Exponential Stability of the Energy of the Wave Equation with Variable Coefficients and a Boundary Distributed Delay

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In this paper, we consider a wave equation with space variable coefficients. Due to physical considerations, a distributed delay damping is acted on the part of the boundary. Under suitable assumptions, we prove the exponential stability of the energy based on the use of Riemannian geometry method, the perturbed energy argument, and some observability inequalities. From the applications point of view, our results may provide some qualitative analysis and intuition for the researchers in fields such as engineering, biophysics, and mechanics. And the method is rather general and can be adapted to other evolution systems with variable coefficients (e. g. elasticity plates) as well.

Key words: Wave Equation; Variable Coefficients; Stabilization; Distributed Delay.
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1. Introduction

In this work, we investigate the following wave equation with space variable coefficients and a distributed delay on the boundary:

$$\begin{aligned}
&u_{tt}(x,t)+\mathcal{A}u(x,t)=0\,,\ \ (x,t)\in\Omega\times(0,\infty)\,,\\
&u(x,t)=0\,,\ \ (x,t)\in\Gamma_0\times[0,\infty)\,,\\
&\frac{\partial u(x,t)}{\partial \nu_{\mathcal{A}}}+a_0u_t(x,t)\\
&+\int_{\tau_1}^{\tau_2}a(s)u_t(x,t-s)\,\mathrm{d}s=0\,,\ \ (x,t)\in\Gamma_1\times[0,\infty)\,,\\
&u(x,0)=u_0(x)\,,\ \ u_t(x,0)=u_1(x)\,,\ \ x\in\Omega\,,\\
&u_t(x,-t)=f_0(x,-t)\,,\ \ (x,t)\in\Gamma_1\times[0,\tau_2)\,,
\end{aligned}\tag{1}$$

where Ω is a bounded domain of $\mathbb{R}^n$ $(n\geq 2)$ with smooth boundary Γ which consists of two closed and disjoint parts: Γ_0 and Γ_1, with Γ_0 nonempty and $\Gamma_0\cup\Gamma_1=\Gamma$; $\mathcal{A}$ is the operator defined by

$$\mathcal{A}u:=-\mathrm{div}(A\nabla u)=-\sum_{i,j=1}^{n}\frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_j}\right)$$

for $A(x)=\big(a_{ij}(x)\big)$, $a_{ij}=a_{ji}\in C^\infty$, and satisfying

$$\begin{aligned}
&\sum_{i,j=1}^{n}a_{ij}(x)\xi_i\xi_j\geq\alpha\sum_{i=1}^{n}\xi_i^2\,,\ \ x\in\mathbb{R}^n\,,\\
&0\neq\xi=\big(\xi_1,\xi_2,\dots\xi_n\big)\in\mathbb{R}^n
\end{aligned}$$

for positive constant α; $\frac{\partial u}{\partial \nu_{\mathcal{A}}}=\sum_{i=1}\big(\sum_{j=1}^{n}a_{ij}(x)\cdot\frac{\partial u}{\partial x_j}\big)\nu_i=(A(x)\nabla u)\cdot\nu$ is the co-normal derivative and $\nu=(\nu_1,\nu_2,\dots\nu_n)$ is the unit outward normal on Γ; a_0 is a positive constant, τ_1 and τ_2 are two real numbers with $0\leq\tau_1<\tau_2$, $a:[\tau_1,\tau_2]\to\mathbb{R}$ is a L^∞ function, $a\geq 0$ a. e.; and the initial datum u_0, u_1, f_0 are given functions belonging to suitable spaces.

The physical applications of the above system is related to the problem of control and suppression in practical applications. A distributed delay acts on the boundary Γ_1, which describes that the rate of change depends upon its past history in a physical or biology system. The coefficients matrix $A(x)$ is related to the material in applications.

The problems of observation, control, and stabilization for the wave equations without delay (i. e., $a(s)\equiv 0$) have been widely studied. In the case where the coefficients are constants (i. e., $\mathcal{A}=-\Delta$), energy decay rates were obtained by [1 – 12] and many other papers. For the case of variable coefficients, the uniform (boundary or internal) stabilization problems have been studied by several authors, by using or extending the Riemannian geometrical method which was introduced by Yao in [13] for the exact controllability of wave equations. For example, Feng and Feng [14] used this method to study the exponential

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decay problem of (1) for the case $a(s) \equiv 0$ by introducing a proper Riemannian manifold. In [15], they extended the results of Zuazua [12] to the variable coefficients case by using the Riemannian geometry method and the integral inequality introduced in [3]. Cavalcanti et al. [16] combined this method with other techniques to establish the uniform stabilization for the damped Cauchy–Ventcel problem by considering a nonlinear feedback and a localized frictional dissipation acting on the system. Recently, Nicaise and Pignotti [17] proved the uniform stabilization for a wave equation with variable coefficients in principal part and memory conditions on the boundary based on the use of differential geometry argument, on the multiplier method, and the introduction of suitable Lyapounov functionals. We refer the readers to [18 – 22] for recent contributions in this direction.

In recent years, the control of partial differential equations (PDEs) with time delay effects has become an active area of research, see for instance [23 – 26, and the references therein]. The presence of delay may be a source of instability. For example, it was proved in [27 – 32] that an arbitrarily small delay may destabilize a system which is uniformly asymptotically stable in the absence of delay unless additional conditions or control terms have been used. In [30], Nicaise and Pignotti examined (1) with $\mathcal{A} = -\Delta$ and the third equation of boundary condition (1) be substituted by

$$\frac{\partial u(x,t)}{\partial \nu} + a_0 u_t(x,t) + a_1 u_t(x,t-\tau) = 0. \qquad (2)$$

Assuming that $0 \le a_1 < a_0$, a stabilization result is given by using a suitable observability estimate and inequalities obtained from Carleman estimates for the wave equation due to Lasiecka et al. in [33]. However, for the opposite case $a_1 \ge a_0$, they were able to construct a sequence of delays for which the corresponding solution is unstable. The same results were obtained for the case when both the damping and the delay act inside the domain (see also [34] for the treatment of this problem in more general abstract form). Recently, Ning and Yan [35] extended the boundary stabilization result of [30] to the variable coefficients case by virtue of the Riemannian geometry method, the energy-perturbed approach, and the multiplier skill. When the boundary condition (2) is replaced by the distributed delay given in the third equation of (1) Nicaise and Pignotti [31] showed an exponential stability result for (1) with $\mathcal{A} = -\Delta$ under the condition $\int_{\tau_1}^{\tau_2} a(s)\,\mathrm{d}s < a_0$.

Motivated by these results, we investigate in this paper problem (1) under suitable assumptions and prove the exponential stability of the solution. Our main contribution is an extension of previous result from [31] to the variable coefficients case and from [35] to the boundary distributed delay case. For our purpose, we introduce the energy functional due to the ideas in [31, 35] and use the Riemannian geometry method and some observability inequalities introduced in [33, 35].

The paper is organized as follows: In Section 2, we present some assumptions and state the main result. The exponential stability result is proved in Section 3.

2. Preliminaries and Main Result

In this section, we present some assumptions and state the main result. We use the standard Lebesgue space $L^2(\Omega)$ and the Sobolev space $H^1(\Omega)$, $H^2(\Omega)$ with their usual scalar products and norms. We denote

$$H^1_{\Gamma_0}(\Omega) := \left\{ u \in H^1(\Omega) : u = 0 \text{ on } \Gamma_0 \right\}.$$

To deal with variable coefficients, we introduce some notations and refer the reader to Yao [13] for further understanding these notations. We define

$$G(x) = \left(g_{ij}(x)\right) = A^{-1}(x), \ \forall x \in \mathbb{R}^n$$

as a Riemannian metric generated by the spatial operator. For each $x \in \mathbb{R}^n$, we define the inner product and the norm on the tangent space $\mathbb{R}^n_x = \mathbb{R}^n$ by

$$g(X,Y) = \langle X,Y\rangle_g = \sum_{i,j=1}^{n} g_{ij}(x)\alpha_i\beta_j, \ |X|_g = \langle X,X\rangle_g^{\frac{1}{2}},$$

$$\forall\, X = \sum_{i=1}^{n} \alpha_i \frac{\partial}{\partial x_i}, \ Y = \sum_{i=1}^{n} \beta_i \frac{\partial}{\partial x_i} \in \mathbb{R}^n_x.$$

Then $(\mathbb{R}^n, g)$ is a Riemannian manifold with Riemannian metric g.

Similar as shown in [13], we give the following assumption.

(H1) There exists a C^1 vector field in $(\mathbb{R}^n, g)$ such that

$$\langle \mathcal{D}_X H, X\rangle_g \ge \rho_0 |X|_g^2, \ \forall X \in \mathbb{R}^n_x, \ x \in \Omega, \qquad (3)$$

for some constant $\rho_0 > 0$. Here $\mathcal{D}$ denotes the Levi–Civita connection in the metric g and $\mathcal{D}_X H$ is the covariant derivative of H with respect to X.

Remark 1. Assumption (H1) has been introduced by Yao in [13] to extend the standard identity with multiplier to the case of variable coefficients. In the case of constant coefficients, we can take as H the standard multiplier $m(x) = x - x_0$. We refer to [13] for examples of function H verifying the assumption in the nonconstant case.

As in [31], we assume that

$$\int_{\tau_1}^{\tau_2} a(s)\,\mathrm{d}s < a_0\,, \tag{4}$$

which implies that there exists a positive constant c_0, such that

$$a_0 - \int_{\tau_1}^{\tau_2} a(s)\,\mathrm{d}s - \frac{c_0}{2}(\tau_2 - \tau_1) > 0\,. \tag{5}$$

Let us introduce the function

$$\begin{aligned} &z(x,\rho,t,s) = u_t(x, t-\rho s)\,,\\ &x \in \Gamma_1\,,\ \ \rho \in (0,1)\,,\ \ s \in (\tau_1,\tau_2)\,,\ \ t > 0\,. \end{aligned}$$

Then, problem (1) is equivalent to

$$\begin{aligned} &u_{tt}(x,t) + \mathcal{A}u(x,t) = 0\,,\ \ (x,t) \in \Omega \times (0,\infty)\,,\\ &u(x,t) = 0\,,\ \ (x,t) \in \Gamma_0 \times [0,\infty)\,,\\ &sz_t(x,\rho,t,s) + z_\rho(x,\rho,t,s) = 0\,,\\ &\quad (x,\rho,t,s) \in \Gamma_1 \times (0,1) \times (0,\infty) \times (\tau_1,\tau_2)\,,\\ &\frac{\partial u}{\partial \nu_{\mathcal{A}}} + a_0 u_t + \int_{\tau_1}^{\tau_2} a(s) z(x,1,t,s)\,\mathrm{d}s = 0\,,\\ &\quad (x,t) \in \Gamma_1 \times [0,\infty)\,,\\ &u(x,0) = u_0(x)\,,\ \ u_t(x,0) = u_1(x)\,,\ \ x \in \Omega\,,\\ &z(x,\rho,0,s) = f_0(x,\rho,s)\,,\\ &\quad (x,\rho,s) \in \Gamma_1 \times (0,1) \times [0,\tau_2)\,. \end{aligned} \tag{6}$$

We now state, without a proof, a well-posedness result, which can be established by combining the arguments of [17, 31].

Lemma 1. *Let (4) be satisfied and (H1) hold true. Then given $u_0 \in H^1_{\Gamma_0}(\Omega)$, $u_1 \in L^2(\Omega)$, $f_0 \in L^2\big(\Gamma_1 \times (0,1) \times (\tau_1,\tau_2)\big)$, there exists a unique weak solution (u,z) of the problem (6) such that*

$$\begin{aligned} &u \in C\big(\mathbb{R}^+, H^1_{\Gamma_0}(\Omega)\big) \cap C^1\big(\mathbb{R}^+, L^2(\Omega)\big)\,,\\ &z \in C\Big(\mathbb{R}^+, L^2\big(\Gamma_1 \times (0,1) \times (\tau_1,\tau_2)\big)\Big)\,. \end{aligned}$$

Inspired by [31, 35], we define the energy functional as

$$\begin{aligned} E(t) := \frac{1}{2}\int_\Omega \Big[u_t^2(x,t) + \sum_{i,j=1}^n a_{ij}(x)\frac{\partial u(x,t)}{\partial x_i}\frac{\partial u(x,t)}{\partial x_j}\Big]\,\mathrm{d}x\\ + \frac{1}{2}\int_{\Gamma_1}\int_{\tau_1}^{\tau_2} s[a(s) + c_0]\int_0^1 u_t^2(x, t-\rho s)\,\mathrm{d}\rho\,\mathrm{d}s\,\mathrm{d}\Gamma\,. \end{aligned} \tag{7}$$

Our main result is the following.

Theorem 1. *Let (4) be satisfied and (H1) hold true such that*

$$\langle H, \nu\rangle \le 0 \ \text{for}\ x \in \Gamma_0\,. \tag{8}$$

Then there exist two positive constants K, k such that, for any solution of problem (1), the energy satisfies

$$E(t) \le K\,\mathrm{e}^{-kt}E(0)\,,\ \ \forall t \ge 0\,. \tag{9}$$

3. Exponential Stability

For $f \in C^1(\overline{\Omega})$, we define the gradient $\nabla_g f$ of f in the Riemannian metric g, via the Riesz representation theorem, by $X(f) = \langle \nabla_g f, X\rangle_g$, where X is any vector field on $(\mathbb{R}^n, g)$. The following lemma provides further relations between the standard dot metric $\langle\cdot,\cdot\rangle$ and the Riemannian metric $\langle\cdot,\cdot\rangle_g$.

Lemma 2. *([13, Lemma 2.1]) Let $x = (x_1,\ldots,x_n)$ be the natural coordinate system in $\mathbb{R}^n$, $f, h \in C^1(\overline{\Omega})$, and H, X be vector fields. Then*

$$\langle H(x), A(x)X(x)\rangle_g = \langle H(x), X(x)\rangle\,,\ \ x \in \mathbb{R}^n\,, \tag{10}$$

$$\nabla_g f = A(x)\nabla f\,,\ \ x \in \mathbb{R}^n\,, \tag{11}$$

$$\nabla_g f(h) = \langle \nabla_g f, \nabla_g h\rangle_g = \langle \nabla f, A(x)\nabla h\rangle\,,\ \ x \in \mathbb{R}^n\,, \tag{12}$$

$$\frac{\partial u}{\partial \nu_{\mathcal{A}}} = \nabla_g u \cdot \nu\,,\ \ x \in \mathbb{R}^n\,, \tag{13}$$

where ∇f is the gradient of f in the standard metric.

Consider the standard energy

$$E_0(t) := \frac{1}{2}\int_\Omega \Big[u_t^2(x,t) + \big|\nabla_g u\big|_g^2\Big]\,\mathrm{d}x\,, \tag{14}$$

then from (7) and Lemma 2, we have

$$\begin{aligned} E(t) = E_0(t) + \frac{1}{2}\int_{\Gamma_1}\int_{\tau_1}^{\tau_2} s\Big[a(s) + c_0\Big]\\ \cdot \int_0^1 u_t^2(x, t-\rho s)\,\mathrm{d}\rho\,\mathrm{d}s\,\mathrm{d}\Gamma\,. \end{aligned} \tag{15}$$

We can prove that the energy $E(t)$ is non-increasing. More precisely, we have the following result.

Lemma 3. *There exists a positive constant C such that for any solution of problem (1), we have*

$$E(S)-E(T)\geq C\Bigg[\int_S^T\int_{\Gamma_1}u_t^2(x,t)\,\mathrm{d}\Gamma\,\mathrm{d}t+\int_S^T\cdot\int_{\Gamma_1}\int_{\tau_1}^{\tau_2}u_t^2(x,t-s)\,\mathrm{d}s\,\mathrm{d}\Gamma\,\mathrm{d}t\Bigg],\tag{16}$$

where $0\leq S\leq T$.

Proof. Differentiating (7), we obtain

$$E'(t)=\int_\Omega\left[u_tu_{tt}+\langle A\nabla u,\nabla u_t\rangle\right]\mathrm{d}x+\int_{\Gamma_1}\int_{\tau_1}^{\tau_2}s\left[a(s)+c_0\right]\cdot\int_0^1u_t(x,t-\rho s)u_{tt}(x,t-\rho s)\,\mathrm{d}\rho\,\mathrm{d}s\,\mathrm{d}\Gamma.$$

Applying Greens formula, referring to the fact that

$$-su_t(x,t-\rho s)=u_\rho(x,t-\rho s),$$
$$s^2u_{tt}(x,t-\rho s)=u_{\rho\rho}(x,t-\rho s),$$

integrating by parts, and using Cauchy–Schwarz's inequality, we arrive at (see [31] for details)

$$\begin{aligned}E'(t)&=-a_0\int_{\Gamma_1}u_t^2(x,t)\,\mathrm{d}\Gamma-\int_{\Gamma_1}u_t(x,t)\left[\int_{\tau_1}^{\tau_2}a(s)\cdot u_t(x,t-s)\,\mathrm{d}s\right]\mathrm{d}\Gamma-\frac12\int_{\Gamma_1}\int_{\tau_1}^{\tau_2}\left[a(s)+c_0\right]\\&\cdot u_t^2(x,t-s)\,\mathrm{d}\Gamma+\frac12\int_{\Gamma_1}u_t^2(x,t)\int_{\tau_1}^{\tau_2}\left[a(s)+c_0\right]\mathrm{d}s\,\mathrm{d}\Gamma\\&\leq-a_0\int_{\Gamma_1}u_t^2(x,t)\,\mathrm{d}\Gamma+\frac12\int_{\Gamma_1}u_t^2(x,t)\left(\int_{\tau_1}^{\tau_2}a(s)\,\mathrm{d}s\right)\mathrm{d}\Gamma\\&+\frac12\int_{\Gamma_1}\int_{\tau_1}^{\tau_2}a(s)u_t^2(x,t-s)\,\mathrm{d}s\,\mathrm{d}\Gamma-\frac12\int_{\Gamma_1}\int_{\tau_1}^{\tau_2}\left[a(s)+c_0\right]\\&\cdot u_t^2(x,t-s)\,\mathrm{d}\Gamma+\frac12\int_{\Gamma_1}u_t^2(x,t)\int_{\tau_1}^{\tau_2}\left[a(s)+c_0\right]\mathrm{d}s\,\mathrm{d}\Gamma\\&=-\left(a_0-\int_{\tau_1}^{\tau_2}a(s)\,\mathrm{d}s-\frac{c_0}{2}(\tau_2-\tau_1)\right)\int_{\Gamma_1}u_t^2(x,t)\,\mathrm{d}\Gamma\\&-\frac{c_0}{2}\int_{\Gamma_1}\int_{\tau_1}^{\tau_2}u_t^2(x,t-s)\,\mathrm{d}s\,\mathrm{d}\Gamma\\&\leq-C\left[\int_{\Gamma_1}u_t^2(x,t)\,\mathrm{d}\Gamma+\int_{\Gamma_1}\int_{\tau_1}^{\tau_2}u_t^2(x,t-s)\,\mathrm{d}s\,\mathrm{d}\Gamma\right],\end{aligned}$$

where

$$C=\min\left\{a_0-\int_{\tau_1}^{\tau_2}a(s)\,\mathrm{d}s-\frac{c_0}{2}(\tau_2-\tau_1),\frac{c_0}{2}\right\}.$$

Then the inequality (16) follows directly from integrating from S to T. □

We need the following observability inequality for our stabilization problem.

Lemma 4. *([33, Theorem 3.5] or [35, Lemma 2.5]) Suppose that all assumptions in Theorem 1 hold true. Let $u(x,t)$ be a solution of problem (1). Then there exits a time $T_0>0$ such that for all times $T>T_0$ and any ε $(0<\varepsilon<\frac12)$, there exits a positive constant $C_{T,\varepsilon}$ (depending on T and ε) for which*

$$E_0(0)\leq C_{T,\varepsilon}\left\{\int_0^T\int_{\Gamma_1}\left[\left(\frac{\partial u}{\partial\nu_{\mathcal{A}}}\right)^2+u_t^2\right]\mathrm{d}\Gamma\,\mathrm{d}t+||u||_{H^{\frac12+\varepsilon}(\Omega\times(0,T))}\right\},\tag{17}$$

where $E_0(t)$ is given in (14).

Then, we can prove a boundary observability estimate for problem (1).

Lemma 5. *Suppose that all assumptions in Theorem 1 hold true. Let $u(x,t)$ be a solution of problem (1). Then there exits a time $T^0>0$ such that for all times $T>T^0$, there exits a positive constant C_T (depending on T) for which*

$$E(0)\leq C_T\int_0^T\int_{\Gamma_1}\left[u_t^2(x,t)+\int_{\tau_1}^{\tau_2}u_t^2(x,t-s)\,\mathrm{d}s\right]\mathrm{d}\Gamma\,\mathrm{d}t.\tag{18}$$

Proof. Let $T_0>0$ be given by Lemma 4. Using the boundary feedback on Γ_1 of the problem (1) in the estimate (17), we have

$$E_0(0)\leq C_{T,\varepsilon}\left\{\int_0^T\int_{\Gamma_1}\left[u_t^2(x,t)+\int_{\tau_1}^{\tau_2}u_t^2(x,t-s)\,\mathrm{d}s\right]\mathrm{d}\Gamma\,\mathrm{d}t+||u||_{H^{\frac12+\varepsilon}(\Omega\times(0,T))}\right\},\tag{19}$$

where we have used the estimations

$$\begin{aligned}\left(\frac{\partial u}{\partial\nu_{\mathcal{A}}}\right)^2&\leq\frac{a_0^2}{2}u_t^2(x,t)+\frac12\left|\int_{\tau_1}^{\tau_2}a(s)u_t(x,t-s)\,\mathrm{d}s\right|^2\\&\leq\frac{a_0^2}{2}u_t^2(x,t)+\frac12\left(\int_{\tau_1}^{\tau_2}a(s)\,\mathrm{d}s\right)\left(\int_{\tau_1}^{\tau_2}a(s)u_t^2(x,t-s)\,\mathrm{d}s\right)^2\end{aligned}$$

and (3).

It follows from (15) that

$$E(t)=E_0(t)+E_B(t),$$

where

$$E_B(t) := \frac{1}{2}\int_{\Gamma_1}\int_{\tau_1}^{\tau_2} s\big[a(s)+c_0\big]\int_0^1 u_t^2(x,t-\rho s)\,\mathrm{d}\rho\,\mathrm{d}s\,\mathrm{d}\Gamma\,.$$

In particular, by a change of variable as in [31], we obtain, for $T \geq \tau_2$,

$$\begin{aligned} E_B(0) &= \frac{1}{2}\int_{\Gamma_1}\int_{\tau_1}^{\tau_2}\big[a(s)+c_0\big]\int_0^s u_t^2(x,t-s)\,\mathrm{d}t\,\mathrm{d}s\,\mathrm{d}\Gamma \\ &\leq C\int_0^T\int_{\Gamma_1}\int_{\tau_1}^{\tau_2} u_t^2(x,t-s)\,\mathrm{d}s\,\mathrm{d}\Gamma\,\mathrm{d}t\,. \end{aligned} \tag{20}$$

Denote by $T^0 := \max\{\tau_2, T_0\}$. Then, from (19) and (20), for any $T > T^0$, we have

$$\begin{aligned} E(0) = E_0(0)+E_B(0) \leq c\Bigg\{ &\int_0^T\int_{\Gamma_1}\Bigg[u_t^2(x,t) \\ &+\int_{\tau_1}^{\tau_2} u_t^2(x,t-s)\,\mathrm{d}s\Bigg]\mathrm{d}\Gamma\,\mathrm{d}t + ||u||_{H^{\frac{1}{2}+\varepsilon}(\Omega\times(0,T))}\Bigg\} \end{aligned}$$

for a suitable positive constant c depending on T.

To obtain (18), we need to absorb the lower order term $||u||_{H^{\frac{1}{2}+\varepsilon}(\Omega\times(0,T))}$. This can be done applying a compactness-uniqueness argument analogously to [30, Proposition 3.2] or [35, Lemma 2.6]. □

Now, we are ready to complete the proof of the exponential stability result.

Proof of Theorem 1. Let $T^0 > 0$ be given by Lemma 5. Then it follows from (18) and (16) that, for $T > T^0$,

$$\begin{aligned} E(0) &\leq C_T\int_0^T\int_{\Gamma_1}\Bigg[u_t^2(x,t)+\int_{\tau_1}^{\tau_2} u_t^2(x,t-s)\,\mathrm{d}s\Bigg]\mathrm{d}\Gamma\,\mathrm{d}t \\ &\leq C_T C^{-1}(E(0)-E(T))\,. \end{aligned}$$

Then

$$E(T) \leq \tilde{C}E(0)\,,$$

where $\tilde{C} = \frac{C_T C^{-1}-1}{C_T C^{-1}}$. Replacing $u(t)$ by $u(kT+t)$ for $k = 1, 2, \ldots$, we get

$$E\big((k+1)T\big) \leq \tilde{C}E(kT) \leq \tilde{C}^{k+1}E(0)\,. \tag{21}$$

Since $\tilde{C} < 1$, problem (1) is invariant by translation and the energy $E(t)$ is non-increasing, (21) yields (9). □

4. Concluding Remarks

In this paper, we have investigated a wave equation with space variable coefficients in a bounded domain, which is related to the problem of control and suppression in practical applications. A distributed delay damping is acted on the part of the boundary, which describes that the rate of change depends upon its past history in a physical or biology system. Under suitable assumptions, we prove the exponential stability of the energy based on the use of the perturbed energy argument due to the ideas in [31, 35], Riemannian geometry method, and some observability inequalities introduced in [33, 35].

From the applications point of view, our results may provide some qualitative analysis and intuition for the researchers in fields such as engineering, biophysics, and mechanics. And the method is rather general and can be adapted to other evolution systems with variable coefficients (e. g. elasticity plates) as well.

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