# Invariance, Conservation Laws, and Exact Solutions of the Nonlinear Cylindrical Fin Equation

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Fins are heat exchange surfaces which are used widely in industry. The partial differential equation arising from heat transfer in a fin of cylindrical shape with temperature dependent thermal diffusivity are studied. The method of multipliers and invariance of the differential equations is employed to obtain conservation laws and perform double reduction.

*Key words:* Nonlinear Cylindrical Fin Equation; Exact Solutions; Conservation Laws.

## 1. Introduction

Fins are extended surfaces used, inter alia, to increase the heat exchange from a hot or cold surface to surrounding areas. Uses of fins include compressors, cooling of computer processors, air-cooled craft engines in air conditioning, etc. The heat transfer by fins of different shapes and profiles with a variety of boundary conditions is described by mathematical models [1]. There have been studies using a number of techniques to discuss the heat transfer through fins of different shapes. For example, [2] discussed the problem

$$\frac{\partial^2\theta}{\partial X^2}+\frac{\partial^2\theta}{\partial Y^2}=0 \tag{1}$$

using separation of variables and a Newton–Raphson method to compute the temperature profiles and heat transfer per fin length. More recently, Pakdemirli and Sahin [3, 4] studied the problem

$$\frac{\partial}{\partial X}\left(k(\theta)\frac{\partial\theta}{\partial X}\right)-N^2 f(X)\theta=\theta_t \tag{2}$$

using the Lie symmetries of the governing partial differential equation. Bokhari et al. [5] further studied the above nonlinear fin equation (2). They considered group theoretic analysis that led to some new exact solutions. In the same series of studies, Moitsheki and Harley [6] considered a two-dimensional pin fin equation with length $L$ and radius $R$ having the form

$$\frac{1}{R}\frac{\partial}{\partial R}\left(Rk(u)\frac{\partial u}{\partial R}\right)+\frac{\partial}{\partial x}\left(k(u)\frac{\partial u}{\partial x}\right)=s(u)\,. \tag{3}$$

Using the Lie symmetry approach, they gave certain solutions of this equation for different cases of $s(u)$. Extending this work in this area, we perform a double reduction of the nonlinear $(2+1)$ fin equation by considering cylindrical fins with nonlinear thermal conductivity and variable heat transfer coefficient described by

$$\operatorname{div}\big(k(u)\operatorname{grad}u\big)-N^2 f(X)u=u_t \tag{4}$$

for which the equivalent cylindrical coordinates of the $(2+1)$ fin equation is given by

$$\begin{aligned}&\frac{1}{R}\frac{\partial}{\partial R}\left(Rk(u)\frac{\partial u}{\partial R}\right)+\frac{1}{R}\frac{\partial}{\partial\theta}\left(\frac{1}{R}k(u)\frac{\partial u}{\partial\theta}\right)\\&+\frac{\partial}{\partial z}\left(k(u)\frac{\partial u}{\partial z}\right)-N^2 f(R)u=u_t\,.\end{aligned} \tag{5}$$

If we assume no variation in axial direction (invariance in the $z$-coordinate), we would ignore the term

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$\frac{\partial}{\partial z}(k(u)\frac{\partial u}{\partial z})$. Then, relabelling the radial coordinate $R$ by $x$ and the angle $\theta$ by $y$, the cylindrical coordinates of the $(2+1)$ fin equation become

$$\frac{1}{x}\frac{\partial}{\partial x}\big(xk(u)u_x\big)+\frac{1}{x}\frac{\partial}{\partial y}\left(\frac{1}{x}k(u)u_y\right) - N^2 f(x)u = u_t \tag{6}$$

so that

$$x^2k(u)u_{xx}+x^2k(u)_u u_x^2+xk(u)u_x+k(u)_u u_y^2 + k(u)u_{yy}-N^2x^2f(x)u-x^2u_t=0\,. \tag{7}$$

## 2. Conservation Laws and Double Reduction

In this section, we derive the conservation laws of the $(2+1)$ fin equation (6) using the method of multipliers and invariance of the differential equation and corresponding conservation under symmetries. The conserved vector $(T^t,T^x,T^y)$ of (7) satisfies the divergence relation

$$D_tT^t+D_xT^x+D_yT^y=Q\Big(x^2k(u)u_{xx}+x^2k(u)_u u_x^2 + xk(u)u_x+k(u)_u u_y^2+k(u)u_{yy}-N^2x^2f(x)u-x^2u_t\Big)\,. \tag{8}$$

Thus,

$$\mathcal{E}_u\left[Q\Big(x^2k(u)u_{xx}+x^2k(u)_u u_x^2+xk(u)u_x + k(u)_u u_y^2+k(u)u_{yy}-N^2x^2f(x)u-x^2u_t\Big)\right]=0\,, \tag{9}$$

where $\mathcal{E}_u$ is the respective Euler–Lagrange operator and $Q$ is called a multiplier. If we suppose $Q$ to be up to second order in derivatives, i.e., $Q=Q(t,x,y,u,u_t,u_x,u_y,u_{tt},u_{xx},u_{yy},u_{tx},u_{ty},u_{xy})$, then application to (9) leads to cumbersome calculations for which the results cannot be presented here. Nevertheless, solving the system, we obtain some nontrivial multipliers $Q$; each one leads to a conserved vector determined by, inter alia, a homotopy operator [7, 8]. We consider the special case $f(x)=c$ where $c$ is an arbitrary constant. In accordance with this choice, we obtain the following forms of $Q$:

$$Q_1=-\frac{\mathrm{e}^{N^2ct}}{x}\,,\quad Q_2=-\frac{\mathrm{e}^{N^2ct}}{x}\ln(x)\,. \tag{10}$$

The corresponding conserved vector of $Q_1$ is given by

$$T^1=(T^t,T^x,T^y) = \left(\mathrm{e}^{N^2t}xu,-\mathrm{e}^{N^2t}xk(u)u_x,\frac{-\mathrm{e}^{N^2t}k(u)u_y}{x}\right), \tag{11}$$

and $Q_2$ is given by

$$T^2=\left(\mathrm{e}^{N^2t}x\ln(x)u,\int_0^1\mathrm{e}^{N^2t}\Big(-x\lambda\ln(x)uk'(u)u_x+k(\lambda u) \cdot(u-x\ln(x)u_x)\Big)\,\mathrm{d}\lambda,\frac{-\mathrm{e}^{N^2t}xk(u)\ln(x)u_y}{x}\right). \tag{12}$$

We perform the double reduction for the particular case $k(u)=\gamma(\alpha u)^{\frac{1}{\alpha}}$ (as an example of a power law nonlinearity), $N^2=1$, and $f(x)=1$.

Equation (7), with the above choice of $k$, $f$, and $N^2$, admits the symmetry generators

$$\begin{aligned}
&\mathbf{X}_1=-\sin(y)\frac{\partial}{\partial x}-\frac{\cos(y)}{x}\frac{\partial}{\partial y}\,,\\
&\mathbf{X}_2=\cos(y)\frac{\partial}{\partial x}-\frac{\sin(y)}{x}\frac{\partial}{\partial y}\,,\\
&\mathbf{X}_3=x\frac{\partial}{\partial x}+2\alpha u\frac{\partial}{\partial u}\,,\ \ \mathbf{X}_4=\mathrm{e}^{\frac{t}{\alpha}}\left(\alpha^2\frac{\partial}{\partial t}-\alpha^2u\frac{\partial}{\partial u}\right),\\
&\mathbf{X}_5=\frac{\partial}{\partial t}\,,\ \ \mathbf{X}_6=\frac{\partial}{\partial y}\,.
\end{aligned} \tag{13}$$

Firstly, we show that $\mathbf{X}_4$ is associated with $T^1$ by using the result

$$\mathbf{X}_4^{[1]}\begin{pmatrix}T^t\\T^x\\T^y\end{pmatrix}-\begin{pmatrix}D_t\tau & D_x\tau & D_y\tau\\ D_t\xi & D_x\xi & D_y\xi\\ D_t\eta & D_x\eta & D_y\eta\end{pmatrix}\begin{pmatrix}T^t\\T^x\\T^y\end{pmatrix} + (D_t\tau+D_x\xi+D_y\eta)\begin{pmatrix}T^t\\T^x\\T^y\end{pmatrix}=0\,. \tag{14}$$

We have

$$\mathbf{X}_4^{[1]}=\mathbf{X}_4-\alpha^2\,\mathrm{e}^{\frac{t}{\alpha}}u_x\frac{\partial}{\partial u_x}-\alpha^2\,\mathrm{e}^{\frac{t}{\alpha}}u_y\frac{\partial}{\partial u_y} - \left(\alpha u\,\mathrm{e}^{\frac{t}{\alpha}}+\left(\alpha^2\,\mathrm{e}^{\frac{t}{\alpha}}-\alpha\,\mathrm{e}^{\frac{t}{\alpha}}u_t\right)\frac{\partial}{\partial u_t}\right). \tag{15}$$

Calculating the above quantities yield

$$\begin{aligned}
&\mathbf{X}_4^{[1]}T^t=0\,,\ \ \mathbf{X}_4^{[1]}T^x=\alpha\,\mathrm{e}^{t-\frac{t}{\alpha}}x(\alpha u)^{\frac{1}{\alpha}}u_x\,,\\
&\mathbf{X}_4^{[1]}T^y=\frac{\alpha\,\mathrm{e}^{t-\frac{t}{\alpha}}x(\alpha u)^{\frac{1}{\alpha}}u_y}{x}\,,
\end{aligned} \tag{16}$$

and we have $\xi=\eta=0$, $\tau=\alpha^2\,\mathrm{e}^{\frac{t}{\alpha}}$, and $\phi=-\alpha^2 u\,\mathrm{e}^{\frac{t}{\alpha}}$. Therefore,

$$\begin{pmatrix} D_t\tau & D_x\tau & D_y\tau \\ D_t\xi & D_x\xi & D_y\xi \\ D_t\eta & D_x\eta & D_y\eta \end{pmatrix} = \begin{pmatrix} \alpha\,\mathrm{e}^{\frac{t}{\alpha}} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{17}$$

and

$$\left(D_t\tau+D_x\xi+D_y\eta\right)\begin{pmatrix} T^t \\ T^x \\ T^y \end{pmatrix} = \alpha\,\mathrm{e}^{\frac{t}{\alpha}}\begin{pmatrix} T^t \\ T^x \\ T^y \end{pmatrix}. \tag{18}$$

Substituting (16), (17), and (18) into (14), we conclude that $\mathbf{X}_4$ is associated with $T^1=(T^t,T^x,T^y)$ with ($Q_1=-\frac{\mathrm{e}^t}{x}$). Thus, we can get a reduced conserved vector by $\mathbf{X}_4$ where $\mathbf{X}_4$ has a canonical form $\mathbf{Y}_4=\frac{\partial}{\partial q}$ when

$$\begin{aligned} \frac{\mathrm{e}^{\frac{-t}{\alpha}}\,\mathrm{d}t}{\alpha^2} &= \frac{\mathrm{d}x}{0} = \frac{\mathrm{d}y}{0} = \frac{\mathrm{e}^{\frac{-t}{\alpha}}\,\mathrm{d}u}{-\alpha^2 u} \\ &= \frac{\mathrm{d}r}{0} = \frac{\mathrm{d}s}{0} = \frac{\mathrm{d}q}{1} = \frac{\mathrm{d}w}{0}. \end{aligned} \tag{19}$$

The invariants of $\mathbf{X}_4$ from (19) are given by

$$\begin{aligned} &b_1=x,\ \ b_2=y,\ \ b_3=\mathrm{e}^t u,\ \ b_4=r, \\ &b_5=s,\ \ b_6+q=-\frac{1}{\alpha}\,\mathrm{e}^{\frac{-t}{\alpha}},\ \ b_7=w, \end{aligned} \tag{20}$$

where $b_4$, $b_5$, $b_6$, and $b_7$ are arbitrary functions all dependent on $b_1$, $b_2$, and $b_3$.

By choosing $b_7=b_3$, $b_5=b_2$, $b_6=0$, and $b_4=b_1$, we obtain the canonical coordinates

$$r=x,\ \ s=y,\ \ q=-\frac{1}{\alpha}\,\mathrm{e}^{\frac{-t}{\alpha}},\ \ w=\mathrm{e}^t u, \tag{21}$$

where $w=w(r,s)$, since $Y_4=\frac{\partial}{\partial q}$.

From (21), the inverse canonical coordinates are given by

$$x=r,\ \ y=s,\ \ t=-\alpha\ln(-\alpha q),\ \ u=\mathrm{e}^{-t}w. \tag{22}$$

In the light of the above similarities, the partial derivatives of $u$ are given by

$$\begin{aligned} &u_x=\mathrm{e}^{-t}w_r,\ \ u_{xx}=\mathrm{e}^{-t}w_{rr},\ \ u_y=\mathrm{e}^{-t}w_s, \\ &u_{yy}=\mathrm{e}^{-t}w_{ss},\ \ u_t=-\mathrm{e}^{-t}w. \end{aligned} \tag{23}$$

Consequently, (7) reduces to

$$\begin{aligned} &r(\alpha w)^{\frac{1}{\alpha}}w_r+r^2(\alpha w)^{\frac{1}{\alpha}-1}w_r^2+r^2(\alpha w)^{\frac{1}{\alpha}}w_{rr} \\ &+(\alpha w)^{\frac{1}{\alpha}-1}w_s^2+(\alpha w)^{\frac{1}{\alpha}}w_{ss}=0. \end{aligned} \tag{24}$$

The inverse $A^{-1}$ is given by

$$A^{-1}=\begin{pmatrix} D_t r & D_t s & D_t q \\ D_x r & D_x s & D_x q \\ D_y r & D_y s & D_y q \end{pmatrix} = \begin{pmatrix} 0 & 0 & \frac{1}{\alpha^2}\,\mathrm{e}^{\frac{-t}{\alpha}} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}. \tag{25}$$

In order to get the reduced conserved form, we use the formula

$$\begin{pmatrix} T^r \\ T^s \\ T^q \end{pmatrix} = J(A^{-1})^{\mathrm{T}}\begin{pmatrix} T^t \\ T^x \\ T^y \end{pmatrix}. \tag{26}$$

We have $J=\det(A)=\alpha^2\,\mathrm{e}^{\frac{t}{\alpha}}$, therefore,

$$\begin{pmatrix} T^r \\ T^s \\ T^q \end{pmatrix} = \alpha^2\,\mathrm{e}^{\frac{t}{\alpha}}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{\alpha^2}\,\mathrm{e}^{\frac{-t}{\alpha}} & 0 & 0 \end{pmatrix}\begin{pmatrix} T^t \\ T^x \\ T^y \end{pmatrix}. \tag{27}$$

Thus, the reduced conserved form is

$$D_r T^r + D_s T^s = 0, \tag{28}$$

where

$$\begin{aligned} T^r &= -\alpha^2 r(\alpha w)^{\frac{1}{\alpha}}w_r, \\ T^s &= \frac{-\alpha^2(\alpha w)^{\frac{1}{\alpha}}}{r}w_s, \\ T^q &= rw. \end{aligned} \tag{29}$$

The reduced conserved form admits the inherited symmetry

$$\widehat{\mathbf{X}}_2=\cos(s)\frac{\partial}{\partial r}-\frac{\sin(s)}{r}\frac{\partial}{\partial s}. \tag{30}$$

This symmetry is associated with the reduced conserved form. To show that this symmetry is associated, from

$$\begin{aligned} \widehat{\mathbf{X}}_2^{[1]} &= \cos(s)\frac{\partial}{\partial r}-\frac{\sin(s)}{r}\frac{\partial}{\partial s}-\frac{\sin(s)}{r^2}w_s\frac{\partial}{\partial w_r} \\ &+\left(\frac{\cos(s)}{r}w_s+\sin(s)w_r\right)\frac{\partial}{\partial w_s}, \end{aligned} \tag{31}$$

we get

$$\begin{aligned} &\widehat{\mathbf{X}}_2^{[1]}\begin{pmatrix} T^r \\ T^s \end{pmatrix}-\begin{pmatrix} D_r\xi^r & D_s\xi^r \\ D_r\xi^s & D_s\xi^s \end{pmatrix}\begin{pmatrix} T^r \\ T^s \end{pmatrix} \\ &+\left(D_r\xi^r+D_s\xi^s\right)\begin{pmatrix} T^r \\ T^s \end{pmatrix}=0. \end{aligned} \tag{32}$$

We transform $\widehat{\mathbf{X}}_2$ to its canonical form $Y = \frac{\partial}{\partial m}$. Thus, the canonical coordinates are

$$n = r\sin(s)\,,\ \ m = r\big(\cos(s)+\sin(s)\big)\,,\ \ v(n) = w\,. \tag{33}$$

Consequently, we have

$$\begin{aligned} A^{-1} &= \begin{pmatrix} D_r n & D_r m \\ D_s n & D_s m \end{pmatrix} \\ &= \begin{pmatrix} \sin(s) & \cos(s)+\sin(s) \\ r\cos(s) & r(-\sin(s)+\cos(s)) \end{pmatrix}. \end{aligned} \tag{34}$$

In accordance with the similarities in (33), the partial derivatives of $w$ are given by

$$\begin{aligned} &w_r = \sin(s)v_n\,,\ \ w_{rr} = \sin^2(s)v_{nn}\,,\ \ w_s = r\cos(s)v_n\,, \\ &w_{ss} = -r\sin(s)v_n + r^2\cos^2(s)v_{nn}\,. \end{aligned} \tag{35}$$

Thus, (24) reduces to

$$(\alpha v)v_{nn} + v_n^2 = 0\,. \tag{36}$$

To obtain the reduced conserved form, we apply the formula

$$\begin{pmatrix} T^n \\ T^m \end{pmatrix} = J(A^{-1})^{\mathrm{T}} \begin{pmatrix} T^r \\ T^s \end{pmatrix}. \tag{37}$$

Thus,

$$\begin{aligned} \begin{pmatrix} T^n \\ T^m \end{pmatrix} &= -r \begin{pmatrix} \sin(s) & r\cos(s) \\ \cos(s)+\sin(s) & r\big(\cos(s)-\sin(s)\big) \end{pmatrix} \\ &\quad \cdot \begin{pmatrix} T^r \\ T^s \end{pmatrix}, \end{aligned} \tag{38}$$

where $J = \det(A) = -r$. Using (35) from which we conclude

$$D_n T^n = 0$$

so that

$$\alpha^2(\alpha v)^{\frac{1}{\alpha}} v_n = C_1\,, \tag{39}$$

where $C_1$ is a constant, $n = x\sin y$, and $v = \mathrm{e}^t u$. Thus,

$$\left(\frac{\alpha^{3+\frac{1}{\alpha}}}{1+\alpha}\right) v^{1+\frac{1}{\alpha}} = C_1 n + C_2\,, \qquad \alpha \neq -1, 0\,,$$

$C_2$ is a constant. Then, we re-cast above in the original coordinates: the exact solution of (7) with $k(u) = (au)^{\frac{1}{\alpha}}$, $f(x) = 1$, and $N^2 = 1$ is then given by

$$u(x,y,t) = \exp(-t)\Big(C_1 x\sin(y) + C_2\Big)^{\frac{\alpha}{1+\alpha}}. \tag{40}$$

## 3. Conclusion

The partial differential equation arising from heat transfer in a fin of cylindrical shape with temperature dependent thermal diffusivity is reduced, and exact solutions are obtained. This was achieved by constructing conservation laws and determining the associated invariances of the underlying differential equations.


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