New Rational Homoclinic Solution and Rogue Wave Solution for the Coupled Nonlinear Schrödinger Equation

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In this work, the rational homoclinic solution (rogue wave solution) can be obtained via the classical homoclinic solution for the nonlinear Schrödinger (NLS) equation and the coupled nonlinear Schrödinger (CNLS) equation, respectively. This is a new way for generating rogue wave comparing with direct constructing method and Darboux dressing technique.

Key words: Coupled Nonlinear Schrödinger Equation; Homoclinic Test; Homoclinic Breather Solution; Rogue Wave.

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1. Introduction

It is well known that the existence of homoclinic and heteroclinic orbits is very important to study the chaotic behavior of nonlinear partial differential equations. In recent years, exact homoclinic and heteroclinic solutions were reported for some nonlinear evolution equations such as the Schrödinger equation [1 – 4], the sine-Gordon equation [5], the coupled Schrödinger equation [6], and the coupled Schrödinger–Boussinesq equation [7]. As a relatively new discovery, rogue waves, as a special type of solitary waves, have been triggered much interest in various physical branches. A rogue wave is a kind of wave that seems abnormal and was first observed in the deep ocean. It always has two to three times amplitude higher than its surrounding waves and generally forms in a short time for which people think that it comes from nowhere. So rogue waves have been the subject of intensive research in oceanography, optical fibres, superfluids, Bose–Einstein condensates, optical cavities, financial markets, and related fields. Exact rogue wave solutions were obtained by using the Darboux dressing technique or the Hirota bilinear method in various integrable equations such as Hirota equation, Sasa–Satsuma equation, Davey–Stewartson equation, coupled Gross–Pitaevskii equation, coupled nonlinear Schrödinger (NLS), Maxwell–Bloch equation, coupled Schrödinger–Boussinesq equation, and so on [8 – 16]. In this work, we consider the coupled Schrödinger equation

$$
\begin{aligned}
iq_{1t} &= q_{1xx} + 2\left[|q_1|^2 + |q_2|^2 - \omega_1^2\right] q_1\,,\\
iq_{2t} &= q_{2xx} + 2\left[|q_1|^2 + |q_2|^2 - \omega_2^2\right] q_2\,.
\end{aligned}
\tag{1}
$$

This is usually called the Manakov model [17], where ω_1, ω_2 are real constants, q_1, q_2 are slowly varying envelopes of the two interacting optical modes, and the variables x and t are the normalized distance and time. The coupled Schrödinger equation constitute an important model which can be used to describe many kinds of nonlinear phenomena or mechanisms in the fields of physics, optical fibers, electric communication, and other engineering sciences [18]. Equation (1) can be exactly solved by the method of the inverse scattering transform [17]. The following properties have been researched: infinitely many local conservation laws, an infinite-dimensional algebra of noncommutative symmetries [19], Lax pair based on the Bäcklund transformation [20], bilinear method, analytical bright multi-soliton solutions, Darboux transformation, and so on [21 – 24]. Recently, various rogue waves from multi-dimensional equations have been investigated and a series of amazing results have been obtained. Multi-rogue waves and rational solutions of the coupled nonlinear Schrödinger (CNLS) equations were obtained by using the iterative algorithm of the Darboux transformation [25]. Three kinds of nonlinear rogue wave propagations have been derived for a $(3+1)$-dimensional nonlinear Schrödinger (NLS) with time-varying coefficients and a harmonic potential by using the similarity transformation and a direct ansätz [26]. The rogue wave solution for the coupled cubic–quintic NLS equations in nonlinear optics was constructed by the Darboux matrix method [27], and a few relevant recent works were reported in the area of NLS equations [28 – 32]. In this work, the homoclinic breather solution is obtained using the homoclinic test approach, and a new rogue wave solution is constructed by taking the limit of period of the homoclinic breather solution approaching infinite. This is a new method that no one has used so far.

2. Homoclinic Breather and Rogue Wave Solution for the NLS Equation

In this section, we firstly consider the NLS equation

$$\mathrm{i}u_t + u_{xx} + |u|^2 u = 0\,, \tag{2}$$

where i is the imaginary unit and $u=u(x,t)$ a complex-valued function of two real variables x, t. This equation describes the evolution of modulations of dispersive waves with weak nonlinearity. It occurs in various area of physics including nonlinear optics, plasma physics, superconductivity, and quantum mechanics. Let

$$u(x,t) = a\mathrm{e}^{\mathrm{i}a^2 t} q(x,t)\,.$$

Equation (2) can be reduced to the form

$$q_t + q_{xx} + a^2(|q|^2 - 1)q = 0\,, \tag{3}$$

where a is a constant and $a \neq 0$.

In [2] and [3], the homoclinic solution for (2) is

$$u = a\,\mathrm{e}^{\mathrm{i}a^2 t}\frac{1 + b_1\cos(px)\,\mathrm{e}^{\Omega t+\gamma} + b_2\,\mathrm{e}^{2\Omega t + 2\gamma}}{1 + b_3\cos(px)\,\mathrm{e}^{\Omega t+\gamma} + b_4\,\mathrm{e}^{2\Omega t + 2\gamma}}\,. \tag{4}$$

The relations among the parameters are as follows:

$$\begin{aligned} b_1 &= \frac{\mathrm{i}\Omega + p^2}{\mathrm{i}\Omega - p^2}b_3\,,\quad b_2 = \left(\frac{\mathrm{i}\Omega + p^2}{\mathrm{i}\Omega - p^2}\right)^2 b_4\,,\\ b_3^2 &= \frac{4\Omega^2}{\Omega^2 + p^4}b_4\,,\quad \Omega^2 = p^2(2a^2 - p^2)\,,\quad p^2 < 2a^2\,. \end{aligned} \tag{5}$$

Equation (4) can be rewritten as

$$u = a\,\mathrm{e}^{\mathrm{i}a^2 t}\frac{2\sqrt{b_2}\cosh(\Omega t + \gamma + \theta) + b_1\cos(px)}{2\sqrt{b_4}\cosh(\Omega t + \gamma + \theta_1) + b_3\cos(px)}\,, \tag{6}$$

where $\theta = \ln\sqrt{b_2}$, $\theta_1 = \ln\sqrt{b_4}$. This is a solution of Abs type [15]. The trajectory of this solution is defined explicitly by $t = -\frac{\gamma+\theta}{\Omega}$. That is, this solution evolves periodically along the straight line parallel to the x-axis. So this solution is Akhmediev breather (space periodic breather solutions) as well.

Let $b_4 = 1$, $\gamma = 0$, substitute $\Omega = \pm p\sqrt{2a^2 - p^2}$ into (5) and (4), thus we get

$$u = a\,\mathrm{e}^{\mathrm{i}a^2 t}\frac{A}{B}\,, \tag{7}$$

where

$$\begin{aligned} A = {} & 1 - 2\left(\frac{\pm\mathrm{i}p\sqrt{2a^2 - p^2} + p^2}{\pm\mathrm{i}p\sqrt{2a^2 - p^2} - p^2}\right)\sqrt{\frac{p^2(2a^2 - p^2)}{p^4 + p^2(2a^2 - p^2)}}\\ & \cdot\cos(px)\,\mathrm{e}^{\pm p\sqrt{2a^2 - p^2}t}\\ & + \left(\frac{\pm\mathrm{i}p\sqrt{2a^2 - p^2} - p^2}{\pm\mathrm{i}p\sqrt{2a^2 - p^2} + p^2}\right)^2 \mathrm{e}^{\pm 2p\sqrt{2a^2 - p^2}t}\,,\\ B = {} & 1 - 2\sqrt{\frac{p^2(2a^2 - p^2)}{p^4 + p^2(2a^2 - p^2)}}\cos(px)\,\mathrm{e}^{\pm p\sqrt{2a^2 - p^2}t}\\ & + \mathrm{e}^{\pm 2p\sqrt{2a^2 - p^2}t}\,. \end{aligned}$$

We take the limit $p \to 0$ in (7) (period $\frac{2\pi}{p} \to \infty$), then we can obtain the rational solution

$$u(x,t) = a\,\mathrm{e}^{\mathrm{i}a^2 t}\frac{-3 - 8\mathrm{i}a^2 t + 4a^4 t^2 + 2a^2 x^2}{1 + 4a^4 t^2 + 2a^2 x^2}\,. \tag{8}$$

The solution represented by (8) is a homoclinic solution with $u(x,t) \to a\,\mathrm{e}^{\mathrm{i}a^2 t}$ as $t \to \pm\infty$.

A typical spatiotemporal structure of the rogue wave is shown in Figure 1. The maximum amplitude of the rogue wave solution $|u(x,t)|$ occurs at point $(0,0)$ and is equal to 3. The minimum amplitude of $|u(x,t)|$ occurs at two points $(x = \pm\frac{\sqrt{6}}{2}, t = 0)$ and is equal to 0.

3. Homoclinic Breather and Rogue Wave Solution for the CNLS Equation

In [6], the homoclinic solution for (1) is

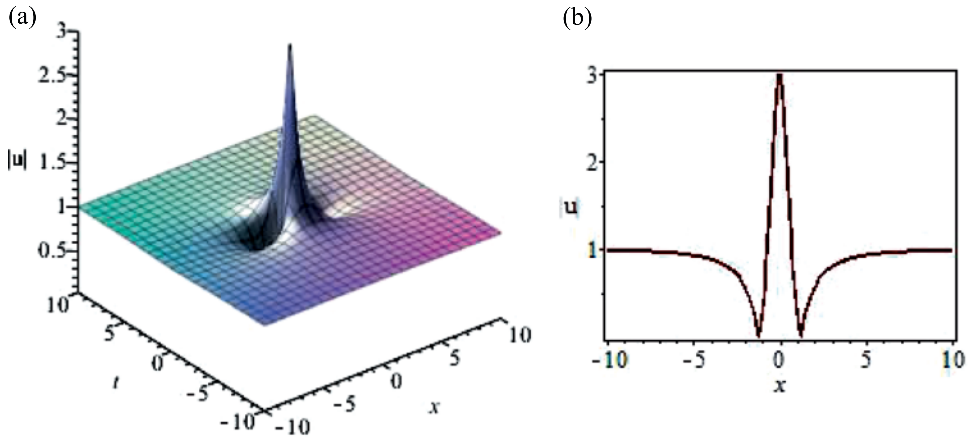

Fig. 1 (colour online). (a) Spatiotemporal structure of $|u(x,t)|$ as $a = 1$ in (8). (b) Plot of the function $|u(x,0)|$.

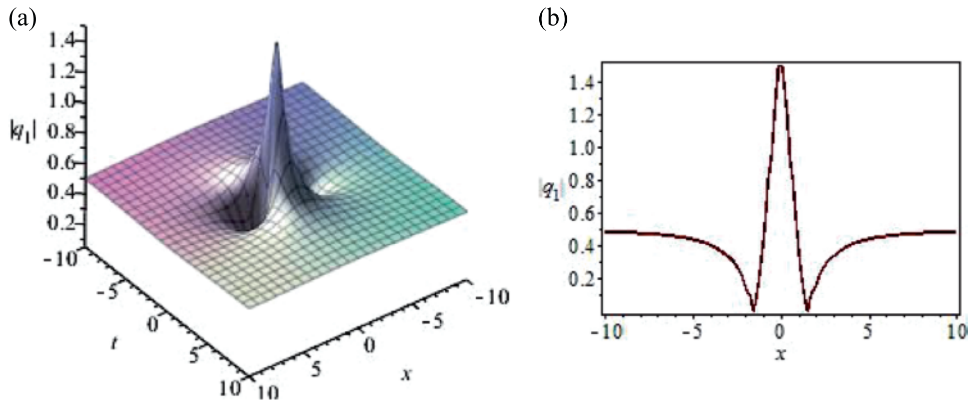

Fig. 2 (colour online). (a) Behaviour of $|q_1|$ in (12). (b) Rogue wave variation in $x - |q_1|$ plane.

$$q_1 = c_1\,\mathrm{e}^{-\mathrm{i}\theta_1(t)}\,\frac{1 + b_1\cos(px)\,\mathrm{e}^{\Omega+\gamma} + b_2\,\mathrm{e}^{2\Omega+2\gamma}}{1 + b_5\cos(px)\,\mathrm{e}^{\Omega+\gamma} + b_6\,\mathrm{e}^{2\Omega+2\gamma}}\,,$$
$$q_2 = c_2\,\mathrm{e}^{-\mathrm{i}\theta_2(t)}\,\frac{1 + b_3\cos(px)\,\mathrm{e}^{\Omega+\gamma} + b_4\,\mathrm{e}^{2\Omega+2\gamma}}{1 + b_5\cos(px)\,\mathrm{e}^{\Omega+\gamma} + b_6\,\mathrm{e}^{2\Omega+2\gamma}}\,, \tag{9}$$

where

$$\theta_1(t) = 2(c_1^2 + c_2^2 - \omega_1^2)t - \gamma,$$
$$\theta_2(t) = 2(c_1^2 + c_2^2 - \omega_2^2)t - \gamma.$$

The relations between the parameters are as follows:

$$b_1 = b_3 = \frac{\mathrm{i}\Omega - p^2}{\mathrm{i}\Omega + p^2} b_5\,, \quad b_2 = b_4 = \left(\frac{\mathrm{i}\Omega - p^2}{\mathrm{i}\Omega + p^2}\right)^2 b_6\,,$$
$$b_5^2 = \frac{4\Omega^2}{\Omega^2 + p^4} b_6\,, \quad \Omega^2 = p^2\big(4(c_1^2 + c_2^2) - p^2\big)\,, \tag{10}$$
$$p^2 < 4\big(c_1^2 + c_2^2\big)\,.$$

From $\Omega^2 > 0$, we have $\Omega = \pm p\sqrt{4(c_1^2 + c_2^2) - p^2}$.

So, we get

$$q_1 = c_1\,\mathrm{e}^{-\mathrm{i}\theta_1(t)}\,\frac{1 + b_1\cos(px)\,\mathrm{e}^{\Omega+\gamma} + b_2\,\mathrm{e}^{2\Omega+2\gamma}}{1 + b_5\cos(px)\,\mathrm{e}^{\Omega+\gamma} + b_6\,\mathrm{e}^{2\Omega+2\gamma}}\,,$$
$$q_2 = c_2\,\mathrm{e}^{-\mathrm{i}\theta_2(t)}\,\frac{1 + b_3\cos(px)\,\mathrm{e}^{\Omega+\gamma} + b_4\,\mathrm{e}^{2\Omega+2\gamma}}{1 + b_5\cos(px)\,\mathrm{e}^{\Omega+\gamma} + b_6\,\mathrm{e}^{2\Omega+2\gamma}}\,.$$

Being similar to the way of dealing with NLS equation in above, we note that q_1, q_2 are also solutions of Abs type. The trajectory of these solutions are defined explicitly by $t = -\frac{\gamma}{\Omega}$. That is, these solutions evolve periodically along a straight line parallel to the x-axis.

Setting $b_6 = 1$, $\gamma = 0$ and substituting $\Omega = \pm p\sqrt{4(c_1^2 + c_2^2) - p^2}$ in (9) and (10), we get

$$q_1 = c_1\,\mathrm{e}^{-\mathrm{i}\theta_1(t)}\frac{A}{B}\,, \quad q_2 = c_2\,\mathrm{e}^{-\mathrm{i}\theta_2(t)}\frac{C}{D}\,, \tag{11}$$

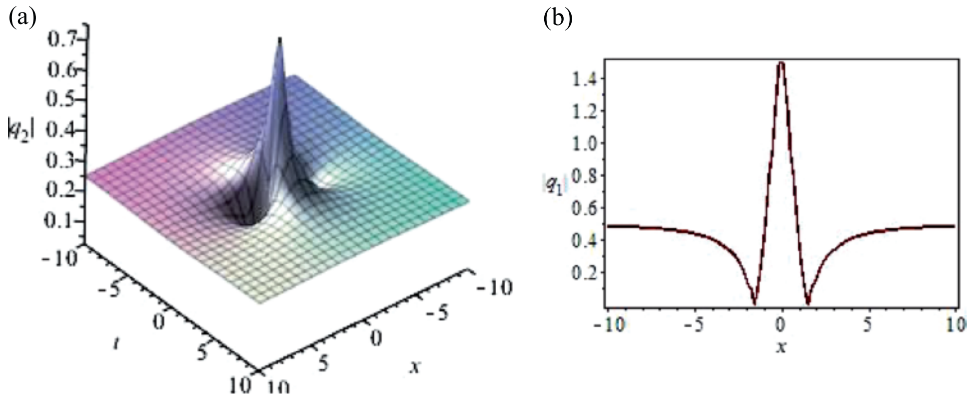

Fig. 3 (colour online). (a) Behaviour of $|q_2|$ in (12). (b) Rogue wave variation in $x-|q_2|$ plane.

where

$$A=C=1-2\left(\frac{\pm \mathrm{i}p\sqrt{\left(4(c_1^2+c_2^2)-p^2\right)}-p^2}{\pm \mathrm{i}p\sqrt{\left(4(c_1^2+c_2^2)-p^2\right)}+p^2}\right)\cdot\sqrt{\frac{p^2\left(4(c_1^2+c_2^2)-p^2\right)}{p^4+p^2\left(4(c_1^2+c_2^2)-p^2\right)}}\cos(px)\,\mathrm{e}^{\pm p\sqrt{4(c_1^2+c_2^2)-p^2}t}+\left(\frac{\pm \mathrm{i}p\sqrt{\left(4(c_1^2+c_2^2)-p^2\right)}-p^2}{\pm \mathrm{i}p\sqrt{\left(4(c_1^2+c_2^2)-p^2\right)}+p^2}\right)^2\mathrm{e}^{\pm 2p\sqrt{4(c_1^2+c_2^2)-p^2}t},$$

$$B=D=1-2\sqrt{\frac{p^2\left(4(c_1^2+c_2^2)-p^2\right)}{p^4+p^2\left(4(c_1^2+c_2^2)-p^2\right)}}\cos(px)\cdot\mathrm{e}^{\pm p\sqrt{4(c_1^2+c_2^2)-p^2}t}+\mathrm{e}^{\pm 2p\sqrt{4(c_1^2+c_2^2)-p^2}t}.$$

Taking the limit $p\to 0$ in (11) (period $\frac{2\pi}{p}\to\infty$), then we can obtain the following rational solutions:

$$q_1=c_1\,\mathrm{e}^{-\mathrm{i}\theta_1(t)}\cdot\frac{-3+\left(16\mathrm{i}t+4x^2\right)\left(c_1^2+c_2^2\right)+16t^2\left(c_1^2+c_2^2\right)^2}{1+4x^2\left(c_1^2+c_2^2\right)+16t^2\left(c_1^2+c_2^2\right)^2},$$

$$q_2=c_2\,\mathrm{e}^{-\mathrm{i}\theta_2(t)}\cdot\frac{-3+\left(16\mathrm{i}t+4x^2\right)\left(c_1^2+c_2^2\right)+16t^2\left(c_1^2+c_2^2\right)^2}{1+4x^2\left(c_1^2+c_2^2\right)+16t^2\left(c_1^2+c_2^2\right)^2}. \tag{12}$$

Giving some special parameters, $c_1=\frac{1}{2}$, $c_2=\frac{1}{4}$, $\omega_1=\frac{1}{2}$, $\omega_2=\frac{1}{4}$, the spatiotemporal structure of the rogue wave can be exhibited as follows.

Typical spatiotemporal structures of $|q_1|$ and $|q_2|$ are shown in Figures 2 and 3. The rogue waves of $|q_1|$ and $|q_2|$ are first-order rogue waves and are concentrated around $(0,0)$. We can observe the changes of $|q_1(x,0)|$ and $|q_2(x,0)|$ in the direction of the x-axes and see that the maximal amplitudes of $|q_1|$ and $|q_2|$ are also at $(0,0)$. They are equal to 1.4 and 0.7, respectively. The minimum amplitudes of $|q_1|$ and $|q_2|$ occur at two points $(x=\pm\frac{\sqrt{6}}{2}, t=0)$ and are both equal to 0. Moreover, (q_1,q_2) is also the rational homoclinic solution and homoclinic to the fixed cycle $(c_1\,\mathrm{e}^{-\mathrm{i}\theta_1(t)}, c_2\,\mathrm{e}^{-\mathrm{i}\theta_2(t)})$ as $t\to\infty$ or $x\to\infty$. In fact,

$$(q_1,q_2)\to(c_1\,\mathrm{e}^{-\mathrm{i}\theta_1(t)}, c_2\,\mathrm{e}^{-\mathrm{i}\theta_2(t)}) \text{ as } t\to\pm\infty.$$

(q_1,q_2) therefore is a rational homoclinic solution of (1) as well.

4. Conclusion

In summary, applying the Hirota bilinear form and homoclinic test method to the Schrödinger and coupled Schrödinger equation, homoclinic solutions have been obtained. Moreover, novel rational homoclinic waves for the Schrödinger and coupled Schrödinger equation are obtained via the limit of the period in homoclinic breather wave solution (from homoclinic breather solution to rogue wave solution), respectively. They are the rational solutions and have the rogue wave features. Here we only got the one-order rogue wave solution by this new method, but second-order or other order rogue wave solutions may be obtained via Darboux transformation or other methods. The result obtained in this work shows the complexity of dynamical behavior and the variety of structure for homoclinic solutions of the Schrödinger and coupled Schrödinger equation. The problem that needed to be further studied is whether other types of nonlinear evolution equations have also this kind of homoclinic solutions which can generate a rogue wave solution or not.

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