Solving Steady Flow of a Third-Grade Fluid in a Porous Half Space via Normal and Modified Rational Christov Functions Collocation Method

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The present study is an attempt to find a solution for steady flow of a third-grade fluid by utilizing spectral methods based on rational Christov functions. This problem is described as a nonlinear two-point boundary value problem. The following method tries to solve the problem on the infinite domain without truncating it to a finite domain and transforms the domain of the problem to a finite domain. Researchers in this try to solve the problem by using anew modified rational Christov functions and normal rational Christov function. Finally, the findings of the current study, i. e., proposal methods, numerical out cames and other methods were compared with each other.

Key words: Non-Newtonian Fluids; Third-Grade Fluid; Collocation Method; Rational Christov Functions; Modified Rational Christov Functions; Nonlinear Ordinary Differential Equations.

1. Introduction

The flow of non-Newtonian fluids can be utilized in many industrial and natural problems, many materials, such as, polymer solutions or melts, drilling mud, elastomers, blood, paints, certain oils and greases, as well as many other emulsions are classified as non-Newtonian fluids. The flow of non-Newtonian fluids has several technical application, especially in the oil recovery, paper and textile industries, and composite manufacturing processes. Different researchers and scholars have paid attention to the fluids of the different types in a way that the fluids of second and third-grade have been studied successfully in various types of flow situations forming a subclass of the fluids of the differential type. Boundary layer theories for fluids similar to a second-grade fluid have been formulated by Rajeswari and Rathna, Bhatnagar, Beard and Waiters and Frater. Rajagopal et al. developed a boundary layer approximation for a second-grade fluid [1]. The present study tackles the problem of non-Newtonian fluids of third grade in a porous half space. The modelling of polymeric flow in porous space has essential focus on the numerical simulation of viscoelastic flows in a specific pore geometry model, for example, capillary tubes, undulating tubes, packs of spheres or cylinders [2 – 5]. It is a well established fact that second-grade fluids exhibit the normal stress effect and do not show the shear-thinning and shear-thickening phenomena [6] which many fluids do. However, third-grade fluids [7] are capable of describing such phenomena. Moreover, the equation of motion in a third-grade fluid is more complicated than the corresponding equation in a second-grade fluid [8].

Recently, some researchers approximate the third-grade fluid equations in a porous half space; for example, Hayat et al. [3] and Ahmad [9] by applying the homotopy analysis method, Kazem et al. [4] by applying the radial basis functions collocation method, Parand and Babolghani [5] by applying the modified generalized Laguerre collocation method, and for more study related to third-grade fluid, see [10 – 15].

Moreover, spectral methods have been successfully applied in the approximation of differential boundary value problems defined in unbounded domains. Different spectral methods can be applied to solve problems in unbounded domains. The first approach using orthogonal functions over the unbounded domains were Sinc, Hermite, and Laguerre polynomials [16 – 22]. The second approach is to reformulate

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original problems in unbounded domains to singular problems in bounded domains by variable transformations, and then to use suitable Jacobi polynomials to approximate the resulting singular problems [23–25]. The third approach is to replace an infinite domain with $[-L,L]$ and a semi-infinite interval with $[0,L]$ by choosing L sufficiently large, named as domain truncation [26]. The fourth approach for solving such problems is based on rational approximations. Christov [27] and Boyd [28, 29] developed some spectral methods on unbounded intervals by using mutually orthogonal systems of rational functions. Boyd [29] defined a new spectral basis, named rational Chebyshev functions on the semi-infinite interval, by suitable mapping to the Chebyshev polynomials. Guo et al. [30] proposed and analyzed a set of Legendre rational functions which are mutually orthogonal in $L^2(0,\infty)$. Boyd et al. [31] applied pseudo-spectral methods on a semi-infinite interval and compared it with rational Chebyshev, Laguerre, and mapped Fourier sine [32] methods.

Moreover, Norbert Wiener [33] defined a different set of complex-valued orthogonal rational functions as transforms of Laguerre functions. These were developed further by Higgins [34]. Nearly thirty years ago, Christov invented a complete orthonormal system of functions in $L^2(-\infty,\infty)$ space [27]. In this paper, the researchers have applied the Christov functions collocation method for solving the steady flow of a third-grade fluid in a porous half space.

The rest of the current study is categorized as follows: in Section 2, the flow of a third-grade fluid is discussed in a porous half space. In Section 3, we describe rational Christov functions and then define modified rational Christov functions in Section 4. In Section 5, the aforementioned method is applied to solve the flow of a third-grade fluid. Section 6 shows the approximate solutions and compare them with other findings. In the last section, we give several concluding remarks.

2. Problem Formulation

In this section, the flow of a third-grade fluid is discussed in a porous half space. For unidirectional flow, the generalized the relation [3] is

$$(\nabla p)_x = -\frac{\mu\varphi}{k}\left(1+\frac{\alpha_1}{\mu}\frac{\partial}{\partial t}\right)u. \tag{1}$$

For a second-grade fluid to the following modified Darcy's Law for a third-grade fluid [3]:

$$(\nabla p)_x = -\frac{\varphi}{k}\left[\mu u+\alpha_1\frac{\partial u}{\partial t}+2\beta_3\left(\frac{\partial t}{\partial y}\right)^2 u\right], \tag{2}$$

where u denotes the fluid velocity, μ is the dynamic viscosity, and p is the pressure. k and φ, respectively, represent the permeability and porosity of the porous half space which occupies the region $y>0$; α_1,β_3 are material constants. Now defining the non-dimensional fluid velocity f and the coordinate z as following [3]:

$$z=\frac{V_0}{\nu}y,\ \ f(z)=\frac{u}{V_0},\ \ V_0=u(0),\ \ \nu=\frac{\mu}{\rho}, \tag{3}$$

where ν and V_0 represent the kinematic viscosities. Then the boundary value problem modelling the steady state flow of a third-grade fluid in a porous half space becomes [9]

$$\frac{\mathrm{d}^2 f}{\mathrm{d}z^2}+b_1\left(\frac{\mathrm{d}f}{\mathrm{d}z}\right)^2\frac{\mathrm{d}^2 f}{\mathrm{d}z^2}-b_2 f\left(\frac{\mathrm{d}f}{\mathrm{d}z}\right)^2-b_3 f=0 \tag{4}$$

with boundary conditions

$$f(0)=1,\ \ f(z)\to 0\ \text{ as } z\to\infty, \tag{5}$$

where b_1, b_2, and b_3 are defined as

$$b_1=\frac{6\beta_3 V_0^4}{\mu\nu^2},\ \ b_2=\frac{2\beta_3\varphi V_0^2}{k\mu},\ \ b_3=\frac{\varphi\nu^2}{kV_0^2}. \tag{6}$$

Above parameters are depended:

$$b_2=\frac{b_1 b_3}{3}. \tag{7}$$

3. Rational Christov Functions

The following system

$$\rho_n=\frac{1}{\sqrt{\pi}}\frac{(\mathrm{i}x-1)^n}{(\mathrm{i}x+1)^{n+1}}\ \ n=0,1,2,\dots,\ \ \mathrm{i}=\sqrt{-1}, \tag{8}$$

was introduced by Wiener [33] as Fourier transform of the Laguerre functions (functions of parabolic cylinder). Higgins [34] defined it also for negative indices n and proved its completeness and orthogonality. Christov invented a new system comprising two real-valued subsequences of odd functions S_n and even functions

C_n with asymptotic behaviour x^{-1} and x^{-2}, respectively, namely [27, 35]:

$$S_n = \frac{\rho_n + \rho_{-n-1}}{\mathrm{i}\sqrt{2}}\,, \quad n = 0,1,2\ldots\,, \tag{9}$$

$$C_n = \frac{\rho_n - \rho_{-n-1}}{\sqrt{2}}\,, \quad n = 0,1,2\ldots\,. \tag{10}$$

Both sequences are orthonormal and each member of (9) is orthogonal to all members of (10); each member of (10) is also orthogonal to all members of (9). It is worth mentioning that (9) and (10) can be defined for negative n through the relations [27]

$$S_{-n} = S_{n-1} \text{ and } C_{-n} = -C_{n-1}\,. \tag{11}$$

The functions S_n and C_n can be easily expressed in an explicit way [27]:

$$S_n = \sqrt{\frac{2}{\pi}}\frac{\sum_{k=1}^{n+1} x^{2k-1}(-1)^{n+k}\binom{2n+1}{2k-1}}{(x^2+1)^{n+1}}\,, \tag{12}$$

$$C_n = \sqrt{\frac{2}{\pi}}\frac{\sum_{k=1}^{n+1} x^{2k-2}(-1)^{n+k+1}\binom{2n+1}{2k-2}}{(x^2+1)^{n+1}}\,. \tag{13}$$

4. Modified Rational Christov Functions

The well-known rational Christov functions are an orthonormal system of functions in $L^2(-\infty,\infty)$ space, meaning that the $\{C_n\}_{n=-\infty}^{n=\infty}$ and $\{S_n\}_{n=-\infty}^{n=\infty}$ sequences are orthonormal with respect to the weight function $w(x) = 1$:

$$\begin{aligned} &\int_{-\infty}^{\infty} C_m(x)C_n(x)w(x)\,\mathrm{d}x = \delta_{mn}\,, \\ &\int_{-\infty}^{\infty} S_m(x)S_n(x)w(x)\,\mathrm{d}x = \delta_{mn}\,, \end{aligned} \tag{14}$$

where δ_{mn} is the Kronecker delta function. Each member of S_n is orthogonal to all members of C_n, also, each member of C_n is orthogonal to all members of S_n with respect to the weight function $w(x) = 1$:

$$\int_{-\infty}^{\infty} S_m(x)C_n(x)w(x)\,\mathrm{d}x = 0\,. \tag{15}$$

Now, we define anew orthonormal, the HCS function, which is defined by

$$\mathrm{HCS}_n = \begin{cases} C_k\,, & n = 2k, \\ S_k\,, & n = 2k+1\,, \end{cases} \quad k = 0,1,2,\ldots\,. \tag{16}$$

The HCS function satisfies the following conditions:

$$\begin{aligned} &\mathrm{HCS}_0 = C_0\,,\ \ \mathrm{HCS}_1 = S_0\,, \\ &\mathrm{HCS}_2 = C_1\,,\ \ \mathrm{HCS}_3 = S_1\,, \\ &\vdots \\ &\mathrm{HCS}_{2n} = C_n\,,\ \ \mathrm{HCS}_{2n+1} = S_n\,,\ \ n = 0,1,2,\ldots\,, \end{aligned}$$

with (14)–(16) the HCS functions are orthonormal with respect to the weight function $w(x) = 1$ in the $L^2(-\infty,\infty)$ space, with the orthonormality property

$$\int_{-\infty}^{\infty} \mathrm{HCS}_m(x)\mathrm{HCS}_n(x)w(x)\,\mathrm{d}x = \delta_{mn}\,. \tag{17}$$

4.1. Function Approximation

Any function f in $L^2(-\infty,\infty)$ can be written as

$$f(x) = \sum_{i=0}^{\infty} a_i\phi_i(x)\,, \tag{18}$$

where ϕ_i is the C_n or S_n or HCS_n function. If the infinite series in (18) is truncated with N terms, then it can be written as [36]

$$f(x) \simeq \sum_{i=0}^{N} a_i\phi_i(x)\,, \tag{19}$$

where ϕ_i is the C_n or S_n or HCS_n function.

5. Solving the Problem

In this section, the steady flow of a third-grade fluid has been solved by utilizing the method describe above. We multiply the operator equation (19) by $\frac{x}{(x+1)}$ and further, we construct a function $p(x)$ to satisfy the boundary conditions (5). This function is given by:

$$p(x) = \exp(-Lx)\,, \tag{20}$$

where L is a constant to be determined [37].

Therefore, the approximate solution of $f(x)$ in (4) with boundary conditions (5) is represented by

$$\hat{f}(x) = \exp(-Lx) + \frac{x}{(x+1)}\sum_{i=0}^{N} a_i\phi_i(x)\,, \tag{21}$$

where ϕ_i is the C_n or HCS_n function, and

$$\hat{f}(x) = \exp(-Lx) + \sum_{i=0}^{N} a_i\phi_i(x)\,, \tag{22}$$

where ϕ_i is the S_n function.

We construct the residual function by substituting $f(x)$ by $\widehat{f}(x)$ in (4):

$$\mathrm{Res}(x) = \frac{\mathrm{d}^2\widehat{f}(x)}{\mathrm{d}x^2} + b_1\left(\frac{\mathrm{d}\widehat{f}(x)}{\mathrm{d}x}\right)^2\frac{\mathrm{d}^2\widehat{f}(x)}{\mathrm{d}x^2} - b_2\widehat{f}(x)\left(\frac{\mathrm{d}\widehat{f}(x)}{\mathrm{d}x}\right)^2 - b_3\widehat{f}(x)\,. \tag{23}$$

A method for forcing the residual function (23) to zero can be defined as collocation algorithm. With collocating $\{x_k\}_{k=0}^{N}$ to residual function (23), we have $N+1$ equations and $N+1$ unknown coefficients (spectral coefficients). In all of the spectral methods, the purpose is to find these coefficients. In shape of algorithmic, for solving (4), we do [38]:

BEGIN
1. Input N.
2. Construct the series (19) by using modified rational Christov functions (HCS$_n$ function)
3. Construct (21) to satisfy boundary conditions (5)
4. Construct the residual function (23) by substituting $f(x)$ by (21) in (4)
5. By choice $\{x_i\}$, $i = 0,1,\ldots,N$, they are roots of S_n Christov function in the interval $[0,\infty)$ as collocation points
6. Substitute collocation points in $\mathrm{Res}(x;a_0,a_1,\ldots,a_n)$, we construct a system containing $N+1$ equations.
7. Solve obtained system of equations in Step 6 via Newton's method [39] and gain the a_n, $n = 0,1,\ldots,N$.
8. Substitute obtained values of these coefficients in (21), we shall approach $f(x)$ by $\widehat{f}(x)$

END.

Now, we have approximated $f(x)$ by $\widehat{f}(x)$. Also we repeated algorithm for solving (4) by using normal rational Christov functions (C_n and S_n functions) in Step 2 of the algorithm. When using S_n functions, we used (22) instead of (21) to satisfy the boundary conditions (5) in Step 3 and to approximate $f(x)$ by $\widehat{f}(x)$ in Step 8 of the algorithm.

6. Result and Discussion

This problem was solved with some typical values of parameters, $b_1 = 0.6$, $b_2 = 0.1$, and $b_3 = 0.5$, by Ahmed [9]. In the current article, researchers have shown the approximate solutions for values of parameters $b_1 = 0.6$, $b_2 = 0.1$, and $b_3 = 0.5$ in the flow of a third-grade fluid problem. $f'(0)$ is important, therefore, the researchers have computed and compared it with other results. Ahmad [9] obtained this value by

x	Present method			Ahmad [9]	
	HCS$_n$, $L = 0.7084$	C_n, $L = 0.7330$	S_n, $L = 0.6541$	HAM	numerical
0.0	1.0000000000	1.0000000000	1.0000000000	1.0000	1.0000
0.2	0.8726085962	0.8726086620	0.8726024966	0.8722	0.8726
0.4	0.7606268056	0.7606268539	0.7606259033	0.7601	0.7606
0.6	0.6624311767	0.6624312833	0.6624320465	0.6619	0.6624
0.8	0.5765022985	0.5765023452	0.5764974757	0.5760	0.5765
1.0	0.5014361651	0.5014360651	0.5014296545	0.5010	0.5014
1.2	0.4359503972	0.4359510788	0.4359590583	0.4356	0.4359
1.6	0.3292020522	0.3292030051	0.3292082443	0.3289	0.3292
2.0	0.2483843480	0.2483834474	0.2483695160	0.2482	0.2484
2.5	0.1745476346	0.1745502969	0.1745627212	0.1744	0.1745
2.7	0.1515571351	0.1515586811	0.1515617949	0.1514	0.1516
3.0	0.1226123864	0.1226112482	0.1225923512	0.1225	0.1226
3.4	0.0924203195	0.0924195474	0.0924021361	0.09234	0.09242
3.6	0.0802364441	0.0802372385	0.0802322198	0.08016	0.08024
4.0	0.0604734182	0.0604774070	0.0604959076	0.06042	0.06047
4.2	0.0524995290	0.0525047033	0.0525272510	0.05245	0.05250
4.4	0.0455767719	0.0455829324	0.0456025472	0.04553	0.04558
4.6	0.0395666601	0.0395737831	0.0395838334	0.03953	0.03957
4.8	0.0343489246	0.0343571460	0.0343521763	0.03432	0.03435
5.0	0.0298191287	0.0298286806	0.0298048359	0.02979	0.02982
$f'(0)$	−0.6783017725	−0.6782997795	−0.6785503287	−0.681835	−0.678301
$\lVert\mathrm{Res}\rVert^2$	$1.9434\cdot10^{-11}$	$4.1150\cdot10^{-9}$	$5.4050\cdot10^{-5}$	–	–

Table 1. Values of $f(x)$ for $b_1 = 0.6$, $b_2 = 0.1$, and $b_3 = 0.5$ with $N = 19$ by utilizing good choice L and comparison between the present method and [9].

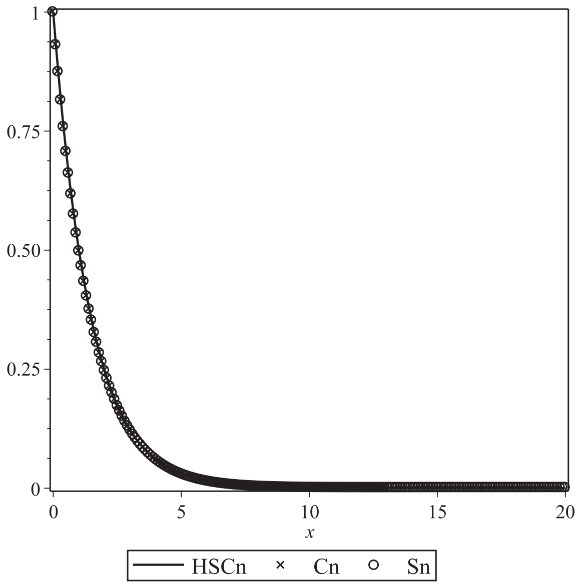

Fig. 1. Graph of approximated $f(x)$ of steady flow equation solution for $N = 19$ with L given in Table 1 and C_n, S_n, HCS_n as basis functions.

the shooting method and founded, corrected to six decimal positions, $f'(0) = 0.678301$. The results of the present method, the numerical solution, and the homotopy analysis method (HAM) [9] were compared with each other in Table 1 and the solutions are presented graphically in Figure 1.

Table 1 contains a shape parameter L that must be specified by the user. But here, by the meaning of residual function, the researchers try to minimize $\|\mathrm{Res}(x)\|^2$

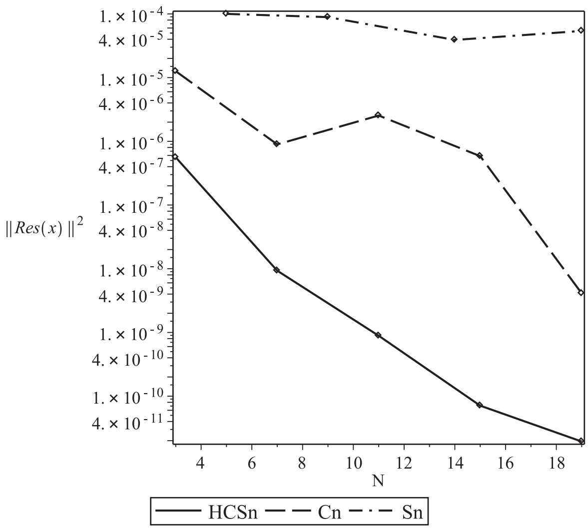

Fig. 2. Graph of residual function $\|\mathrm{Res}(x)\|^2$ of steady flow equation solution for $N = 19$ with L given in Table 1 and C_n, S_n, HCS_n as basis functions.

Table 2. Representation coefficients of the steady flow of a third-grade fluid in a porous half space in the present method for $b_1 = 0.6$, $b_2 = 0.1$, and $b_3 = 0.5$ with $N = 19$.

a_i	HCS_n	C_n	S_n
a_0	−0.3034057599	+0.0883269282	+0.0449654451
a_1	−0.5157958598	+0.0260781880	−0.0037423346
a_2	−0.5127993477	+0.0068536114	−0.0155629546
a_3	+0.1015286518	−0.0007219310	−0.0160477354
a_4	−0.0754752206	−0.0031338618	−0.0131281977
a_5	+0.3706201926	−0.0036252570	−0.0097461745
a_6	+0.1840661630	−0.0032746906	−0.0068343375
a_7	+0.1273668763	−0.0026914871	−0.0046120327
a_8	+0.0908736009	−0.0020725487	−0.0030232797
a_9	−0.0581185540	−0.0015356649	−0.0019406513
a_{10}	−0.0076409658	−0.0010947110	−0.0012298998
a_{11}	−0.0409946252	−0.0007590050	−0.0007791214
a_{12}	−0.0123142515	−0.0005078320	−0.0005012772
a_{13}	−0.0023509153	−0.0003302030	−0.0003337489
a_{14}	−0.0014488306	−0.0002051383	−0.0002327572
a_{15}	+0.0023589458	−0.0001224031	−0.0001696032
a_{16}	+0.0002485239	−0.0000671652	−0.0001260487
a_{17}	+0.0003153593	−0.0000337625	−0.0000915710
a_{18}	+0.0000273397	−0.0000130791	−0.0000604473
a_{19}	−0.0000091592	−0.0000035209	−0.0000302436

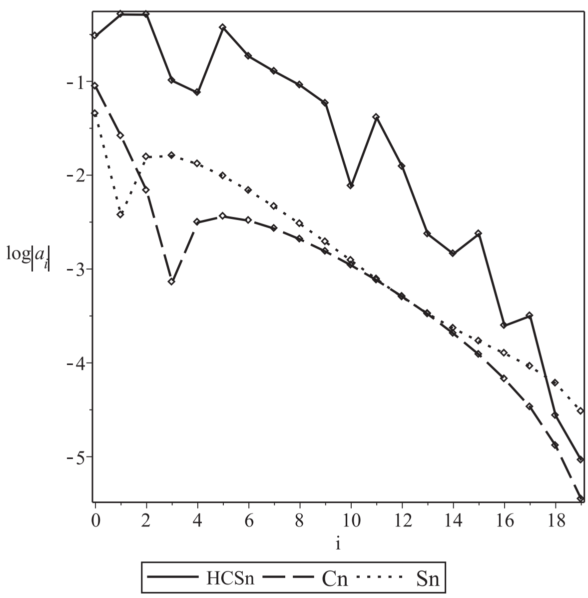

Fig. 3. Graph of $\log|a_i|$ of steady flow equation solution for $N = 19$ with L given in Table 1 and C_n, S_n, HCS_n as basis functions.

by choosing a good shape parameter L [40]. We define $||\mathrm{Res}(x)||^2$ [4] as

$$||\mathrm{Res}(x)||^2 = \int_0^b \mathrm{Res}^2(x)\,\mathrm{d}x,$$

where b is the biggest collocation node. We present the minimum of $||\mathrm{Res}(x)||^2$ obtained with shape parameter L in Table 1. The graphs of $||\mathrm{Res}(x)||^2$ are shown in Figure 2.

Table 2 represents the coefficients of the modified rational Christov function and the rational Christov function obtained by the present method for $N = 19$ of the steady flow of a third-grade fluid in a porous half space. The logarithmic graph of absolute coefficients $|a_i|$ of modified rational Christov function and rational Christov function in the approximate solutions is shown in Figure 3. The graph illustrates that the method has an appropriate convergence rate [41].

7. Conclusions

The method presented in this paper uses a set of modified rational Christov functions and normal rational Christov functions to solve the aforementioned problem on the infinite domain without truncating it to a finite domain. Several strategies can be applied to solve problems in unbounded domains. We have created a new strategy to solve them. The validity of the method is based on the assumption that it converges by increasing the number of collocation points. It is worth mentioning that it was confirmed by the theorem and by logarithmic figures of absolute coefficients that this approach has an exponentially convergence rate. In total, an important concern of spectral methods is the choice of basis functions; the basis functions have three properties: easy computation, rapid convergence, and completeness, i. e., any solution can be represented to arbitrarily high accuracy by taking the truncation N sufficiently large [41]. In the present method, Figure 1 reveals that by x tending to ∞ $f(x)$ tends to zero. As shown in Figure 2, by increasing N, the error tends to zero. Figure 3 shows that the absolute coefficients in this approach have an exponential convergence. In the flow of a third-grade fluid problem, $f'(0)$ is important, therefore, the researchers have computed and compared it with other results. By comparing modified rational Christov functions and normal rational Christov functions with considering Figures 1 – 3 and Table 1, the researchers believe that the modified rational Christov function has a better convergence than normal rational Christov functions. The results presented indicate that the method provides another powerful tool to solve nonlinear ordinary differential equations.

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