,r$*:*

$$\Psi = \begin{bmatrix} \Omega & \sqrt{\frac{\tau}{l}}L_1 & \sqrt{\frac{h}{m}}L_2 & W_{N_1}^{\mathrm{T}}NH_i & \varepsilon_{1i}E_i^{\mathrm{T}} \\ * & -M_1 & 0 & 0 & 0 \\ * & * & -M_2 & 0 & 0 \\ * & * & * & -\varepsilon_{1i}I & 0 \\ * & * & * & * & -\varepsilon_{1i}I \end{bmatrix} < 0, \tag{6}$$

$$M_j < R_{j2}, \quad j = 1,2, \tag{7}$$

where

$$\Omega = W_Q^{\mathrm{T}}QW_Q + W_{R_1}^{\mathrm{T}}R_1W_{R_1} + W_{R_2}^{\mathrm{T}}R_2W_{R_2} + W_Z^{\mathrm{T}}ZW_Z + \mathrm{sym}\left(W_{P_1}^{\mathrm{T}}\bar{P}W_{P_2} + LW_L + W_{N_1}^{\mathrm{T}}NW_{N_2}\right),$$

$$W_{P_1} = \begin{bmatrix} I_n & 0_{n,n(l+m+4)} & \\ 0_{n,n(l+m+2)} & I_n & 0_{n,2n} \\ 0_{n,n(l+1)} & I_n & 0_{n,n(m+3)} \end{bmatrix},$$

$$W_{P_2} = \begin{bmatrix} 0_{n,n(l+m+4)} & I_n & & \\ I_n & -I_n & 0_{n,n(l+m+3)} & \\ I_n & 0_{n,n(l+m+2)} & -I_n & 0_n \end{bmatrix},$$

$$\bar{P} = \begin{bmatrix} P_{11} & P_{12} & P_{13} \\ * & P_{22} & P_{23} \\ * & * & P_{33} \end{bmatrix},$$

$$W_Q = \begin{bmatrix} I_{n,ln} & 0_{n,n(m+5)} & \\ 0_n & I_{n,ln} & 0_{n,n(m+4)} \\ 0_{n,n(l+1)} & I_{n,mn} & 0_{n,4n} \\ 0_{n,n(l+2)} & I_{n,mn} & 0_{n,3n} \end{bmatrix},$$

$$N = \begin{bmatrix} N_1 \\ N_2 \\ N_3 \\ N_4 \end{bmatrix},$$

$$W_{R_1} = \begin{bmatrix} \sqrt{\frac{\tau}{l}}I_n & 0_{n,n(l+m+4)} & \\ 0_{n,n(l+m+2)} & \sqrt{\frac{l}{\tau}}I_n & 0_{n,2n} \\ 0_{n,n(l+m+4)} & \sqrt{\frac{\tau}{l}}I_n & \end{bmatrix},$$

$$W_{R_2} = \begin{bmatrix} \sqrt{\frac{h}{m}}I_n & 0_{n,n(l+m+4)} & \\ 0_{n,n(l+m+3)} & \sqrt{\frac{m}{h}}I_n & 0_n \\ 0_{n,n(l+m+4)} & \sqrt{\frac{h}{m}}I_n & \end{bmatrix},$$

$$W_Z = \begin{bmatrix} 0_{n,n(l+m+4)} & \frac{1}{\sqrt{2}}\frac{\tau}{l}I_n & & \\ 0_{n,n(l+m+4)} & \frac{1}{\sqrt{2}}\frac{h}{m}I_n & & \\ I_n & 0_{n,n(l+m+1)} & -\sqrt{2}\frac{l}{\tau}I_n & 0_{n,2n} \\ I_n & 0_{n,nl} & -\sqrt{2}\frac{m}{h}I_n & 0_{n,n(m+3)} \end{bmatrix},$$

$$W_{N_1} = \begin{bmatrix} I_{n,ln} & 0_{n,n(m+5)} & \\ 0_{n,ln} & I_n & 0_{n,n(m+4)} \\ 0_{n,n(l+1)} & I_{n,mn} & 0_{n,4n} \\ 0_{n,n(l+m+4)} & I_n & \end{bmatrix},$$

$$Q = \mathrm{diag}\{Q_1, -Q_1, Q_2, -Q_2\},$$

$$R_1 = \mathrm{diag}\{R_{11}, -R_{11}, R_{12}\},$$

$$R_2 = \mathrm{diag}\{R_{21}, -R_{21}, R_{12}\},$$

$$Z = \mathrm{diag}\{Z_1, Z_2, -Z_1, -Z_2,\}, \; L = [L_1 \; L_2],$$

$$W_L = \begin{bmatrix} I_n & -I_n & 0_{n,n(l+m+3)} \\ I_n & 0_{n,n(l+m+2)} & -I_n \quad 0_n \end{bmatrix},$$

$$W_{N_2} = \begin{bmatrix} A_i & 0_{n,n(l-1)} & A_{di} & \overbrace{A_{hi}\ldots A_{hi}}^{m} & 0_{3n} & -I_n \end{bmatrix},$$

$$E_i = \begin{bmatrix} E_{1i} & 0_{n,(l-1)n} & E_{2i} & \overbrace{E_{3i},\ldots,E_{3i}}^{m} & 0_{n,4n} \end{bmatrix}.$$

Proof. In order to prove the stability result, we consider the following Lyapunov–Krasovskii functional for model (5) based on the delay fractioning idea:

$$V(t,x(t),x_t(\theta)) = \sum_{j=1}^{3} V_j(t,x(t),x_t(\theta)), \tag{8}$$

where

$$V_1(t,x(t),x_t(\theta)) = \zeta_1^{\mathrm{T}}(t)\begin{bmatrix} P_{11} & P_{12} & P_{13} \\ * & P_{22} & P_{23} \\ * & * & P_{33} \end{bmatrix}\zeta_1(t) + \int_{t-\frac{\tau}{l}}^{t}\gamma_1^{\mathrm{T}}(s)Q_1\gamma_1(s)\,\mathrm{d}s + \int_{t-\frac{h}{m}}^{t}\gamma_2^{\mathrm{T}}(s)Q_2\gamma_2(s)\,\mathrm{d}s,$$

$$V_2(t,x(t),x_t(\theta)) = \int_{-\frac{\tau}{l}}^{0}\int_{t+\theta}^{t}x^{\mathrm{T}}(s)R_{11}x(s)\,\mathrm{d}s\,\mathrm{d}\theta + \int_{-\frac{\tau}{l}}^{0}\int_{t+\theta}^{t}\dot{x}^{\mathrm{T}}(s)R_{12}\dot{x}(s)\,\mathrm{d}s\,\mathrm{d}\theta + \int_{-\frac{h}{m}}^{0}\int_{t+\theta}^{t}x^{\mathrm{T}}(s)R_{21}x(s)\,\mathrm{d}s\,\mathrm{d}\theta + \int_{-\frac{h}{m}}^{0}\int_{t+\theta}^{t}\dot{x}^{\mathrm{T}}(s)R_{22}\dot{x}(s)\,\mathrm{d}s\,\mathrm{d}\theta,$$

$$V_3(t,x(t),x_t(\theta)) = \int_{-\frac{\tau}{l}}^{0}\int_{\theta}^{0}\int_{t+\lambda}^{t}\dot{x}^{\mathrm{T}}(s)Z_1\dot{x}(s)\,\mathrm{d}s\,\mathrm{d}\lambda\,\mathrm{d}\theta + \int_{-\frac{h}{m}}^{0}\int_{\theta}^{0}\int_{t+\lambda}^{t}\dot{x}^{\mathrm{T}}(s)Z_2\dot{x}(s)\,\mathrm{d}s\,\mathrm{d}\lambda\,\mathrm{d}\theta,$$

where $x_t(\theta) = x(t+\theta), \forall\theta \in [\bar{\tau},0]$,

$$\zeta_1(t) = \left[x^{\mathrm{T}}(t) \int_{t-\frac{\tau}{l}}^{t}x^{\mathrm{T}}(s)\,\mathrm{d}s \int_{t-\frac{h}{m}}^{t}x^{\mathrm{T}}(s)\,\mathrm{d}s\right]^{\mathrm{T}},$$

$$\gamma_1(t)=\left[x^{\mathrm{T}}(t)\ x^{\mathrm{T}}\left(t-\frac{\tau}{l}\right)\ \ldots\ x^{\mathrm{T}}\left(t-\frac{(l-1)\tau}{l}\right)\right]^{\mathrm{T}}\in\mathbb{R}^{ln},$$

and $\gamma_2(t)=\left[\int_{t-\frac{h}{m}}^{t}x^{\mathrm{T}}(s)\,\mathrm{d}s\ \ldots\right.$

$$\left.\ldots\int_{t-\frac{(m-1)h}{m}}^{t-\frac{(m-2)h}{m}}x^{\mathrm{T}}(s)\,\mathrm{d}s\ \int_{t-h}^{t-\frac{(m-1)h}{m}}x^{\mathrm{T}}(s)\,\mathrm{d}s\right]^{\mathrm{T}}\in\mathbb{R}^{mn},$$

here $l,m\geq 1$ are integers.

By evaluating the derivative of $V(t,x(t))$ along the trajectories of system (5), we obtain

$$\begin{aligned}\dot{V}_1(t,x(t),x_t(\theta))&=2\left[x^{\mathrm{T}}(t)\int_{t-\frac{\tau}{l}}^{t}x^{\mathrm{T}}(s)\,\mathrm{d}s\int_{t-\frac{h}{m}}^{t}x^{\mathrm{T}}(s)\,\mathrm{d}s\right]\\&P\left[\dot{x}^{\mathrm{T}}(t)\ \ x^{\mathrm{T}}(t)-x^{\mathrm{T}}\left(t-\frac{\tau}{l}\right)\ \ x^{\mathrm{T}}(t)-x^{\mathrm{T}}\left(t-\frac{h}{m}\right)\right]^{\mathrm{T}}\\&+\gamma_1^{\mathrm{T}}(t)Q_1\gamma_1(t)-\gamma_1^{\mathrm{T}}\left(t-\frac{\tau}{l}\right)Q_1\gamma_1\left(t-\frac{\tau}{l}\right)\\&+\gamma_2^{\mathrm{T}}(t)Q_2\gamma_2(t)-\gamma_2^{\mathrm{T}}\left(t-\frac{h}{m}\right)Q_2\gamma_2\left(t-\frac{h}{m}\right),\end{aligned}\tag{9}$$

$$\begin{aligned}\dot{V}_2(t,x(t),x_t(\theta))&=x^{\mathrm{T}}(t)\left(\frac{\tau}{l}R_{11}+\frac{h}{m}R_{21}\right)x(t)\\&-\int_{t-\frac{\tau}{l}}^{t}x^{\mathrm{T}}(s)R_{11}x(s)\,\mathrm{d}s-\int_{t-\frac{h}{m}}^{t}x^{\mathrm{T}}(s)R_{21}x(s)\,\mathrm{d}s\\&+\dot{x}^{\mathrm{T}}(t)\left(\frac{\tau}{l}R_{12}+\frac{h}{m}R_{22}\right)\dot{x}(t)-\int_{t-\frac{\tau}{l}}^{t}\dot{x}^{\mathrm{T}}(s)R_{12}\dot{x}(s)\,\mathrm{d}s\\&-\int_{t-\frac{h}{m}}^{t}\dot{x}^{\mathrm{T}}(s)R_{22}\dot{x}(s)\,\mathrm{d}s,\end{aligned}\tag{10}$$

$$\begin{aligned}\dot{V}_3(t,x(t),x_t(\theta))&=\dot{x}^{\mathrm{T}}(t)\left(\frac{1}{2}\left(\frac{\tau}{l}\right)^2Z_1+\frac{1}{2}\left(\frac{h}{m}\right)^2Z_2\right)\dot{x}(t)\\&-\int_{-\frac{\tau}{l}}^{0}\int_{t+\theta}^{t}\dot{x}^{\mathrm{T}}(s)Z_1\dot{x}(s)\,\mathrm{d}s\,\mathrm{d}\theta\\&-\int_{-\frac{h}{m}}^{0}\int_{t+\theta}^{t}\dot{x}^{\mathrm{T}}(s)Z_2\dot{x}(s)\,\mathrm{d}s\,\mathrm{d}\theta.\end{aligned}\tag{11}$$

By applying Lemma 2 in each integral term of (10) and (11), we obtain

$$\begin{aligned}-\int_{t-\frac{\tau}{l}}^{t}x^{\mathrm{T}}(s)R_{11}x(s)\,\mathrm{d}s&\leq-\frac{l}{\tau}\int_{t-\frac{\tau}{l}}^{t}x^{\mathrm{T}}(s)\,\mathrm{d}sR_{11}\\&\quad\cdot\int_{t-\frac{\tau}{l}}^{t}x(s)\,\mathrm{d}s,\end{aligned}\tag{12}$$

$$\begin{aligned}-\int_{t-\frac{h}{m}}^{t}x^{\mathrm{T}}(s)R_{21}x(s)\,\mathrm{d}s&\leq-\frac{m}{h}\int_{t-\frac{h}{m}}^{t}x^{\mathrm{T}}(s)\,\mathrm{d}sR_{21}\\&\quad\cdot\int_{t-\frac{h}{m}}^{t}x(s)\,\mathrm{d}s,\end{aligned}\tag{13}$$

$$\begin{aligned}&-\int_{-\frac{\tau}{l}}^{0}\int_{t+\theta}^{t}\dot{x}^{\mathrm{T}}(s)Z_1\dot{x}(s)\,\mathrm{d}s\,\mathrm{d}\theta\leq\\&-2\left(\frac{l}{\tau}\right)^2\int_{-\frac{\tau}{l}}^{0}\int_{t+\theta}^{t}\dot{x}^{\mathrm{T}}(s)\,\mathrm{d}s\,\mathrm{d}\theta Z_1\int_{-\frac{\tau}{l}}^{0}\int_{t+\theta}^{t}\dot{x}(s)\,\mathrm{d}s\,\mathrm{d}\theta\\&\leq-2\left(\frac{l}{\tau}\right)^2\left[\frac{\tau}{l}x^{\mathrm{T}}(t)-\int_{t-\frac{\tau}{l}}^{t}x^{\mathrm{T}}(s)\,\mathrm{d}s\,\mathrm{d}\theta\right]\\&\quad\cdot Z_1\left[\frac{\tau}{l}x(t)-\int_{t-\frac{\tau}{l}}^{t}x(s)\,\mathrm{d}s\,\mathrm{d}\theta\right],\end{aligned}\tag{14}$$

$$\begin{aligned}&-\int_{-\frac{h}{m}}^{0}\int_{t+\theta}^{t}\dot{x}^{\mathrm{T}}(s)Z_2\dot{x}(s)\,\mathrm{d}s\,\mathrm{d}\theta\leq\\&-2\left(\frac{m}{h}\right)^2\int_{-\frac{h}{m}}^{0}\int_{t+\theta}^{t}\dot{x}^{\mathrm{T}}(s)\,\mathrm{d}s\,\mathrm{d}\theta Z_2\int_{-\frac{h}{m}}^{0}\int_{t+\theta}^{t}\dot{x}(s)\,\mathrm{d}s\,\mathrm{d}\theta\\&\leq-2\left(\frac{m}{h}\right)^2\left[\frac{h}{m}x^{\mathrm{T}}(t)-\int_{t-\frac{h}{m}}^{t}x^{\mathrm{T}}(s)\,\mathrm{d}s\right]\\&\quad\cdot Z_2\left[\frac{h}{m}x(t)-\int_{t-\frac{h}{m}}^{t}x(s)\,\mathrm{d}s\right].\end{aligned}\tag{15}$$

Substituting (9)–(15) in (8), we have

$$\begin{aligned}\dot{V}(t,x(t),x_t(\theta))&=2\left[x^{\mathrm{T}}(t)\ \int_{t-\frac{\tau}{l}}^{t}x^{\mathrm{T}}(s)\,\mathrm{d}s\ \int_{t-\frac{h}{m}}^{t}x^{\mathrm{T}}(s)\,\mathrm{d}s\right]\\&\cdot P\left[\dot{x}^{\mathrm{T}}(t)\ x^{\mathrm{T}}(t)-x^{\mathrm{T}}\left(t-\frac{\tau}{l}\right)\ x^{\mathrm{T}}(t)-x^{\mathrm{T}}\left(t-\frac{h}{m}\right)\right]^{\mathrm{T}}\\&+\gamma_1^{\mathrm{T}}(t)Q_1\gamma_1(t)-\gamma_1^{\mathrm{T}}\left(t-\frac{\tau}{l}\right)Q_1\gamma_1\left(t-\frac{\tau}{l}\right)\\&+\gamma_2^{\mathrm{T}}(t)Q_2\gamma_2(t)-\gamma_2^{\mathrm{T}}\left(t-\frac{h}{m}\right)Q_2\gamma_2\left(t-\frac{h}{m}\right)\\&+x^{\mathrm{T}}(t)\left(\frac{\tau}{l}R_{11}+\frac{h}{m}R_{21}\right)x(t)+\dot{x}^{\mathrm{T}}(t)\left(\frac{\tau}{l}R_{12}+\frac{h}{m}R_{22}\right.\\&\left.+\frac{1}{2}\left(\frac{\tau}{l}\right)^2Z_1+\frac{1}{2}\left(\frac{h}{m}\right)^2Z_2\right)\dot{x}(t)\\&-\frac{l}{\tau}\int_{t-\frac{\tau}{l}}^{t}x^{\mathrm{T}}(s)\,\mathrm{d}sR_{11}\int_{t-\frac{\tau}{l}}^{t}x(s)\,\mathrm{d}s\\&-\frac{m}{h}\int_{t-\frac{h}{m}}^{t}x^{\mathrm{T}}(s)\,\mathrm{d}sR_{21}\int_{t-\frac{h}{m}}^{t}x(s)\,\mathrm{d}s-\int_{t-\frac{\tau}{l}}^{t}\dot{x}^{\mathrm{T}}(s)R_{12}\dot{x}(s)\,\mathrm{d}s\\&-\int_{t-\frac{h}{m}}^{t}\dot{x}^{\mathrm{T}}(s)R_{22}\dot{x}(s)\,\mathrm{d}s-2\left(\frac{l}{\tau}\right)^2\left[\frac{\tau}{l}x^{\mathrm{T}}(t)\right.\\&\left.-\int_{t-\frac{\tau}{l}}^{t}x^{\mathrm{T}}(s)\,\mathrm{d}s\,\mathrm{d}\theta\right]Z_1\left[\frac{\tau}{l}x(t)-\int_{t-\frac{\tau}{l}}^{t}x(s)\,\mathrm{d}s\,\mathrm{d}\theta\right]\\&-2\left(\frac{m}{h}\right)^2\left[\frac{h}{m}x^{\mathrm{T}}(t)-\int_{t-\frac{h}{m}}^{t}x^{\mathrm{T}}(s)\,\mathrm{d}s\right]Z_2\left[\frac{h}{m}x(t)\right.\\&\left.-\int_{t-\frac{h}{m}}^{t}x(s)\,\mathrm{d}s\right].\end{aligned}\tag{16}$$

On the other hand, for any appropriately dimensioned matrices L_1, L_2 and for matrices M_j satisfying $M_j < R_{j2}$, $j = 1,2$, we have

$$2\zeta^{\mathrm{T}}(t)L_1\left[x(t)-x\left(t-\frac{\tau}{l}\right)-\int_{t-\frac{\tau}{l}}^{t}\dot{x}(s)\,\mathrm{d}s\right]=0, \quad (17)$$

$$2\zeta^{\mathrm{T}}(t)L_2\left[x(t)-x\left(t-\frac{h}{m}\right)-\int_{t-\frac{h}{m}}^{t}\dot{x}(s)\,\mathrm{d}s\right]=0, \quad (18)$$

$$\frac{\tau}{l}\zeta^{\mathrm{T}}(t)L_1M_1^{-1}L_1^{\mathrm{T}}\zeta(t)-\int_{t-\frac{\tau}{l}}^{t}\zeta^{\mathrm{T}}(t)L_1R_{12}^{-1}L_1^{\mathrm{T}}\zeta(t)\,\mathrm{d}s \geq 0, \quad (19)$$

$$\frac{h}{m}\zeta^{\mathrm{T}}(t)L_2M_2^{-1}L_2^{\mathrm{T}}\zeta(t)-\int_{t-\frac{h}{m}}^{t}\zeta^{\mathrm{T}}(t)L_2R_{22}^{-1}L_2^{\mathrm{T}}\zeta(t)\,\mathrm{d}s \geq 0, \quad (20)$$

where $\zeta^{\mathrm{T}}(t)=\left[\gamma_1^{\mathrm{T}}(t)\ x^{\mathrm{T}}(t-\tau)\ \gamma_2^{\mathrm{T}}(t)\ \int_{t-\frac{(m+1)h}{m}}^{t-\frac{h}{m}}x^{\mathrm{T}}(s)\,\mathrm{d}s\right.$ $\left.\int_{t-\frac{\tau}{l}}^{t}x^{\mathrm{T}}(s)\,\mathrm{d}s\ x^{\mathrm{T}}\left(t-\frac{h}{m}\right)\ \dot{x}^{\mathrm{T}}(t)\right]$.

Also for $\zeta_2^{\mathrm{T}}(t)=\left[\gamma_1^{\mathrm{T}}(t)\ \ x^{\mathrm{T}}(t-\tau)\ \ \gamma_2^{\mathrm{T}}(t)\ \ \dot{x}^{\mathrm{T}}(t)\right]$ and any matrix $N=[N_1^{\mathrm{T}}\ N_2^{\mathrm{T}}\ N_3^{\mathrm{T}}\ N_4^{\mathrm{T}}]^{\mathrm{T}}$, in which $N_1\in\mathbb{R}^{ln\times ln}$, $N_2\in\mathbb{R}^{n\times n}$, $N_3\in\mathbb{R}^{mn\times mn}$, and $N_4\in\mathbb{R}^{n\times n}$, the following equality holds:

$$2\sum_{i=1}^{r}\eta_i(\theta(t))\zeta_2^{\mathrm{T}}(t)N\left[A_i(t)x(t)+A_{di}(t)x(t-\tau)+A_{hi}(t)\int_{t-h}^{t}x(s)\,\mathrm{d}s-\dot{x}(t)\right]=0, \quad (21)$$

$$\Rightarrow\sum_{i=1}^{r}\eta_i(\theta(t))\zeta^{\mathrm{T}}(t)\left(\mathrm{sym}(W_{N_1}^{\mathrm{T}}NW_{N_2})+\varepsilon_{1i}^{-1}W_{N_1}^{\mathrm{T}}NH_iH_i^{\mathrm{T}}N^{\mathrm{T}}W_{N_1}+\varepsilon_{1i}E^{\mathrm{T}}E\right)\zeta(t)=0. \quad (22)$$

It follows from (16) and (17) – (22) that

$$\dot{V}(t,x(t),x_t(\theta)) \leq \sum_{i=1}^{r}\eta_i(\theta(t))\zeta^{\mathrm{T}}(t)\left[\Omega+\frac{\tau}{l}L_1M_1^{-1}L_1^{\mathrm{T}}+\frac{h}{m}L_2M_2^{-1}L_2^{\mathrm{T}}+\varepsilon_{1i}^{-1}W_{N_1}^{\mathrm{T}}NH_iH_i^{\mathrm{T}}N^{\mathrm{T}}W_{N_1}+\varepsilon_{1i}E^{\mathrm{T}}E\right]\zeta(t)-\int_{t-\frac{\tau}{l}}^{t}\left[\zeta^{\mathrm{T}}(t)L_1+\dot{x}^{\mathrm{T}}(s)R_{12}\right]R_{12}^{-1}\cdot\left[L_1^{\mathrm{T}}\zeta(t)+R_{12}\dot{x}(s)\right]\mathrm{d}s \quad (23)$$

$$-\int_{t-\frac{h}{m}}^{t}\left[\zeta^{\mathrm{T}}(t)L_2+\dot{x}^{\mathrm{T}}(s)R_{22}\right]R_{22}^{-1}\left[L_2^{\mathrm{T}}\zeta(t)+R_{22}\dot{x}(s)\right]\mathrm{d}s \leq \sum_{i=1}^{r}\eta_i(\theta(t))\zeta^{\mathrm{T}}(t)\left[\Omega+\frac{\tau}{l}L_1M_1^{-1}L_1^{\mathrm{T}}+\frac{h}{m}L_2M_2^{-1}L_2^{\mathrm{T}}+\varepsilon_{1i}^{-1}W_{N_1}^{\mathrm{T}}NH_iH_i^{\mathrm{T}}N^{\mathrm{T}}W_{N_1}+\varepsilon_{1i}E^{\mathrm{T}}E\right]\zeta(t).$$

Then, by using Lemma 1 (Schur complement) in (23), it is easy to obtain Ψ in (7). Hence, if $\Psi<0$ then $\dot{V}(t)<0$. Therefore, by the Lyapunov–Krasovski theorem [27], the closed loop fuzzy system (5) is robustly asymptotically stable. This completes the proof. □

Next we present the controller design method for the nonlinear fuzzy system (4) through the following Theorem:

Theorem 2. *Let Assumptions 1 and 2 hold. For given integers $m,l\geq 1$, closed loop system (4) is robustly asymptotically stable, if there exist symmetric positive definite matrices $\tilde{R}_{1j}>0$, $\tilde{R}_{2j}>0$, $\tilde{Z}_j>0$, $j=1,2$, $3n\times 3n$ matrix $\tilde{P}>0$, $ln\times ln$ matrices $\tilde{Q}_1>0$, $\tilde{Q}_2>0$, any appropriately dimensioned matrices X, $\tilde{M}_1,\tilde{M}_2$, $\tilde{L}_1,\tilde{L}_2$ and scalars n_j, $j=1,2,\ldots,(l+m+5)$, $\varepsilon_{ji}>0$, $j=1,2$, such that the following LMIs hold for $i,k=1,2,\ldots,r$, $(k\leq i)$:*

$$\tilde{\Psi}=\begin{bmatrix}\tilde{\Omega} & \sqrt{\frac{\tau}{l}}\tilde{L}_1 & \sqrt{\frac{h}{m}}\tilde{L}_2 & \varepsilon_{1i}W_{N_1}^{\mathrm{T}}\hat{n}H_i & X\bar{E}^{\mathrm{T}} & \sqrt{2}X\bar{G}^{\mathrm{T}} & \varepsilon_{2i}W_{N_1}^{\mathrm{T}}\hat{n}A_{fi}\\ * & -\tilde{M}_1 & 0 & 0 & 0 & 0 & 0\\ * & * & -\tilde{M}_2 & 0 & 0 & 0 & 0\\ * & * & * & -\varepsilon_{1i}I & 0 & 0 & 0\\ * & * & * & * & -\varepsilon_{1i}I & 0 & 0\\ * & * & * & * & * & -\varepsilon_{2i}I & 0\\ * & * & * & * & * & * & -\varepsilon_{2i}I\end{bmatrix}<0, \quad (24)$$

where $\hat{n}=\mathrm{col}(n_j)$,

$$\tilde{\Omega}=W_Q^{\mathrm{T}}\tilde{Q}W_Q+W_{R_1}^{\mathrm{T}}\tilde{R}_1W_{R_1}+W_{R_2}^{\mathrm{T}}\tilde{R}_2W_{R_2}+W_Z^{\mathrm{T}}\tilde{Z}W_Z+\mathrm{sym}\left(W_{P_1}^{\mathrm{T}}\tilde{P}W_{P_2}+\tilde{L}W_L+W_{N_1}^{\mathrm{T}}\hat{n}X\bar{W}_{N_2}\right),$$

$$\tilde{P} = \begin{bmatrix} \tilde{P}_{11} & \tilde{P}_{12} & \tilde{P}_{13} \\ * & \tilde{P}_{22} & \tilde{P}_{23} \\ * & * & \tilde{P}_{33} \end{bmatrix},$$
$$\tilde{Q} = \mathrm{diag}\{\tilde{Q}_1, -\tilde{Q}_1, \tilde{Q}_2, -\tilde{Q}_2\}, \ \tilde{L} = [\tilde{L}_1 \ \tilde{L}_2],$$
$$\tilde{R}_1 = \mathrm{diag}\{\tilde{R}_{11}, -\tilde{R}_{11}, \tilde{R}_{12}\},$$
$$\tilde{R}_2 = \mathrm{diag}\{\tilde{R}_{21}, -\tilde{R}_{21}, \tilde{R}_{12}\},$$
$$\tilde{Z} = \mathrm{diag}\{\tilde{Z}_1, \tilde{Z}_2, -\tilde{Z}_1, -\tilde{Z}_2\},$$
$$\bar{W}_{N_2} = \Big[A_i + B_i K_k \ \ 0_{n,(l-1)n} \ \ A_{di} \ \ \overbrace{A_{hi} \ldots A_{hi}}^{m} \ \ 0_{3n} \ \ -I_n\Big],$$
$$\bar{G} = \Big[G_{1i} \ \ 0_{n,(l-1)n} \ \ G_{2i} \ \ 0_{n,(m+4)n}\Big],$$
$$\bar{E} = \Big[E_{1i} + E_{4i}K \quad 0_{n,(l-1)n} \quad E_{2i} \quad \overbrace{E_{3i}, \ldots, E_{3i}}^{m} \quad 0_{n,4n}\Big],$$

and other parameters are same as given in Theorem 1.

Proof. By following the similar steps of Theorem 1 for the nonlinear fuzzy system (4), from the equality (21), we get

$$2\sum_{i=1}^{r}\eta_i(\theta(t))\sum_{k=1}^{r}\eta_k(\theta(t))\zeta_2^{\mathrm{T}}(t)N\Big[\big(A_i(t) + B_i(t)K_k\big)x(t) + A_{di}(t)x(t-\tau) + A_{hi}(t)\int_{t-h}^{t}x(s)\,\mathrm{d}s + A_{f_i}f_i(x(t),x(t-\tau),t) - \dot{x}(t)\Big] = 0. \tag{25}$$

For any positive scalar ε_{2i} and from Assumption 2, we have

$$2\sum_{i=1}^{r}\eta_i(\theta(t))\zeta_2^{\mathrm{T}}(t)NA_{fi}f_i(x(t),x(t-\tau)) \le \sum_{i=1}^{r}\eta_i(\theta(t))\varepsilon_{2i}\zeta_2^{\mathrm{T}}(t)NA_{fi}A_{fi}^{\mathrm{T}}N^{\mathrm{T}}\zeta_2(t) + \varepsilon_{2i}^{-1}\Big(|G_{1i}x(t)| + |G_{2i}x(t-\tau)|\Big)^2 \le \sum_{i=1}^{r}\eta_i(\theta(t))\varepsilon_{2i}\zeta_2^{\mathrm{T}}(t)NA_{fi}A_{fi}^{\mathrm{T}}N^{\mathrm{T}}\zeta_2(t) + 2\varepsilon_{2i}^{-1}\Big(x^{\mathrm{T}}(t)G_{1i}^{\mathrm{T}}G_{1i}x(t) + x^{\mathrm{T}}(t-\tau)G_{2i}^{\mathrm{T}}G_{2i}x(t-\tau)\Big).$$

Then (25) becomes

$$\sum_{i=1}^{r}\eta_i(\theta(t))\sum_{k=1}^{r}\eta_k(\theta(t))\zeta^{\mathrm{T}}(t)\Big[\mathrm{sym}(W_{N_1}^{\mathrm{T}}NW_{N_2}) + \varepsilon_{1i}W_{N_1}^{\mathrm{T}}NH_iH_i^{\mathrm{T}}N^{\mathrm{T}}W_{N_1} + \varepsilon_{1i}^{-1}\bar{E}^{\mathrm{T}}\bar{E} + \varepsilon_{2i}W_{N_1}^{\mathrm{T}}NA_{fi}A_{fi}^{\mathrm{T}}N^{\mathrm{T}}W_{N_1}^{\mathrm{T}} + 2\varepsilon_{2i}^{-1}\bar{G}^{\mathrm{T}}\bar{G}\Big]\zeta(t) = 0. \tag{26}$$

It follows from (8)–(20) and (26) that

$$\dot{V}(t,x(t),x_t(\theta)) \le \sum_{i=1}^{r}\eta_i(\theta(t))\sum_{k=1}^{r}\eta_k(\theta(t))\Big[\overline{\Omega} + \frac{\tau}{l}L_1M_1^{-1}L_1^{\mathrm{T}} + \frac{h}{m}L_2M_2^{-1}L_2^{\mathrm{T}} + \varepsilon_{1i}^{-1}W_{N_1}^{\mathrm{T}}NH_iH_i^{\mathrm{T}}N^{\mathrm{T}}W_{N_1} + \varepsilon_{1i}\bar{E}^{\mathrm{T}}\bar{E} + \varepsilon_{2i}W_{N_1}^{\mathrm{T}}NA_{fi}A_{fi}^{\mathrm{T}}N^{\mathrm{T}}W_{N_1}^{\mathrm{T}} + 2\varepsilon_{2i}^{-1}\bar{G}^{\mathrm{T}}\bar{G}\Big]\zeta(t), \tag{27}$$

where

$$\overline{\Omega} = W_{P_1}^{\mathrm{T}}\tilde{P}W_{P_2} + W_Q^{\mathrm{T}}QW_Q + W_{R_1}^{\mathrm{T}}R_1W_{R_1} + W_{R_2}^{\mathrm{T}}R_2W_{R_2} + W_Z^{\mathrm{T}}ZW_Z + \mathrm{sym}\left(LW_L + W_{N_1}^{\mathrm{T}}N\bar{W}_{N_2}\right).$$

By using Lemma 1 (Schur's complement), (27) becomes

$$\dot{V}(t) \le \sum_{i=1}^{r}\eta_i(\theta(t))\sum_{k=1}^{r}\eta_k(\theta(t))\zeta^{\mathrm{T}}(t)\bar{\Psi}\zeta(t), \tag{28}$$

where

$$\bar{\Psi} = \begin{bmatrix} \overline{\Omega} & \sqrt{\frac{\tau}{l}}L_1 & \sqrt{\frac{h}{m}}L_2 & \varepsilon_{1i}W_{N_1}^{\mathrm{T}}NH_i & \bar{E}^{\mathrm{T}} & \sqrt{2}\bar{G}^{\mathrm{T}} & \varepsilon_{2i}W_{N_1}^{\mathrm{T}}NA_{fi} \\ * & -M_1 & 0 & 0 & 0 & 0 & 0 \\ * & * & -M_2 & 0 & 0 & 0 & 0 \\ * & * & * & -\varepsilon_{1i}I & 0 & 0 & 0 \\ * & * & * & * & -\varepsilon_{1i}I & 0 & 0 \\ * & * & * & * & * & -\varepsilon_{2i}I & 0 \\ * & * & * & * & * & * & -\varepsilon_{2i}I \end{bmatrix}. \tag{29}$$

For any scalars $n_j > 0$, let $N_j = n_j\bar{N}$, $j = 1,2,\ldots,(l+m+5)$ and applying the congruent transformation to (29) with $\hat{N} = \mathrm{diag}\{\tilde{N},I,I,I,I\}$, where $\tilde{N} = \mathrm{diag}\{\bar{N}^{-1},\ldots,\bar{N}^{-1}\} \in \mathbb{R}^{(l+m+7)}$. Furthermore, for simplicity we choose $X = \bar{N}^{-1}$, $XPX^{\mathrm{T}} = \tilde{P}$, $XQX^{\mathrm{T}} = \tilde{Q}$, $XR_1X^{\mathrm{T}} = \tilde{R}_1$, $XR_2X^{\mathrm{T}} = \tilde{R}_2$, $XZX^{\mathrm{T}} = \tilde{Z}$ etc., we obtain $\tilde{\Psi}$ in (24). If $\tilde{\Psi} < 0$ then we get $\dot{V}(t) < 0$ and hence closed loop system (4) is robustly asymptotically sta-

ble. Thus, the fuzzy nonlinear system (2) is robustly asymptotically stabilizable by the state feedback controller given by (3) with gain matrices $K_k = Y_k X^{-1}$, $k = 1,2,\ldots,r$. The proof is completed. □

4. Numerical Simulation

In this section, we present two numerical examples to demonstrate the effectiveness of the proposed method.

Example 1. Consider the following uncertain TS fuzzy system together with the ith rule given as follows:

Plant Rule 1 : **IF** $\theta_1(t) = x_2(t)/0.5$ is about 0 **THEN**

$$\dot{x}(t) = A_1(t)x(t) + A_{d1}(t)x(t-\tau) + A_{h1}(t)\int_{t-h}^{t} x(s)\,\mathrm{d}s.$$

Plant Rule 2 : **IF** $\theta_1(t) = x_2(t)/0.5$ is less than or about $-\pi$ **or** greater than or about π **THEN**

$$\dot{x}(t) = A_2(t)x(t) + A_{d2}(t)x(t-\tau) + A_{h2}(t)\int_{t-h}^{t} x(s)\,\mathrm{d}s.$$

The membership functions are shown in Figure 1. Then the final output of the uncertain TS fuzzy system is inferred as given below:

$$\dot{x}(t) = \sum_{i=1}^{r} \eta_i(\theta(t)) \Bigg[A_i(t)x(t) + A_{di}(t)x(t-\tau) + A_{hi}(t)\int_{t-h}^{t} x(s)\,\mathrm{d}s \Bigg], \tag{30}$$

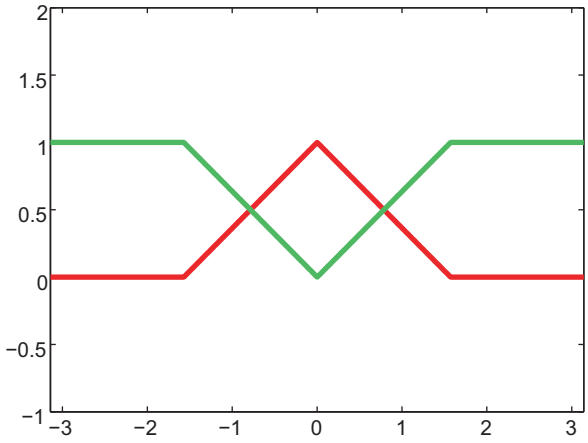

Fig. 1 (colour online). Membership functions for Example 1.

where

$$\eta_1(x_2(t)) = \begin{cases} 0 & \text{for } |x_2| > \pi/2, \\ \frac{2}{\pi}x_2 + 1 & \text{for } -\pi/2 \le x_2 < 0, \\ -\frac{2}{\pi}x_2 + 1 & \text{for } 0 \le x_2 \le \pi/2, \end{cases}$$

$$\eta_2(x_2(t)) = \begin{cases} 1 & \text{for } |x_2| > \pi/2, \\ -\frac{2}{\pi}x_2 & \text{for } -\pi/2 \le x_2 < 0, \\ \frac{2}{\pi}x_2 & \text{for } 0 \le x_2 \le \pi/2, \end{cases}$$

and the parameters are

$$A_1 = \begin{bmatrix} -2 & 0 \\ 0 & -0.9 \end{bmatrix}, A_2 = \begin{bmatrix} -2 & 0 \\ 0 & -1.4 \end{bmatrix},$$

$$A_{d1} = \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix}, A_{d2} = \begin{bmatrix} -1 & 0 \\ -1 & -1.5 \end{bmatrix},$$

$$A_{h1} = A_{h2} = \begin{bmatrix} \delta & 0 \\ 0 & \delta \end{bmatrix}, H_i = I,$$

$$F_i(t) = \begin{bmatrix} \sin(t) & -\cos(t) \\ -\cos(t) & -\sin(t) \end{bmatrix},$$

$$E_{1i} = E_{2i} = E_{3i} = \beta I, \quad i = 1,2.$$

If the partitioning number l and m are given, by solving LMI conditions in Theorem 1 via MATLAB LMI toolbox, one can easily obtain the feasible solutions. If we take the values for distributed delay same as discrete

Table 1. Calculated upper bounds of $\tau = h$ for $\beta = 0$.

δ	0.1	0.3	0.5	0.7
when $m = l = 1$	1.6659	1.2455	1.0565	0.9386
when $m = l = 2$	2.4067	1.6987	1.4033	1.2254
when $m = l = 3$	2.8376	1.9603	1.6054	1.3939
when $m = l = 4$	3.1859	2.1859	1.7832	1.5198

Table 2. Calculated upper bounds of $\tau = h$ for $\delta = 0.3$

β	0.1	0.3	0.5
when $m = l = 1$	1.0817	0.8029	0.5968
when $m = l = 2$	1.4606	1.0609	0.7652
when $m = l = 3$	1.6465	1.1919	0.8346
when $m = l = 4$	1.8451	1.3353	0.9269

Table 3. Calculated upper bounds of τ for $\delta = 0$.

β	0	0.3
In [20]	2.2615	0.8168
In [26]	2.3675	1.0658
Theorem 1 when $m = l = 1$	2.4483	1.2859
Theorem 1 when $m = l = 2$	4.1320	2.0374
Theorem 1 when $m = l = 3$	5.2511	2.8298
Theorem 1 when $m = l = 4$	6.1416	2.9812

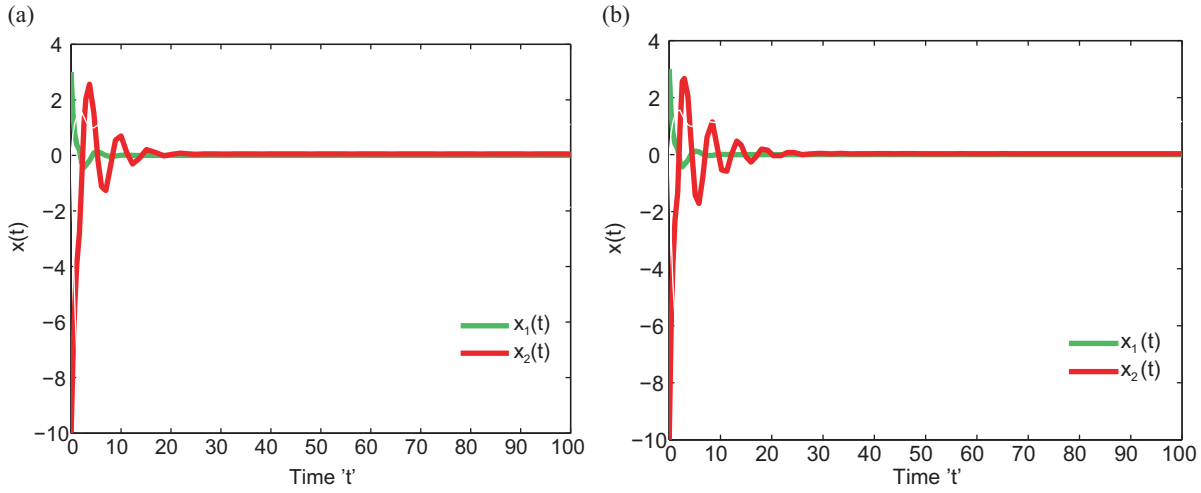

Fig. 2 (colour online). Trajectories of $x(t)$ of the fuzzy system in Example 1 for (a) $i=1$ and (b) $i=2$.

delay (i.e. $h=\tau$), the obtained delay upper bounds for $h=\tau$ are given in Tables 1 and 2 for various values of δ when $\beta=0$ and for various values of β when $\delta=0.3$. Also the obtained delay upper bounds of $h=\tau$ for $\delta=0$ by taking $\beta=0$ and $\beta=0.3$ are given in Table 3, which are compared with the results obtained in [20, 26]. From Table 3 it is observed that our results are less conservative than the existing results even if there is no partitioning in both discrete and distributed delay terms of the fuzzy system. It is shown that the calculated upper bound of $h=\tau$ increases as the partitioning numbers l and m increase. This reveals that our results are much less conservative when the partitioning on both discrete and distributed delay terms are taken into account. However, it is noted that the computational complexity also increases when the delay partitioning number increases. More over the state responses of the TS fuzzy system (30) with initial condition $(3,-10)$ for the time delay upper bound $\tau=h=1.8451$ are given in Figure 2. The simulation results reveal that the considered TS fuzzy system is robustly asymptotically stabilizable.

Example 2. Consider the following uncertain fuzzy nonlinear systems together with the ith rule given as follows:

Plant Rule 1 : **IF** $\theta_1(t)=x_1(t)$ is about 0 **THEN**

$$\dot{x}(t)=(A_1(t)+B_1(t)K_1)x(t)+A_{d1}(t)x(t-\tau)+A_{h1}(t)\int_{t-h}^{t}x(s)\,\mathrm{d}s+A_{f1}(t)f_1(x(t),x(t-\tau),t)\,.$$

Plant Rule 2 : **IF** $\theta_1(t)=x_1(t)$ is less than or about -1 **or** greater than or about 1 **THEN**

$$\dot{x}(t)=(A_2(t)+B_2(t)K_2)x(t)+A_{d2}(t)x(t-\tau)+A_{h2}(t)\int_{t-h}^{t}x(s)\,\mathrm{d}s+A_{f2}(t)f_2(x(t),x(t-\tau),t)\,.$$

The membership functions are shown in Figure 3. Then the final output of the uncertain TS fuzzy nonlinear system is inferred as follows:

$$\begin{aligned}\dot{x}(t)=\sum_{i=1}^{r}\eta_i(\theta(t))\sum_{k=1}^{r}\eta_k(\theta(t))\big[&(A_i(t)+B_i(t)K_k)x(t)+A_{di}(t)x(t-\tau)\\&+A_{hi}(t)\int_{t-h}^{t}x(s)\,\mathrm{d}s+A_{fi}(t)f_i(x(t),x(t-\tau),t)\big]\,,\end{aligned}\tag{31}$$

where

$$\eta_1(x_1(t))=\begin{cases}0 & \text{for } |x_1|>1\,,\\ x_1+1 & \text{for } -1\le x_1<0\,,\\ -x_1+1 & \text{for } 0\le x_1\le 1\,,\end{cases}$$

$$\eta_2(x_1(t))=\begin{cases}1 & \text{for } |x_1|>1\,,\\ -x_1 & \text{for } -1\le x_1<0\,,\\ x_1 & \text{for } 0\le x_1\le 1\,,\end{cases}$$

$$A_1=\begin{bmatrix}-2.1 & 0.1\\ 0.1 & 1\end{bmatrix},\ A_2=\begin{bmatrix}-2 & 1\\ 0 & -1.4\end{bmatrix},$$

$$A_{d1}=\begin{bmatrix}-1 & 0\\ -1.1 & -1\end{bmatrix},\ A_{d2}=\begin{bmatrix}-1 & 0.2\\ -1 & -1.5\end{bmatrix},$$

$$A_{h1}=\begin{bmatrix}0.2 & -0.1\\ -0.2 & 0.2\end{bmatrix},\ A_{h2}=\begin{bmatrix}0.6 & 0.1\\ 0.2 & 0.3\end{bmatrix},$$

$$A_{f1} = \begin{bmatrix} 0.1 & 0.2 \\ -0.05 & 0.6 \end{bmatrix}, A_{f2} = \begin{bmatrix} 0.01 & -0.02 \\ 0.1 & -1 \end{bmatrix},$$
$$B_i = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, E_{4i} = \begin{bmatrix} 0.3 \\ 0.3 \end{bmatrix}, G_{1i} = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix},$$
$$G_{2i} = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}, F_i(t) = \begin{bmatrix} \sin(t) & 0 \\ 0 & \cos(t) \end{bmatrix},$$

with $i = 1,2$, and the parameter uncertainties are described as $H_i = 0.1I$, $E_{1i} = E_{2i} = E_{3i} = 0.3I$, $n_1 = 0.5$, $n_2 = 0.4$, $n_3 = 0.45$, $n_4 = -0.35$, $n_5 = -0.26$, $n_6 = 0.33$, $n_7 = 0.1$, $n_8 = 0.2$, $n_9 = -0.02$, $n_{10} = 0.78$.

Solving LMIs in Theorem 2 via MATLAB LMI toolbox, the allowable upper bounds for the discrete and distributed delays ($h = \tau$) satisfying the LMI in (24) can be calculated for $l = m = 1,2,3,4$ and are given in Table 4. It reveals that the calculated upper bound for $h = \tau$ increases as the partitioning numbers l and m increase. More over, the corresponding stabilizing state feedback controller gain matrices for the obtained upper bounds of $h = \tau$ are presented in Table 4.

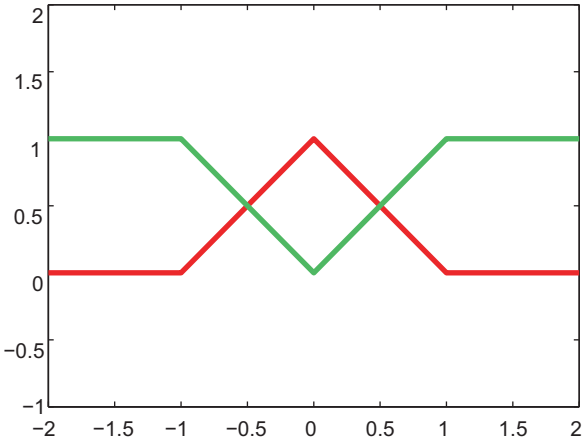

Fig. 3 (colour online). Membership functions for Example 2.

Also the state responses of the open loop TS fuzzy system, that is without the state feedback controller and closed loop TS fuzzy system, that is with the state feedback controller when the initial condition is $(-4,5)$ for

Table 4. Calculated upper bound of h and τ and corresponding gain matrices.

	$h = \tau$	K_1	K_2
Theorem 2 $l = m = 1$	1.2153	$[-1.6201 \quad -7.7032]$	$[-1.7295 \quad -5.6124]$
Theorem 2 $l = m = 2$	1.3974	$[-2.0174 \quad -5.8471]$	$[-2.1754 \quad -4.4592]$
Theorem 2 $l = m = 3$	1.9812	$[-1.6551 \quad -4.4751]$	$[-1.6814 \quad -2.9557]$
Theorem 2 $l = m = 4$	2.3690	$[-0.8692 \quad -7.0779]$	$[-1.0902 \quad -4.7188]$

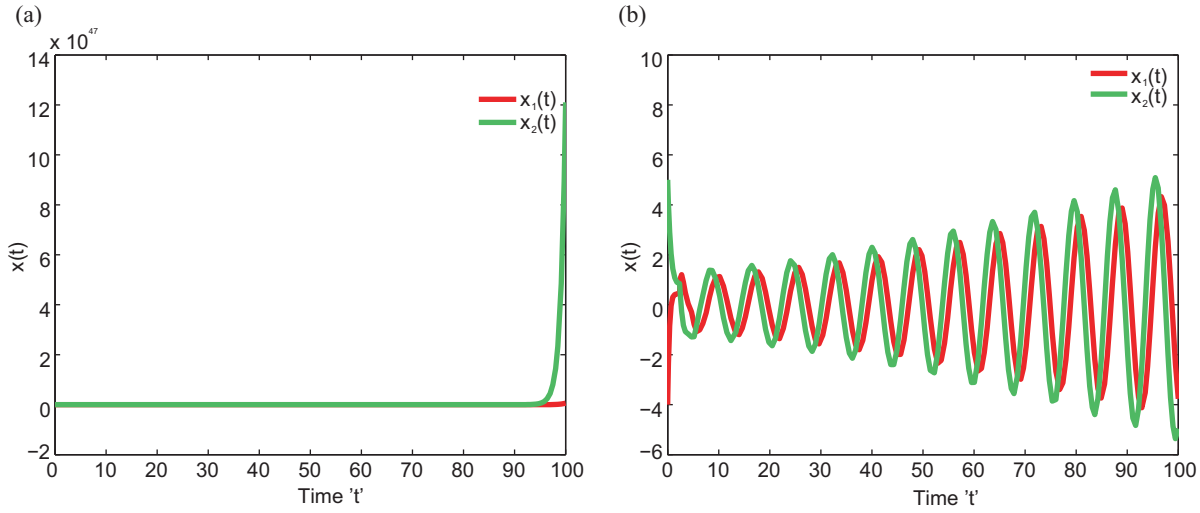

Fig. 4 (colour online). Trajectories of $x(t)$ of the open-loop (control deactivated) fuzzy system in Example 2 for (a) $i = 1$ and (b) $i = 2$.

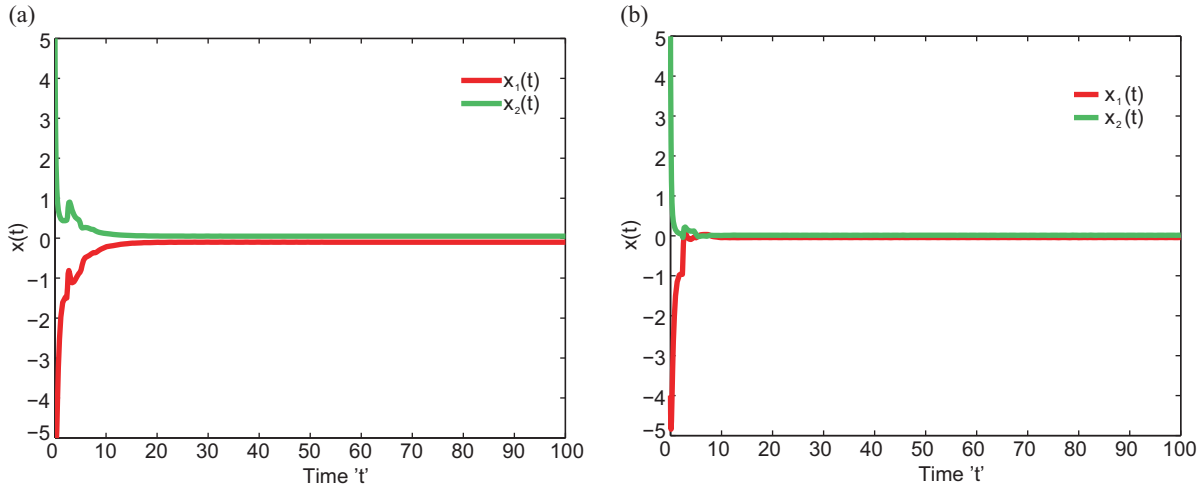

Fig. 5 (colour online). Trajectories of $x(t)$ of closed-loop (control activated) fuzzy system in Example 2 for (a) $i = 1$ and (b) $i = 2$.

the time delay upper bound $\tau = h = 2.3690$ are given in Figures 4 and 5. The simulation results reveal that the obtained controller gain matrices make the considered uncertain fuzzy nonlinear system robustly asymptotically stable.

Remark 1. In the proposed TS fuzzy model, mixed time delays and nonlinear perturbations are taken into consideration in obtaining the required result. It is worth pointing out that the LKF in (8) is constructed with some augmented terms and triple integrals, the reason to choose such a LKF with several positive definite matrices is to utilize the complete information about the discrete and distributed delay occurring in the TS fuzzy model. More precisely, we proposed a new Lyapunov–Krasovskii functional involving some triple integral terms which leads to less conservative results than some existing works. Moreover, the derived conditions for the existence of the robust controller are established in terms of LMIs which can be easily solved by the MATLAB LMI toolbox. Finally, the feasibility and effectiveness of the developed method have been demonstrated through numerical examples. The proposed result in this paper has been compared with some existing works through numerical examples which show that the criteria derived here are considerably less conservative. On the other hand, it can be seen from Table 3 that the upper bound of delay increases compared with the results obtained in [20, 26] and hence our stability criteria are less conservative than some existing ones.

Remark 2. In this paper, the delay fractioning approach is used to derive the delay-dependent stabilization criteria for the considered TS fuzzy system. The decision variables used in this paper depend on the fractioning numbers 'l' and 'm'. That is, if the fractioning numbers increase, the dimensions of the decision variables are increased accordingly with appropriate dimensions. Simply, it is assumed that the matrices which are not explicitly stated are compatible for matrix multiplication. But, it is noted that the computational complexity also increases when the delay partitioning number increases.

5. Conclusion

In this paper, we have investigated the robust stability and stabilization problem for a class of TS fuzzy nonlinear uncertain systems with mixed time delays and norm bounded uncertainties. By constructing a novel Lyapunov–Krasovskii functional involving triple integrals and delay fractioning technique, a new set of sufficient conditions is derived in terms of linear matrix inequalities for ensuring the robust asymptotic stability of the closed loop uncertain TS fuzzy system. The main contribution of this paper is to design state feedback robust controller interms of solutions to a set of LMIs. More precisely, the main results are expressed in the form of LMI constraints and the control gains can be obtained by solving the set of LMIs. Finally, numerical examples with simulation results are provided to demonstrate the effectiveness and applicability of the proposed design techniques.

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