Pseudospin and Spin Symmetric Solutions of the Dirac Equation: Hellmann Potential, Wei–Hua Potential, Varshni Potential

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Approximate analytical solutions of the Dirac equation are obtained for the Hellmann potential, the Wei–Hua potential, and the Varshni potential with any κ-value for the cases having the Dirac equation pseudospin and spin symmetries. Closed forms of the energy eigenvalue equations and the spinor wave functions are obtained by using the Nikiforov–Uvarov method and some tables are given to see the dependence of the energy eigenvalues on different quantum number pairs (n, κ).

Key words: Hellmann Potential; Wei–Hua Potential; Varshni Potential; Dirac Equation; Nikiforov–Uvarov Method; Spin Symmetry; Pseudospin Symmetry.
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1. Introduction

The pseudospin and spin symmetric solutions of the Dirac equation have been of great interest in literature for the last decades [1 – 3]. The Dirac equation with vector, $V(r)$, and scalar, $S(r)$, potentials has pseudospin (spin) symmetry when the difference $V(r) - S(r)$ (the sum $V(r) + S(r)$) of the potentials is constant, which means $\frac{\mathrm{d}}{\mathrm{d}r}[V(r) - S(r)] = 0$ (or $\frac{\mathrm{d}}{\mathrm{d}r}[V(r) + S(r)] = 0$). It is pointed out that these symmetries can explain degeneracies in single-particle energy levels in nuclei or in some heavy meson-spectra within the contexts of relativistic mean-field theories [1 – 3]. In the relativistic domain, these symmetries were used in the context of deformation and superdeformation in nuclei, magnetic moment interpretation, and identical bands [4]. In the non-relativistic domain, performing a helicity unitary transformation to a single-particle Hamiltonian maps the normal state onto the pseudo-state [5]. Moreover, the Dirac Hamiltonian has not only a spin symmetry but also a $U(3)$ symmetry for the case $V(r) = S(r)$ while not only a pseudospin symmetry but also a pseudo-$U(3)$ symmetry with vector and scalar harmonic oscillator potentials [6, 7]. Because of these investigations, the solutions of the Dirac equation having spin and pseudospin symmetry have received great attention for different type of potentials such as Morse, Eckart, the modified Pöschl–Teller, the Manning–Rosen potentials, and the symmetrical well potential [8 – 15].

Throughout the paper, we use the following approximation instead of the spin–orbit coupling term to obtain the analytical solutions of the Hellmann, Wei–Hua, and Varshni potentials [16 – 23]

$$\frac{1}{r^2} \approx \beta^2 \frac{1}{(1 - \mathrm{e}^{-\beta r})^2}\,, \tag{1}$$

where β is a parameter related with the above potentials.

The potentials studied in the present work and also some other exponential-type potentials such as a ring-shaped Hùlthen, Yukawa, and Tietz–Hua potentials have been analyzed in details by using different methods [24 – 30]. We intend to use the Nikiforov–Uvarov method (NU) to analyze the bound states of the Dirac equation for the cases of pseudospin and spin symmetries. This method is a powerful tool to solve a second-order differential equation and has been used to find the bound states of different potentials in literature [31, 32].

The organization of this work is as follows. In Section 2, we briefly give the Dirac equation with attractive scalar and repulsive vector potentials for the cases where the Dirac equation has pseudospin and spin symmetries, respectively. In Section 3, we present the NU method and the parameters required within the

method. In Section 4, we find the analytical energy eigenvalue equations for the bound states and the two-component spinor wave functions of the above potentials by using an approximation instead of the spin–orbit coupling term. In Section 5, we give our results and discussions. The last section includes our conclusions.

2. Dirac Equation

The free particle Dirac equation is given by ($\hbar = c = 1$)

$$\left(\mathrm{i}\gamma^{\mu}\partial_{\mu} - M\right)\Psi(\mathbf{r},t) = 0\,. \tag{2}$$

Taking the total wave function as $\Psi(\mathbf{r},t) = \mathrm{e}^{-\mathrm{i}Et}\psi(\mathbf{r})$ for time-independent potentials, where E is the relativistic energy and M the particle mass, the Dirac equation with spherical symmetric vector and scalar potentials is written as

$$\left[\boldsymbol{\alpha}\cdot\mathbf{P} + \beta(M + S(r))\right]\psi(\mathbf{r}) = \left[E - V(r)\right]\psi(\mathbf{r})\,. \tag{3}$$

Here α and β are usual 4×4 matrices. For spherical nuclei, the angular momentum $\mathbf{J}$ and the operator $\hat{K} = -\beta(\hat{\sigma}.\hat{L} + 1)$ with eigenvalues $\kappa = \pm(j + 1/2)$ commute with the Dirac Hamiltonian, where $\hat{L}$ is the orbital angular momentum. By using the radial eigenfunctions for upper and lower components of the Dirac eigenfunction $F(r)$ and $G(r)$, respectively, the wave function is written as [31]

$$\psi(\mathbf{r}) = \frac{1}{r}\begin{pmatrix} F(r)Y^{(1)}(\theta,\phi) \\ \mathrm{i}G(r)Y^{(2)}(\theta,\phi)\end{pmatrix}, \tag{4}$$

where $Y^{(1)}(\theta,\phi)$ and $Y^{(2)}(\theta,\phi)$ are the pseudospin and spin spherical harmonics, respectively. They correspond to angular and spin parts of the wave function given by

$$\begin{aligned} &Y^{(1),(2)}(\theta,\phi) \\ &= \sum_{m_\ell m_s}\left\langle \ell m_\ell \frac{1}{2} m_s \middle| \ell \frac{1}{2} jm \right\rangle Y_\ell m_\ell(\theta,\phi)\chi_{\frac{1}{2}m_s}\,, \\ &j = |\kappa| - \frac{1}{2}\,,\ \ \ell = \kappa(\kappa > 0)\,;\ \ \ell = -(\kappa+1)(\kappa < 0)\,. \end{aligned} \tag{5}$$

Here, $Y_{\ell m_\ell}(\theta,\phi)$ denotes the spherical harmonics and m_ℓ and m_s are related magnetic quantum numbers.

Substituting (4) into (3) gives us the following coupled differential equations:

$$\left(\frac{\mathrm{d}}{\mathrm{d}r} + \frac{\kappa}{r}\right)F(r) = \left[E + M - \Gamma(r)\right]G(r)\,, \tag{6a}$$

$$\left(\frac{\mathrm{d}}{\mathrm{d}r} - \frac{\kappa}{r}\right)G(r) = \left[M - E + \Lambda(r)\right]F(r)\,, \tag{6b}$$

where $\Gamma(r) = V(r) - S(r)$ and $\Lambda(r) = V(r) + S(r)$. Using the expression $G(r)$ in (6a) and inserting it into (6b), we get the second-order differential equation

$$\begin{aligned} &\left[\frac{\mathrm{d}^2}{\mathrm{d}r^2} - \frac{\kappa(\kappa+1)}{r^2} + \varepsilon^{(1)}(r)\right]F(r) \\ &= -\left[\frac{\mathrm{d}\Gamma(r)/\mathrm{d}r}{\left[E + M - \Gamma(r)\right]}\right]F(r)\,, \end{aligned} \tag{7}$$

where $\varepsilon^{(1)}(r) = [E + M - \Gamma(r)][E - M - \Lambda(r)]$. By similar steps, we write the second-order differential equation for $G(r)$ as

$$\begin{aligned} &\left[\frac{\mathrm{d}^2}{\mathrm{d}r^2} - \frac{\kappa(\kappa-1)}{r^2} + \varepsilon^{(2)}(r)\right]G(r) \\ &= \left[\frac{\mathrm{d}\Lambda(r)/\mathrm{d}r}{\left[M - E + \Lambda(r)\right]}\right]G(r)\,, \end{aligned} \tag{8}$$

where $\varepsilon^{(2)}(r) = [E - M - \Lambda(r)][E + M - \Gamma(r)]$. If the Dirac equation has spin symmetry which means that $\Gamma(r) = A_1$ ($\mathrm{d}\Gamma(r)/\mathrm{d}r = 0$) is a constant, (7) has the following form:

$$\begin{aligned} &\left\{\frac{\mathrm{d}^2}{\mathrm{d}r^2} - \frac{\kappa(\kappa+1)}{r^2} + \left[E + M - A_1\right]\right. \\ &\left.\cdot\left[E - M - \Lambda(r)\right]\right\}F(r) = 0\,, \end{aligned} \tag{9}$$

and if the Dirac equation has pseudospin symmetry which means that $\Lambda(r) = A_2$ ($\mathrm{d}\Lambda(r)/\mathrm{d}r = 0$) is a constant, (8) becomes

$$\begin{aligned} &\left\{\frac{\mathrm{d}^2}{\mathrm{d}r^2} - \frac{\kappa(\kappa-1)}{r^2} + \left[E - M - A_2\right]\right. \\ &\left.\cdot\left[E + M - \Gamma(r)\right]\right\}G(r) = 0\,. \end{aligned} \tag{10}$$

3. Nikiforov–Uvarov Method

The Nikiforov–Uvarov method could be used to solve a second-order differential equation of the hypergeometric-type which can be transformed by using an appropriate coordinate transformation into the following form:

$$\sigma^2(z)\frac{\mathrm{d}^2\Psi(z)}{\mathrm{d}z^2}+\sigma(z)\tilde{\tau}(z)\frac{\mathrm{d}\Psi(z)}{\mathrm{d}z} + \tilde{\sigma}(z)\Psi(z)=0\,, \tag{11}$$

where $\sigma(z)$, and $\tilde{\sigma}(z)$ are polynomials, at most, second degree, and $\tilde{\tau}(z)$ is a first-degree polynomial. By taking the solution as

$$\Psi(z)=\psi(z)\varphi(z) \tag{12}$$

gives (11) as a hypergeometric type equation [32]

$$\frac{\mathrm{d}^2\varphi(z)}{\mathrm{d}z^2}+\frac{\tau(z)}{\sigma(z)}\frac{\mathrm{d}\varphi(z)}{\mathrm{d}z}+\frac{\lambda}{\sigma(z)}\varphi(z)=0\,, \tag{13}$$

where $\psi(z)$ is defined by using the equation [32]

$$\frac{1}{\psi(z)}\frac{\mathrm{d}\psi(z)}{\mathrm{d}z}=\frac{\pi(z)}{\sigma(z)}\,, \tag{14}$$

and the other part of the solution in (12) is given by

$$\varphi_n(z)=\frac{a_\mathrm{n}}{\rho(z)}\frac{\mathrm{d}^n}{\mathrm{d}z^n}\left[\sigma^n(z)\rho(z)\right], \tag{15}$$

where a_n is a normalization constant, and $\rho(z)$ is the weight function, which satisfies the following equation [32]:

$$\frac{\mathrm{d}\sigma(z)}{\mathrm{d}z}+\frac{\sigma(z)}{\rho(z)}\frac{\mathrm{d}\rho(z)}{\mathrm{d}z}=\tau(z)\,. \tag{16}$$

The function $\pi(z)$ and the parameter λ in the above equation are defined as

$$\pi(z)=\frac{1}{2}\left[\frac{\mathrm{d}}{\mathrm{d}z}\sigma(z)-\tilde{\tau}(z)\right] \pm\left\{\frac{1}{4}\left[\frac{\mathrm{d}}{\mathrm{d}z}\sigma(z)-\tilde{\tau}(z)\right]^2-\tilde{\sigma}(z)+k\sigma(z)\right\}^{1/2}, \tag{17}$$

$$\lambda=k+\frac{\mathrm{d}}{\mathrm{d}z}\pi(z)\,. \tag{18}$$

In the NU method, the square root in (17) must be the square of a polynomial, so the parameter k can be determined. Thus, a new eigenvalue equation becomes

$$\lambda=\lambda_n=-n\frac{\mathrm{d}}{\mathrm{d}z}\tau(z)-\frac{1}{2}(n^2-n)\frac{\mathrm{d}^2}{\mathrm{d}z^2}\sigma(z)\,, \tag{19}$$

and the derivative of the function $\tau(z)=\tilde{\tau}(z)+2\pi(z)$ should be negative.

4. Bound State Solutions

4.1. Hellmann Potential

The Hellmann potential having the form

$$V(r)=-\frac{a}{r}+\frac{b}{r}\mathrm{e}^{-\beta r} \tag{20}$$

has been used to explain the electron–ion [33] or electron–core interaction [34], alkali hydride molecules and to study the inner-shell ionisation problem [35]. We present the plot of the above potential in Figure 1 to see the variation with position coordinate.

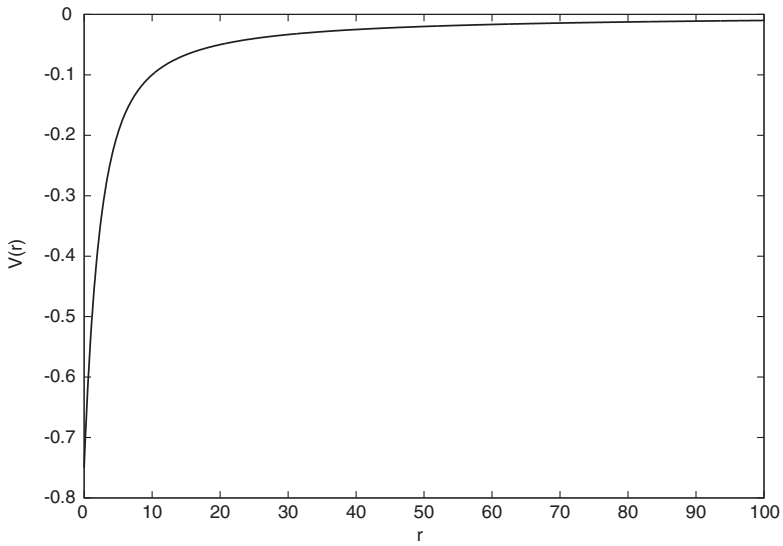

Fig. 1. Hellman potential.

4.1.1. Spin Symmetric Solutions

Inserting (20) into (9) and using the approximation given in (1) instead of the spin–orbit coupling term, we obtain

$$\left\{\frac{\mathrm{d}^2}{\mathrm{d}r^2}-\frac{\beta^2\kappa(\kappa+1)}{(1-\mathrm{e}^{-\beta r})^2}+\frac{\beta}{1-\mathrm{e}^{-\beta r}}\cdot\left(a-b\,\mathrm{e}^{-\beta r}\right)+\varepsilon_{\mathrm{H}}^{\mathrm{SS}}\right\}F(r)=0\,, \tag{21}$$

where H stands for the Hellmann potential and $\varepsilon_{\mathrm{H}}^{\mathrm{SS}}=(E+M-A_1)(E-M)$. Defining a new variable $z=\mathrm{e}^{-\beta r}$ and using the abbreviations

$$a_1^2=\kappa(\kappa+1)-\frac{1}{\beta^2}\left(a\beta+\varepsilon_{\mathrm{H}}^{\mathrm{SS}}\right), \tag{22a}$$

$$a_2^2=\frac{1}{\beta^2}\left[\beta(a+b)+2\varepsilon_{\mathrm{H}}^{\mathrm{SS}}\right], \tag{22b}$$

$$a_3^2=-\frac{1}{\beta^2}\left[b\beta+\varepsilon_{\mathrm{H}}^{\mathrm{SS}}\right], \tag{22c}$$

we write (21) as

$$\frac{\mathrm{d}^2F(z)}{\mathrm{d}z^2}+\frac{1-z}{z(1-z)}\frac{\mathrm{d}F(z)}{\mathrm{d}z}+\frac{1}{z^2(1-z)^2}\left[-a_1^2-a_2^2z-a_3^2z^2\right]F(z)=0\,. \tag{23}$$

Comparing the last equation with (11), we have

$$\tilde{\tau}(z)=1-z,\;\;\sigma(z)=z(1-z)\,,\qquad \tilde{\sigma}(z)=-a_1^2z^2-a_2^2z-a_3^2\,. \tag{24}$$

The function $\pi(z)$ is obtained from (17) as

$$\pi(z)=-\frac{1}{2}z\mp\sqrt{\left(\frac{1}{4}+a_3^2-k\right)z^2+\left(a_2^2+k\right)z+a_1^2}\,. \tag{25}$$

The constant k is determined by imposing a condition such that the discriminant under the square root should be zero. The roots of k are $k_{1,2}=-a_2^2-2a_1^2\mp a_1(1+2\kappa)$. Substituting the value of $k_1=-a_2^2-2a_1^2+a_1(1+2\kappa)$ into (25), we get for $\pi(z)$

$$\pi(z)(k\to k_1)=-(a_1-\kappa)z+a_1-(1+\kappa-a_1)z-a_1\,. \tag{26}$$

Now we calculate the polynomial $\tau(z)$ from $\pi(z)$ such that its derivative with respect to z must be negative. Thus we obtain $\tau(z)$ for the second choice in last equation as

$$\tau(z)=(2a_1-1)z-(1+2\kappa+2a_1)\,. \tag{27}$$

The constant λ in (18) becomes

$$\lambda=-a_2^2-2a_1^2+a_1(1+2\kappa)-a_1+\kappa\,, \tag{28}$$

and (19) gives us

$$\lambda_n=n(n-2a_1)\,. \tag{29}$$

Substituting the values of the parameters given by (22), and setting $\lambda=\lambda_n$, one can find the energy eigenvalues for the Hellmann potential as

$$E=\frac{1}{2}\left[A_1\mp\sqrt{A_1^2-4(MA_1-M^2-N)}\right], \tag{30}$$

where N is a parameter written in terms of the quantum numbers n and κ as

$$N=-\frac{\beta^2}{4(n+\kappa)^2}\left[\frac{1}{\beta}(a-b)-(n^2-\kappa^2)-\kappa(\kappa+1)\right]^2-a\beta+\beta^2\kappa(\kappa+1)\,. \tag{31}$$

Now we get the upper component of the Dirac wave function. We first compute the weight function from (16) with the help of (27),

$$\rho(z)=z^{-2(1+\kappa+a_1)}(1-z)^{(1+2\kappa)}\,, \tag{32}$$

and then obtain from (15)

$$\varphi_n(z)\sim z^{-2(1+\kappa+a_1)}(1-z)^{(1+2\kappa)}\cdot\frac{\mathrm{d}^n}{\mathrm{d}z^n}\left[z^{n-\kappa-a_1-2}(1-z)^{n-2\kappa-1}\right]. \tag{33}$$

The polynomial solutions can be written in terms of the Jacobi polynomials [36]

$$\varphi_n(z)\sim P_n^{(-2(1+\kappa+a_1),-(1+2\kappa))}(1-2z)\,. \tag{34}$$

The other part of the wave function is obtained from (15) as

$$\psi(z)=z^{-a_1}(1-z)^{1-\kappa}\,. \tag{35}$$

Thus we write the upper component for the Hellmann potential in (4) as

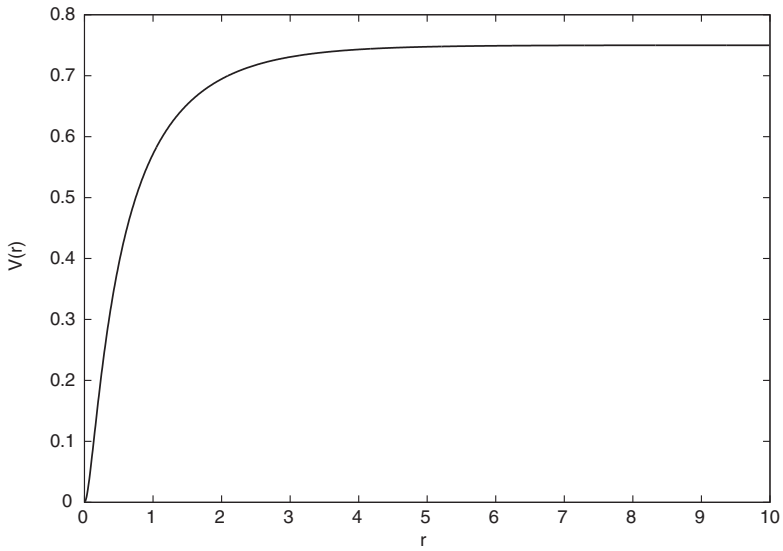

Fig. 2. Wei–Hua potential.

$$F(z) \sim z^{-a_1}(1-z)^{1-\kappa} \cdot P_n^{(-2(1+\kappa+a_1),-(1+2\kappa))}(1-2z)\,. \tag{36}$$

By using (6a) and the identity for derivative of the Jacobi polynomials given as $\frac{\mathrm{d}}{\mathrm{d}x}P_n^{(p,q)(x)} = \frac{1}{2}(n+p+q+1)$ $P_{n-1}^{(p+1,q+1)}(x)$ [36], we obtain the other component for the Hellmann potential as

$$G(z) \sim \frac{z^{-a_1}(1-z)^{1-\kappa}}{E+M-A}\left[\beta\left(\frac{1}{a_1}-\frac{\kappa}{\ln z}\right) \cdot P_n^{(-2(1+\kappa+a_1),-(1+2\kappa))}(1-2z) - \frac{1}{4}(n-2a_1)P_n^{(-(1+2\kappa+2a_1),-(2+2\kappa))}(1-2z)\right]. \tag{37}$$

4.1.2. Pseudospin Symmetric Solutions

Inserting (20) into (10) and using the approximation given in (1), we obtain

$$\left\{\frac{\mathrm{d}^2}{\mathrm{d}r^2} - \frac{\beta^2\kappa(\kappa-1)}{(1-\mathrm{e}^{-\beta r})^2} + \frac{\beta}{1-\mathrm{e}^{-\beta r}} \cdot \left(a - b\,\mathrm{e}^{-\beta r}\right) + \varepsilon_{\mathrm{H}}^{\mathrm{PSS}}\right\}F(r) = 0\,, \tag{38}$$

where $\varepsilon_{\mathrm{H}}^{\mathrm{PSS}} = (E-M-A_2)(E+M)$. Using the same variable and the abbreviations

$$a_1^2 = \kappa(\kappa-1) - \frac{1}{\beta^2}\left(a\beta + \varepsilon_{\mathrm{H}}^{\mathrm{PSS}}\right), \tag{39a}$$

$$a_2^2 = \frac{1}{\beta^2}\left[\beta(a+b) + 2\varepsilon_{\mathrm{H}}^{\mathrm{PSS}}\right], \tag{39b}$$

$$a_3^2 = -\frac{1}{\beta^2}\left[b\beta + \varepsilon_{\mathrm{H}}^{\mathrm{PSS}}\right], \tag{39c}$$

we obtain

$$\frac{\mathrm{d}^2G(z)}{\mathrm{d}z^2} + \frac{1-z}{z(1-z)}\frac{\mathrm{d}G(z)}{\mathrm{d}z} + \frac{1}{z^2(1-z)^2} \cdot \left[-a_1^2 - a_2^2 z - a_3^2 z^2\right]G(z) = 0\,. \tag{40}$$

Following the same steps as the in previous section, we write the energy eigenvalues for the Hellmann potential for the case of pseudospin symmetry as

$$E = \frac{1}{2}\left[A_2 \mp \sqrt{A_2^2 + 4(MA_2 + M^2 + N)}\right], \tag{41}$$

where N is given by

$$N = -\frac{\beta^2}{4(n+\kappa)^2}\left[-\frac{1}{\beta}(a-b) + n^2 + \kappa^2 + \kappa(\kappa-3)\right]^2 - a\beta + \beta^2\kappa(\kappa-1)\,, \tag{42}$$

and the lower component can be written as

$$G(z) \sim z^{-a_1}(1-z)^{1-\kappa} \cdot P_n^{(-2(1+\kappa+a_1),-(1+2\kappa))}(1-2z)\,. \tag{43}$$

Using (6b) gives us the other component as

$$F(z) \sim \frac{z^{-a_1}(1-z)^{1-\kappa}}{M-E+A}\left[\beta\left(\frac{1}{a_1}-\frac{\kappa}{\ln z}\right)\right.$$
$$\cdot P_n^{(-2(1+\kappa+a_1),-(1+2\kappa))}(1-2z)-\frac{1}{4}(n-2a_1)$$
$$\left.\cdot P_n^{(-(1+2\kappa+2a_1),-(2+2\kappa))}(1-2z)\right]. \tag{44}$$

4.2. Wei–Hua Potential

The Wei–Hua potential has the form

$$V(r)=D\left[\frac{1-\mathrm{e}^{-\beta r}}{1-a\mathrm{e}^{-\beta r}}\right]^2, \tag{45}$$

which is proposed for bond-stretching vibration of diatomic molecules [37]. We present the plot of the Wei–Hua potential in Figure 2.

4.2.1. Spin Symmetric Solutions

Inserting the last equation and (1) into (9), we obtain

$$\left\{\frac{\mathrm{d}^2}{\mathrm{d}r^2}-\frac{\beta^2\kappa(\kappa+1)}{(1-\mathrm{e}^{-\beta r})^2}\right.$$
$$\left.-D\left[\frac{1-\mathrm{e}^{-\beta r}}{1-a\mathrm{e}^{-\beta r}}\right]^2+\varepsilon_{\mathrm{WH}}^{\mathrm{SS}}\right\}F(r)=0, \tag{46}$$

where WH stands for the Wei–Hua potential and $\varepsilon_{\mathrm{WH}}^{\mathrm{SS}}=(E+M-A_1)(E-M)$. Defining a new variable $z=a\mathrm{e}^{-\beta r}$, using the abbreviations

$$a_1^2=\kappa(\kappa+1)-\frac{1}{\beta^2}\left(\varepsilon_{\mathrm{WH}}^{\mathrm{SS}}-D\right), \tag{47a}$$
$$a_2^2=-\frac{1}{\beta^2}\left[\frac{2D}{a}-2\varepsilon_{\mathrm{WH}}^{\mathrm{SS}}\right], \tag{47b}$$
$$a_3^2=-\frac{1}{\beta^2}\left[\varepsilon_{\mathrm{WH}}^{\mathrm{SS}}-\frac{D}{a^2}\right], \tag{47c}$$

and following the same procedure as in the above section for the Hellmann potential, we write the energy eigenvalues of the Wei–Hua potential for the case of spin symmetry as

$$E=\frac{1}{2}\left[A_1\mp\sqrt{A_1^2-4(MA_1-M^2-N)}\right], \tag{48}$$

where N is a parameter written in terms of the quantum numbers n and κ,

$$N=-\frac{\beta^2}{4(n+\kappa)^2}\left[n^2+\kappa^2+\kappa(\kappa+1)\right.$$
$$\left.-\frac{2D}{\beta^2}\left(\frac{1}{a}-1\right)\right]^2+D+\beta^2\kappa(\kappa+1). \tag{49}$$

The lower component for the Wei–Hua potential is

$$F(z)\sim z^{-a_1}(1-z)^{1-\kappa}$$
$$\cdot P_n^{(-2(1+\kappa+a_1),-(1+2\kappa))}(1-2z). \tag{50}$$

By using (6a), we obtain the other component to

$$G(z)\sim\frac{z^{-a_1}(1-z)^{1-\kappa}}{E+M-A}\left[\beta\left(\frac{1}{a_1}-\frac{\kappa}{\ln z}\right)\right.$$
$$\cdot P_n^{(-2(1+\kappa+a_1),-(1+2\kappa))}(1-2z)-\frac{1}{4}(n-2a_1)$$
$$\left.\cdot P_n^{(-(1+2\kappa+2a_1),-(2+2\kappa))}(1-2z)\right]. \tag{51}$$

4.2.2. Pseudospin Symmetric Solutions

Inserting (45) and (1) into (10), we obtain

$$\left\{\frac{\mathrm{d}^2}{\mathrm{d}r^2}-\frac{\beta^2\kappa(\kappa-1)}{(1-\mathrm{e}^{-\beta r})^2}\right.$$
$$\left.-D\left[\frac{1-\mathrm{e}^{-\beta r}}{1-a\mathrm{e}^{-\beta r}}\right]^2+\varepsilon_{\mathrm{WH}}^{\mathrm{PSS}}\right\}F(r)=0 \tag{52}$$

with $\varepsilon_{\mathrm{WH}}^{\mathrm{PSS}}=(E-M-A_2)(E+M)$. Using the same variable z for the Hellmann potential, defining the abbreviations

$$a_1^2=\kappa(\kappa-1)-\frac{1}{\beta^2}\left(\varepsilon_{\mathrm{WH}}^{\mathrm{PSS}}-D\right), \tag{53a}$$
$$a_2^2=-\frac{1}{\beta^2}\left[\frac{2D}{a}-2\varepsilon_{\mathrm{WH}}^{\mathrm{PSS}}\right], \tag{53b}$$
$$a_3^2=-\frac{1}{\beta^2}\left[\varepsilon_{\mathrm{WH}}^{\mathrm{PSS}}-\frac{D}{a^2}\right], \tag{53c}$$

and following the same procedure as in the above section for the Hellmann potential, we get the energy eigenvalues of the Wei–Hua potential for the case of pseudospin symmetry as

$$E=\frac{1}{2}\left[A_2\mp\sqrt{A_2^2+4(MA_2+M^2+N)}\right], \tag{54}$$

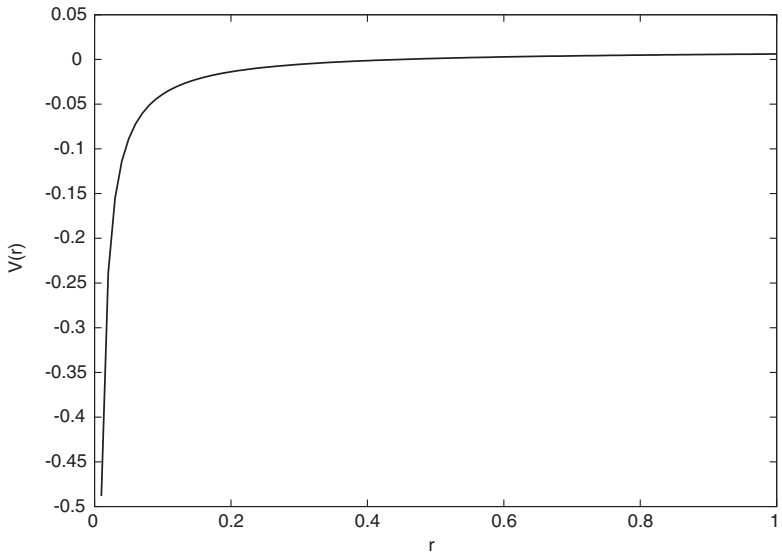

Fig. 3. Varshni potential.

where N is a parameter written in terms of the quantum numbers n and κ:

$$N = -\frac{\beta^2}{4(n+\kappa)^2}\left[n^2-\kappa^2+\kappa(\kappa-1) - \frac{2D}{\beta^2}\left(\frac{1}{a}-1\right)\right]^2 + D + \beta^2\kappa(\kappa-1)\,. \tag{55}$$

The upper component for the Wei–Hua potential is

$$G(z) \sim z^{-a_1}(1-z)^{1-\kappa} \cdot P_n^{(-2(1+\kappa+a_1),-(1+2\kappa))}(1-2z)\,. \tag{56}$$

The other component can be obtained from (6b) as

$$F(z) \sim \frac{z^{-a_1}(1-z)^{1-\kappa}}{M-E+A}\left[\beta\left(\frac{1}{a_1}-\frac{\kappa}{\ln z}\right) \cdot P_n^{(-2(1+\kappa+a_1),-(1+2\kappa))}(1-2z) - \frac{1}{4}(n-2a_1) \cdot P_n^{(-(1+2\kappa+2a_1),-(2+2\kappa))}(1-2z)\right]. \tag{57}$$

4.3. Varshni Potential

Varshni, for the first time, proposed the potential function

$$V(r) = a\left[1-\frac{b}{r}\mathrm{e}^{-\beta r}\right] \tag{58}$$

to study diatomic molecules [38]. It is clearly seen that the potential is very similar to the Hellmann potential which could be seen in Figure 3. All figures show that the form of the potentials presented in this work are very similar.

Now we tend to study the spin and pseudospin symmetric solutions of the Dirac equation for the above potential.

4.3.1. Spin Symmetric Solutions

Inserting (58) and (1) into (9), we obtain

$$\left\{\frac{\mathrm{d}^2}{\mathrm{d}r^2} - \frac{\beta}{1-\mathrm{e}^{-\beta r}}\left(\frac{\beta\kappa(\kappa+1)}{1-\mathrm{e}^{-\beta r}} - ab\,\mathrm{e}^{-\beta r}\right) + \varepsilon_{\mathrm{V}}^{\mathrm{SS}}\right\}F(r) = 0\,, \tag{59}$$

where V stands for the Varshni potential and $\varepsilon_{\mathrm{V}}^{\mathrm{SS}} = (E+M-A_1)(E-M)-a$. Defining a new variable $z = \mathrm{e}^{-\beta r}$, using the abbreviations

$$a_1^2 = \kappa(\kappa+1) - \frac{\varepsilon_{\mathrm{V}}^{\mathrm{SS}}}{\beta^2}\,, \tag{60a}$$

$$a_2^2 = -\frac{1}{\beta^2}\left[ab\beta - 2\varepsilon_{\mathrm{V}}^{\mathrm{SS}}\right], \tag{60b}$$

$$a_3^2 = -\frac{1}{\beta^2}\left[\varepsilon_{\mathrm{V}}^{\mathrm{SS}} - ab\beta\right], \tag{60c}$$

and following the same procedure as in the above sections, we write the energy eigenvalues of the Varshni potential for the case of spin symmetry as

$$E = \frac{1}{2}\left[A_1 \mp \sqrt{A_1^2 - 4(MA_1 - M^2 - N)}\right], \quad (61)$$

where

$$N = -\frac{\beta^2}{4(n+\kappa)^2}\left[-\frac{ab}{\beta} + n^2 + \kappa^2 + \kappa(\kappa+1)\right]^2 + \beta^2\kappa(\kappa+1) + a\,. \quad (62)$$

The lower component for the Varshni potential is

$$F(z) \sim z^{-a_1}(1-z)^{1-\kappa} \cdot P_n^{(-2(1+\kappa+a_1),-(1+2\kappa))}(1-2z)\,. \quad (63)$$

By using (6a), we obtain the other component as

$$G(z) \sim \frac{z^{-a_1}(1-z)^{1-\kappa}}{E+M-A}\left[\beta\left(\frac{1}{a_1} - \frac{\kappa}{\ln z}\right) \cdot P_n^{(-2(1+\kappa+a_1),-(1+2\kappa))}(1-2z) - \frac{1}{4}(n-2a_1) \cdot P_n^{(-(1+2\kappa+2a_1),-(2+2\kappa))}(1-2z)\right]. \quad (64)$$

4.3.2. Pseudospin Symmetric Solutions

Inserting (58) and (1) into (10), we obtain

$$\left\{\frac{\mathrm{d}^2}{\mathrm{d}r^2} - \frac{\beta}{1-\mathrm{e}^{-\beta r}}\left(\frac{\beta\kappa(\kappa-1)}{1-\mathrm{e}^{-\beta r}} - ab\,\mathrm{e}^{-\beta r}\right) + \varepsilon_{\mathrm{V}}^{\mathrm{PSS}}\right\}F(r) = 0\,, \quad (65)$$

$\varepsilon_{\mathrm{V}}^{\mathrm{PSS}} = (E-M-A_2)(E+M)$. Using the same variable z for the Hellmann potential, defining the abbreviations

$$a_1^2 = \kappa(\kappa-1) - \frac{\varepsilon_{\mathrm{V}}^{\mathrm{PSS}}}{\beta^2}\,, \quad (66a)$$

$$a_2^2 = -\frac{1}{\beta^2}\left[ab\beta - 2\varepsilon_{\mathrm{V}}^{\mathrm{PSS}}\right], \quad (66b)$$

$$a_3^2 = -\frac{1}{\beta^2}\left[\varepsilon_{\mathrm{V}}^{\mathrm{PSS}} - ab\beta\right], \quad (66c)$$

and following the same procedure as in the above sections, we get the energy eigenvalues of the Varshni potential for the case of pseudospin symmetry as

$$E = \frac{1}{2}\left[A_2 \mp \sqrt{A_2^2 + 4(MA_2 + M^2 + N)}\right], \quad (67)$$

where

$$N = -\frac{\beta^2}{4(n+\kappa)^2}\left[n^2 + \kappa^2 + \kappa(\kappa-3) - \frac{ab}{\beta}\right]^2 - \beta^2\kappa(\kappa-1) + a\,. \quad (68)$$

The upper component for the Varshni potential is

$$G(z) \sim z^{-a_1}(1-z)^{1-\kappa} \cdot P_n^{(-2(1+\kappa+a_1),-(1+2\kappa))}(1-2z)\,. \quad (69)$$

Using (6b) gives the other component to

$$F(z) \sim \frac{z^{-a_1}(1-z)^{1-\kappa}}{M-E+A}\left[\beta\left(\frac{1}{a_1} - \frac{\kappa}{\ln z}\right) \cdot P_n^{(-2(1+\kappa+a_1),-(1+2\kappa))}(1-2z) - \frac{1}{4}(n-2a_1) \cdot P_n^{(-(1+2\kappa+2a_1),-(2+2\kappa))}(1-2z)\right]. \quad (70)$$

5. Results and Discussions

We have listed some numerical values for energy eigenvalues in Tables 1–6 separately for the cases of spin and pseudospin symmetries using the same parameter values in both cases for the Hellmann potential; this is valid also for the Varshni potential. But the values of the parameters for the Wei–Hua potential are different for the cases of spin and pseudospin symmetries. It could be seen that the dependence of the bound states for the Wei–Hua potential are more sensitive. It also should be stressed that the spin (and pseudospin) doublets, i. e., $(0,-2)$ and $(0,1)$ states or $(1,-2)$ and $(1,1)$ states, etc. are given up to fourth decimal in energy eigenvalues.

6. Conclusions

We have studied the approximate bound state solutions of the Dirac equation for the Hellmann potential, the Wei–Hua potential, and the Varshni potential, which have an exponential form depending on the spatially coordinate r, for the cases where the Dirac equation has pseudospin and spin symmetry, respectively.

Table 1. Energy eigenvalues of the Hellmann potential for the case of spin symmetry for $a = 0.25$, $b = 0.20$, $\beta = 0.02$, and $A_1 = M = 10$.

ℓ	n	κ	$E > 0$	n	κ	$E > 0$
1	0	−2	9.9995294	0	1	9.9995575
2		−3	9.9997604		2	9.9997394
3		−4	9.9999536		3	9.9999700
4		−5	10.0002770		4	10.0002900
1	1	−2	9.9994575	1	1	9.9995700
2		−3	9.9996894		2	9.9997300
3		−4	9.9999464		3	9.9999700
4		−5	10.0002740		4	10.0002900

Table 2. Energy eigenvalues of the Hellmann potential for the case of pseudospin symmetry for $a = 0.25$, $b = 0.20$, $\beta = 0.02$, and $A_2 = M = 10$.

ℓ	n	κ	$E > 0$	n	κ	$E > 0$
1	0	−2	9.9998031	0	1	9.9997925
2		−3	9.9997710		2	9.9998598
3		−4	9.9997412		3	9.9998977
4		−5	9.9997125		4	9.9992950
1	1	−2	9.9993925	1	1	9.9998281
2		−3	9.9994514		2	9.9998599
3		−4	9.9994310		3	9.9999016
4		−5	9.9993933		4	9.9999477

Table 3. Energy eigenvalues of the Wei–Hua potential for the case of spin symmetry for $a = 0.10$, $D = 0.0001$, $\beta = 0.01$, $A_1 = 2$, and $M = 0.001$.

ℓ	n	κ	$E > 0$	n	κ	$E > 0$
1	0	−2	1.9986997	0	1	1.9963313
2		−3	1.9993378		2	1.9991501
3		−4	1.9995723		3	1.9996379
4		−5	1.9996860		4	1.9997974
1	1	−2	1.9976352	1	1	1.9985369
2		−3	1.9993378		2	1.9996379
3		−4	1.9994823		3	1.9996379
4		−5	1.9994378		4	1.9998700

Table 4. Energy eigenvalues of the Wei–Hua potential for the case of pseudospin symmetry for $a = 0.25$, $D = 0.01$, $\beta = 0.10$, $A_2 = 10$, and $M = 1$.

ℓ	n	κ	$E > 0$	n	κ	$E > 0$
1	0	−2	1.0049979	0	1	0.9956234
2		−3	1.0056224		2	1.0024995
3		−4	1.0057785		3	1.0039570
4		−5	1.0058222		4	1.0045295
1	1	−2	1.0006250	1	1	1.0000000
2		−3	0.9974994		2	1.0024763
3		−4	0.9952527		3	1.0045295
4		−5	0.9932780		4	1.0064216

Table 5. Energy eigenvalues of the Varshni potential for the case of spin symmetry for $a = b = 0.15$, $\beta = 0.001$, and $A_1 = M = 5$.

ℓ	n	κ	$E > 0$	n	κ	$E > 0$
1	0	−2	4.9999970	0	1	4.9999814
2		−3	5.0000009		2	4.9999992
3		−4	5.0000023		3	5.0000024
4		−5	5.0000030		4	5.0000034
1	1	−2	4.9999884	1	1	4.9999961
2		−3	5.0000070		2	5.0000050
3		−4	5.0000022		3	5.0000024
4		−5	5.0000023		4	5.0000036

Table 6. Energy eigenvalues of the Varshni potential for the case of pseudospin symmetry for $a = b = 0.15$, $\beta = 0.001$, and $A_2 = M = 5$.

ℓ	n	κ	$E > 0$	n	κ	$E > 0$
1	0	−2	5.0000001	0	1	4.9999908
2		−3	5.0000008		2	4.9999984
3		−4	5.0000009		3	5.0000001
4		−5	5.0000008		4	5.0000008
1	1	−2	4.9999995	1	1	4.9999979
2		−3	5.0000007		2	4.9999994
3		−4	5.0000004		3	5.0000002
4		−5	5.0000000		4	5.0000008

The variation of the above potentials according to coordinate r are given in Figures 1 – 3. We have obtained the energy eigenvalue equations and the related two-component spinor wave functions with the help of the Nikiforov–Uvarov method and summarized the numerical results for the bound states in Tables 1 – 6. It is seen that the Nikiforov–Uvarov method is a suitable method to study the bound state solutions of the above potentials.

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