

# Laplace Transformation Approach to the Spin Symmetry of the Mie-Type Potential with a Coulomb Tensor Interaction

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The Dirac equation is solved exactly under the condition of spin symmetry for a spin  $1/2$  particle in the field of Mie-type potential and a Coulomb-like tensor interaction via the Laplace transform approach (LTA). The Dirac bound state energy equation and the corresponding normalized wave functions are obtained in closed forms with any spin-orbit coupling term  $\kappa$ . The effects of the tensor interaction and the potential parameters on the bound states are also studied. It is noticed that the tensor interaction removes degeneracy between two states in spin doublets. Some numerical results are given and a few special cases of interest are presented. We have demonstrated that in the non-relativistic limit, the solutions of the Dirac system converges to that of the Schrödinger system. The nonrelativistic limits of the present solutions are compared with the ones obtained by findings of other methods. Our results are sufficiently accurate for practical purpose.

*Key words:* Dirac Equation; Schrödinger Equation; Mie-Type Potential; Laplace Transformation Approach; Tensor Interaction.

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## 1. Introduction

The exact analytical solutions of the Dirac equation with spin and pseudo-spin ( $p$ -spin) symmetry have been extensively studied over the past years [1, 2]. Nonetheless, solving this equation is still a very challenging problem even if it has been derived and been utilized profusely more than 80 years ago. This equation is very useful while studying relativistic effects [3]. In the relativistic treatment of nuclear phenomena, the Dirac equation is used to describe the behaviour of nucleons in nuclei and also in solving many problems of high-energy physics and chemistry. For this reason, it has been used extensively in many fields such as relativistic heavy ion collisions, heavy ion spectroscopy and in laser–matter interaction (cf. [4] and references therein) and condensed matter physics [5].

The Mie-type potential which is an exactly solvable potential model [6] is also used in the study of the di-

atomic molecules [7]. The potential is used in the determination of molecular structure. It has recently received much attention in literature [8]. Various methods have been used to exactly solve the Schrödinger-like equation for a system bound by the Mie-type potentials. This method include the Laplace transform approach (LTA) [9], asymptotic iteration method [10], Lie algebraic approach (LAA) [11], and shape invariant algebra (SIA) method [12], etc.

The LAA was applied to the Hulthen potential to calculate the energy eigenvalues and the corresponding eigenfunctions. The Ladder operators were obtained and found to satisfy the SU(2) commutation relation [11]. The SIA was used to map the shape invariant potentials by the point canonical transformation (PCT) [12]. It was found that the Coulomb and Kratzer potentials can be mapped to the Morse potential with SIA belongs to SU(1,1) while the Pöschl–Teller type I belongs to the same sub class SU(2) as the Hulthen potential [12]. The exact solution of the Klein–Gordon

equation for charged particle in a magnetic field was found with the SIA method [13].

Recently, the relativistic bound states of spin-1/2 particles in the presence of a Mie-type potential have been found under spin and  $p$ -spin symmetry [1] by means of the asymptotic iteration method.

In the present work, we attempt to solve the Dirac equation for the Mie-type potential [1, 14] by using the LTA which is an integral transform and comprehensively useful in physics and engineering [15]. This method has been used by many authors to solve the Schrödinger-like equation for different potential models [16–21].

For example, the one-dimensional Morse potential [16],  $D$ -dimensional harmonic oscillator [17], non-central potentials [18], Morse-like potential [19], one-dimensional harmonic oscillator [20], double Dirac delta [21], and so on.

It could be a nearly new formalism in the literature and serves as a powerful algebraic treatment for solving the second-order differential equations. As a result, the LTA describes a simple way for solving of radial and one-dimensional differential equations. Another advantage of this approach is that a second-order differential equation can be converted into a simpler form whose solution may be obtained easily [16].

The Mie-type potential is one of the diatomic potentials having the form [1]

$$V(r) = \frac{a}{r^2} - \frac{b}{r} + c, \quad (1)$$

which is essentially a Coulomb potential plus an inversely square potential. Moreover, the Mie-type potential is essentially a Coulomb potential since the potential parameter  $a$  in (1) can be combined to the centrifugal term.

We consider the tensor interaction potential in the Coulomb form as

$$U(r) = -\frac{T}{r}, \quad T = \frac{Z_a Z_b e^2}{4\pi\epsilon_0}, \quad r \geq R_c, \quad (2)$$

where  $R_c$  is the Coulomb radius,  $Z_a$  and  $Z_b$  stand for the charges of the projectile particle  $a$  and the target nucleus  $b$ , respectively [22]. The tensor potential is a central linear, quadratic, Coulomb or Yukawa form central potentials introduced into the Dirac equation with a simple substitution  $\mathbf{p} \rightarrow \mathbf{p} - im\omega\boldsymbol{\beta} \cdot \hat{\mathbf{r}}U(r)$  [23]. The tensor couplings which are radically dependent potential functions have been used widely in the studies of

nuclear properties [24] and it has various physical applications [25].

The aim of the present work is to apply the LTA to the Dirac equation with the spin symmetry in the presence of tensor interaction. The present solution is being reduced to its nonrelativistic one under some appropriate transformations. In the practical case, we apply the present nonrelativistic model to study the energy states of some diatomic molecules.

For example, Wei and Dong [26–28] examined the spin symmetry in the Dirac equation for deformed generalized Pöschl–Teller potential, Manning–Rosen potential, and relativistic symmetrical well potential including a proper approximation to the spin–orbit coupling term. Setare and Nazari [29, 30] also studied the pseudo-spin symmetry in deformed nuclei with triaxial [29] and axially [30] symmetric harmonic oscillator potential. Setare and Haidari [31] obtained the exact analytical solutions of the  $S$ -wave Dirac equation with the reflectionless type Rosen–Morse and Manning–Rosen potentials under the condition of spin symmetry in the framework of the Nikiforov–Uvarov method. Overmore, the Dirac equation with scalar and vector Coulomb potential has been discussed in  $D$ -dimensions [32]. The group theory approach to the Dirac equation with a Coulomb plus scalar potential was used in  $D + 1$  dimensions [33, 34].

This paper is organized as follows. In Section 2, we present the general Dirac equation with scalar and vector Mie-type potentials and a Coulomb-like tensor potential. We then obtain the energy eigenvalues and normalized wave functions of this equation for  $\Sigma = S(r) + V(r) = V_{\text{Mie}}(r)$  and  $\Delta = V(r) - S(r) = C_s = \text{constant}$  within the framework of the LTA. In Section 3, we study some special cases like pure Coulomb, Kratzer–Fues, and modified Kratzer potentials and solutions in the absence of the tensor potential. Furthermore, we find the nonrelativistic limits and apply our results to two diatomic molecules. Numerical discussions are presented in Section 4. Finally, we give a brief conclusion in Section 4.

## 2. Dirac Equation with Scalar-Vector Mie-Type Potentials and Tensor Coupling

The Dirac equation which describes a nucleon with mass  $M$  moving in repulsive vector  $V(r)$  and attractive scalar  $S(r)$  potentials including a Coulomb-like tensor

potential  $U(r)$  reads (in units  $\hbar = c = 1$ ) [22, 35]

$$\left[ \boldsymbol{\alpha} \cdot \mathbf{p} + \boldsymbol{\beta} (M + S(r)) - i\boldsymbol{\beta}\boldsymbol{\alpha} \cdot \hat{r}U(r) \right] \boldsymbol{\psi}(\mathbf{r}) = (3) \\ [E - V(r)] \boldsymbol{\psi}(\mathbf{r}),$$

where  $E$  is the relativistic energy,  $M$  is the fermion mass,  $\mathbf{p} = -i\nabla$  is the momentum operator, and  $\boldsymbol{\alpha}$  and  $\boldsymbol{\beta}$  are  $4 \times 4$  matrices. The total angular momentum operator  $\mathbf{J}$  and spin-orbit coupling operator  $K = (\boldsymbol{\sigma} \cdot \mathbf{L} + 1)$ , where  $\mathbf{L}$  is orbital angular momentum, of the spherical nucleons commute with the Dirac Hamiltonian. The eigenvalues of spin-orbit coupling operator are  $\kappa = (j + 1/2) > 0$  and  $\kappa = -(j + 1/2) < 0$  for unaligned spin  $j = l - 1/2$  and the aligned spin  $j = l + 1/2$ , respectively.  $(H, K, J^2, J_z)$  can be taken as the complete set of the conservative quantities. Thus, the spinor wave functions can be classified according to their angular momentum  $j$ , spin-orbit quantum number  $\kappa$ , and the radial quantum number  $n$ , and can be written as follows:

$$\boldsymbol{\psi}_{n\kappa}(\mathbf{r}) = \begin{pmatrix} f_{n\kappa}(\mathbf{r}) \\ g_{n\kappa}(\mathbf{r}) \end{pmatrix} \\ = \frac{1}{r} \begin{pmatrix} F_{n\kappa}(r) Y_{jm}^l(\theta, \phi) \\ iG_{n\kappa}(r) Y_{jm}^{\tilde{l}}(\theta, \phi) \end{pmatrix}, \quad (4)$$

where  $f_{n\kappa}(\mathbf{r})$  is the upper (large) component and  $g_{n\kappa}(\mathbf{r})$  is the lower (small) component of the Dirac spinors.  $Y_{jm}^l(\theta, \phi)$  and  $Y_{jm}^{\tilde{l}}(\theta, \phi)$  are spin and  $p$ -spin spherical harmonics, respectively, and  $m$  is the projection of the angular momentum on the  $z$ -axis. Substituting (4) into (3) and using the relations [22], we obtain the following two Schrödinger-like differential equations for the upper and lower radial spinor components:

$$\left[ \frac{d^2}{dr^2} - \frac{\kappa(\kappa+1)}{r^2} + \frac{2\kappa}{r}U(r) - \frac{dU(r)}{dr} - U^2(r) \right] F_{n\kappa}(r) \\ + \left[ \frac{d\Delta(r)/dr}{M + E_{n\kappa} - \Delta(r)} \left( \frac{d}{dr} + \frac{\kappa}{r} - U(r) \right) \right. \\ \left. - (M + E_{n\kappa} - \Delta(r))(M - E_{n\kappa} + \Sigma(r)) \right] F_{n\kappa}(r) = 0 \quad (5a)$$

and

$$\left[ \frac{d^2}{dr^2} - \frac{\kappa(\kappa-1)}{r^2} + \frac{2\kappa}{r}U(r) + \frac{dU(r)}{dr} - U^2(r) \right] G_{n\kappa}(r) \\ - \left[ \frac{d\Sigma(r)/dr}{M - E_{n\kappa} + \Sigma(r)} \left( \frac{d}{dr} - \frac{\kappa}{r} + U(r) \right) \right. \\ \left. - [(M - E_{n\kappa})(M + E_{n\kappa} - C_s)] \right] G_{n\kappa}(r) = 0, \quad (5b)$$

$$+ (M + E_{n\kappa} - \Delta(r))(M - E_{n\kappa} + \Sigma(r)) \left. \right] G_{n\kappa}(r) = 0,$$

where  $\kappa(\kappa-1) = \tilde{l}(\tilde{l}+1)$ ,  $\kappa(\kappa+1) = l(l+1)$ ,  $\Delta(r) = V(r) - S(r)$ , and  $\Sigma(r) = V(r) + S(r)$ .

These radial wave functions should satisfy the asymptotic behaviours at the boundaries, i.e., must be finite at both  $r = 0$  and  $r \rightarrow \infty$ . The spin-orbit quantum number  $\kappa$  is related to the quantum numbers for spin symmetry  $l$  and  $p$ -spin symmetry  $\tilde{l}$  as

$$\kappa = \begin{cases} -(l+1) = -(j+1/2) & (s_{1/2}, p_{3/2}, \text{etc.}) \\ j = l + 1/2 & \text{for aligned spin } (\kappa < 0), \\ +l = +(j+1/2) & (p_{1/2}, d_{3/2}, \text{etc.}) \\ j = l - 1/2 & \text{for unaligned spin } (\kappa > 0), \end{cases} \quad (6)$$

and the quasi-degenerate doublet structure can be expressed in terms of a  $p$ -spin angular momentum  $\tilde{s} = 1/2$  and pseudo-orbital angular momentum  $\tilde{l}$ , which is defined as

$$\kappa = \begin{cases} -\tilde{l} = -(j+1/2) & (s_{1/2}, p_{3/2}, \text{etc.}) \\ j = \tilde{l} - 1/2 & \text{for aligned } p\text{-spin } (\kappa < 0), \\ +(\tilde{l}+1) = +(j+1/2) & (d_{3/2}, f_{5/2}, \text{etc.}) \\ j = \tilde{l} + 1/2 & \text{for unaligned } p\text{-spin } (\kappa > 0), \end{cases} \quad (7)$$

where  $\kappa = \pm 1, \pm 2, \dots$ . For example,  $(1s_{1/2}, 0d_{3/2})$  and  $(1p_{3/2}, 0f_{5/2})$  can be considered as  $p$ -spin doublets. We will look at the so-called spin symmetric case. This has notable applications in the study of deformation, superdeformation, identical bands, and magnetic moment in the nuclear structure [36]. Equation (5a) can not be solved analytically because of the  $\frac{d\Delta(r)/dr}{M + E_{n\kappa} - \Delta(r)} \left( \frac{d}{dr} + \frac{\kappa}{r} - U(r) \right)$  term. Following [37], taking a Mie-type potential for  $\Sigma(r)$ , and assuming that  $\frac{d\Delta(r)}{dr} = 0$  or alternatively  $\Delta(r) = C_s = \text{const.}$ , we have

$$\Sigma(r) = \frac{a}{r^2} - \frac{b}{r} + c, \quad \Delta(r) = C_s, \quad U(r) = -\frac{T}{r}. \quad (8)$$

In the presence of spin symmetry, (5a) with potential forms in (8) becomes

$$\left\{ \frac{d^2}{dr^2} - \frac{(\kappa+T)(\kappa+T+1)}{r^2} \right. \\ \left. - (M + E_{n\kappa} - C_s) \left( \frac{a}{r^2} - \frac{b}{r} + c \right) \right. \\ \left. - [(M - E_{n\kappa})(M + E_{n\kappa} - C_s)] \right\} F_{n\kappa}(r) = 0, \quad (9)$$

where we have introduced the following parameters for simplicity:

$$\begin{aligned}\lambda^2 &= (\kappa + T)(\kappa + T + 1) + a(M + E_{n\kappa} - C_s), \\ \delta^2 &= -b(M + E_{n\kappa} - C_s), \\ \varepsilon^2 &= (M - E_{n\kappa} + c)(M + E_{n\kappa} - C_s),\end{aligned}\quad (10)$$

and thus (9) can be rewritten as

$$\left\{ \frac{d^2}{dr^2} - \varepsilon^2 - \frac{\delta^2}{r} - \frac{\lambda^2}{r^2} \right\} F_{n\kappa}(r) = 0. \quad (11)$$

At this stage, we will use the Laplace transform approach to solve (17) [14–19]. It is an integral transform and comprehensively useful in physics and engineering: Recently, LTA has been used by many authors to solve the Schrödinger equation for different potential models [14–19]. The advantage of LTA is that it converts the second-order differential equation into a first-order differential equation whose solutions may be obtained easily [38, 39].

Now by defining a new function  $F_{n\kappa}(r) = \sqrt{r}\varphi(r)$ , equation (11) turns into

$$\left\{ r^2 \frac{d^2}{dr^2} + r \frac{d}{dr} - \left( \frac{\gamma^2}{r^2} + \frac{\delta^2}{r} + \varepsilon^2 \right) r^2 \right\} \varphi(r) = 0, \quad (12a)$$

$$\gamma^2 = \lambda^2 + \frac{1}{4}. \quad (12b)$$

Furthermore, by setting  $\varphi(r) = r^\tau \chi(r)$ , where  $\tau$  is a constant, (12a) becomes

$$\left\{ r^2 \frac{d^2}{dr^2} + (2\tau + 1)r \frac{d}{dr} - (\gamma^2 + \delta^2 r + \varepsilon^2 r^2 - \tau^2) \right\} \chi(r) = 0. \quad (13)$$

Now, to obtain a finite wave function when  $r \rightarrow \infty$ , we must take  $\tau = -\gamma$  in above (13), and then we obtain

$$\left\{ r \frac{d^2}{dr^2} - (2\gamma - 1) \frac{d}{dr} - (\delta^2 + \varepsilon^2 r) \right\} \chi(r) = 0. \quad (14)$$

Using the LTA defined as [14–19, 38, 39]

$$L\{\chi(r)\} = f(t) = \int_0^\infty dr e^{-tr} \chi(r), \quad (15)$$

(14) becomes

$$(t^2 - \varepsilon^2) \frac{df(t)}{dt} + [(2\gamma + 1)t + \delta^2] f(t) = 0, \quad (16)$$

which is a first-order ordinary differential equation and its solution is simply given as

$$f(t) = N(t + \varepsilon)^{-(2\gamma+1)} \left( \frac{t - \varepsilon}{t + \varepsilon} \right)^{-\frac{\delta^2}{2\varepsilon} - \frac{(2\gamma+1)}{2}}, \quad (17)$$

where  $N$  is a integral constant. The wave functions must be single-valued which required that

$$-\frac{\delta^2}{2\varepsilon} - \frac{(2\gamma+1)}{2} = n, \quad n = 0, 1, 2, \dots \quad (18)$$

We can easily obtain the energy eigenvalue of the radial part with inserting the parameters in (10) into (18). Taking into account this requirement and expanding (17) into series, we have

$$f(t) = N' \sum_{k=0}^n \frac{(-1)^k n! (2\varepsilon)^k (t + \varepsilon)^{-(2\gamma+1+k)}}{(n-k)! k!}, \quad (19)$$

where  $N'$  is a constant. Using the inverse Laplace transformation [40], we immediately obtain the solution of (19) as

$$\begin{aligned}\chi(r) &= N'' r^{2\gamma} e^{-\varepsilon r} \\ &\cdot \sum_{k=0}^n \frac{(-1)^k n!}{(n-k)! k!} \frac{\Gamma(2\gamma+1)}{\Gamma(2\gamma+1+k)} (2\varepsilon r)^k.\end{aligned}\quad (20)$$

Finally, by using  $\varphi(r) = r^{-\gamma} \chi(r)$ , we obtain

$$\begin{aligned}\varphi(r) &= N''' r^\gamma e^{-\varepsilon r} \\ &\cdot \sum_{k=0}^n \frac{(-1)^k n!}{(n-k)! k!} \frac{\Gamma(2\gamma+1)}{\Gamma(2\gamma+1+k)} (2\varepsilon r)^k,\end{aligned}\quad (21)$$

where  $N'''$  is a constant. On the other hand, the confluent hyper-geometric function is defined as a series expansion [40]

$${}_1F_1(-n; \sigma; z) = \sum_{m=0}^n \frac{(-1)^m n!}{(n-m)! m!} \frac{\Gamma(\sigma)}{\Gamma(\sigma+m)} z^m. \quad (22)$$

So, on comparing (22) with (21), we obtain the upper spinor component of the wave function:

$$F_{n\kappa}(r) = N r^{\gamma+1/2} e^{-\varepsilon r} {}_1F_1(-n; 2\gamma+1; 2\varepsilon r), \quad (23)$$

where  $N$  is a normalization constant;  $\gamma$  and  $\varepsilon$  are defined as  $\gamma = \sqrt{(\kappa + T + 1/2)^2 + a(M + E_{n\kappa} - C_s)}$ ,  $\varepsilon = \sqrt{(M + E_{n\kappa} - C_s)(M - E_{n\kappa} + c)}$ , respectively. The normalization condition is given in the Appendix.

Further, the lower spinor component of the wave function can be calculated as

$$G_{n\kappa}(r) = N \frac{1}{(M + E_{n\kappa} - C_s)} \left( \frac{d}{dr} + \frac{\kappa + T}{r} \right) \cdot r^{\gamma+1/2} e^{-\varepsilon r} {}_1F_1(-n; 2\gamma + 1; 2\varepsilon r), \quad (24)$$

where  $E_{n\kappa} \neq -M$  when  $C_s = 0$ , which means that only Dirac valence energy states are permissible for a normalizable and well-defined wave function [41]. Hence, there are no Dirac hole states [42]. By inserting the parameters in (10) into (18), one can obtain the energy equation, with spin symmetry, as

$$\frac{b\sqrt{M + E_{n\kappa} - C_s}}{\sqrt{M - E_{n\kappa} + c}} = 1 + 2n + 2\sqrt{(\kappa + T + 1/2)^2 + a(E_{n\kappa} + M - C_s)}, \quad (25)$$

$$M + c > E_{n\kappa},$$

where  $n = 0, 1, 2, \dots$ , which is identical to (52) of [1] when  $T = 0$ .

### 3. Some Special Cases

In this section, we study some special cases of interest in Section 2 and also calculate numerical results for the valence energy states. Further, we compare these results with the ones obtained by other methods.

#### 3.1. The Coulomb Potential ( $a = c = 0$ )

The eigensolutions of the pure Coulomb potential, in the presence of exact spin symmetry, can be found via (23), (24), and (25) as

$$b\sqrt{E_{n\kappa} + M - C_s} = 2(n + \kappa + T + 1)\sqrt{M - E_{n\kappa}}, \quad (26a)$$

$$M > E_{n\kappa}, \quad n = 0, 1, 2, \dots, \quad (26b)$$

$$F_{n\kappa}(r) = N r^{\gamma+1/2} e^{-\varepsilon r} {}_1F_1(-n; 2\gamma + 1; 2\varepsilon r), \quad (26b)$$

$$G_{n\kappa}(r) = N \frac{1}{(M + E_{n\kappa} - C_s)} \left( \frac{d}{dr} + \frac{\kappa + T}{r} \right) \cdot r^{\gamma+1/2} e^{-\varepsilon r} {}_1F_1(-n; 2\gamma + 1; 2\varepsilon r), \quad (26c)$$

where  $\varepsilon = \sqrt{(M - E_{n\kappa})(M + E_{n\kappa} - C_s)}$ , and  $\gamma = \kappa + T + 1/2$ .

In the nonrelativistic limit ( $E_{n\kappa} + M \rightarrow 2\mu$ ,  $E_{n\kappa} - M \rightarrow E_{nl}$ ,  $T = 0$ ,  $F_{n,\kappa}(r) \rightarrow R_{n,l}(r)$ ,  $\kappa \rightarrow l$ ), we have the eigensolutions [35, 43]

$$E_{nl} = -\frac{\mu b^2}{2(n + l + 1)^2}, \quad (27a)$$

$$n = 0, 1, 2, \dots, \quad l = 0, 1, 2, \dots,$$

$$R_{nl}(r) = C_{nl} r^{l+1} e^{-\sqrt{-2\mu E_{nl}} r} L_n^{2l+1} \left( 2\sqrt{-2\mu E_{nl}} r \right), \quad (27b)$$

where  $E_{nl} < 0$  and the normalization constant  $C_{nl}$  is given in the Appendix.

#### 3.2. The Kratzer–Fues potential

The Kratzer–Fues potential can be derived as a simple example from the Mie-type potential by setting  $a = D_e r_e^2$ ,  $b = 2D_e r_e$ , and  $c = 0$  [1, 14]:

$$V_{KF}(r) = -D_e \left( \frac{2r_e}{r} - \frac{r_e^2}{r^2} \right), \quad (28)$$

where  $D_e$  is the dissociation energy, and  $r_e$  is the equilibrium inter-nuclear length.

The energy equation (25) becomes

$$\frac{2D_e r_e \sqrt{E_{n\kappa} + M - C_s}}{\sqrt{(M - E_{n\kappa} + c)}} = 1 + 2n + 2\sqrt{(\kappa + T + 1/2)^2 + D_e r_e^2 (E_{n\kappa} + M - C_s)}, \quad (29)$$

where  $n = 0, 1, 2, \dots$  and the wave function is as (29) with

$$\gamma = \sqrt{(\kappa + T + 1/2)^2 + D_e r_e^2 (E_{n\kappa} + M - C_s)}, \quad (30)$$

$$\varepsilon = \sqrt{M^2 - E_{n\kappa}^2 + C_s(E_{n\kappa} - M)}.$$

The numerical results for this case are given in Table 1.

#### 3.3. The Modified Kratzer Potential

Another example is the modified Kratzer potential which is obtained by setting  $a = D_e r_e^2$ ,  $b = 2D_e r_e$ , and

Table 1. Valence bound states of the Kratzer-Fues and the modified Kratzer potentials in the exact spin symmetry for several values of  $n$  and  $\kappa$  with  $M = 5.0 \text{ fm}^{-1}$ ,  $D_e = 1.25 \text{ fm}^{-1}$ , and  $r_e = 0.35 \text{ fm}$ .

$l$	$n, \kappa < 0$	$(l, j = l + 1/2)$	potential	$E_{n, \kappa < 0}$		
				$T = 0$	$T = 10$	$T = 15$
1	0, -2	$0p_{3/2}$	Kratzer-Fues modified Kratzer	4.68657 5.91047	4.97689 6.22406	4.99032 6.23912
2	0, -3	$0d_{5/2}$	Kratzer-Fues modified Kratzer	4.82569 6.05779	4.97092 6.21739	4.98879 6.23741
3	0, -4	$0f_{7/2}$	Kratzer-Fues modified Kratzer	4.89317 6.13124	4.96235 6.20781	4.98687 6.23525
4	0, -5	$0g_{9/2}$	Kratzer-Fues modified Kratzer	4.92879 6.17051	4.94943 6.19342	4.98441 6.23249
1	1, -2	$1p_{3/2}$	Kratzer-Fues modified Kratzer	4.84067 6.0756	4.98123 6.22893	4.99156 6.24052
2	1, -3	$1d_{5/2}$	Kratzer-Fues modified Kratzer	4.89691 6.13581	4.97695 6.22413	4.99033 6.23913
3	1, -4	$1f_{7/2}$	Kratzer-Fues modified Kratzer	4.93000 6.17201	4.97103 6.21752	4.98880 6.23742
—	$n, \kappa > 0$	$(l, j = l - 1/2)$		$E_{n, \kappa > 0}$		
1	0, 1	$0p_{1/2}$	Kratzer-Fues modified Kratzer	4.68657 5.91047	4.98687 6.23525	4.99342 6.2426
2	0, 2	$0d_{3/2}$	Kratzer-Fues modified Kratzer	4.82569 6.05779	4.98879 6.23741	4.99412 6.24339
3	0, 3	$0f_{5/2}$	Kratzer-Fues modified Kratzer	4.89317 6.13124	4.99032 6.23912	4.99472 6.24407
4	0, 4	$0g_{7/2}$	Kratzer-Fues modified Kratzer	4.92879 6.17051	4.99156 6.24051	4.99524 6.24464
1	1, 1	$1p_{1/2}$	Kratzer-Fues modified Kratzer	4.84067 6.0756	4.9888 6.23742	4.99413 6.2434
2	1, 2	$1d_{3/2}$	Kratzer-Fues modified Kratzer	4.89691 6.13581	4.99033 6.23913	4.99472 6.24407
3	1, 3	$1f_{5/2}$	Kratzer-Fues modified Kratzer	4.93000 6.17201	4.99156 6.24052	4.99524 6.24464
5	1, 5	$1h_{9/2}$	Kratzer-Fues modified Kratzer	4.96256 6.20808	4.99342 6.2426	4.99606 6.24557

$c = D_e$  [1, 14]. Thus we can have

$$V_{m\kappa}(r) = D_e \left( \frac{r - r_e}{r} \right)^2. \quad (31)$$

The relativistic energy equation can be easily obtained as

$$\frac{2D_e r_e \sqrt{E_{n\kappa} + M - C_s}}{\sqrt{M - E_{n\kappa} + D_e}} = 1 + 2n + 2\sqrt{(\kappa + T + 1/2)^2 + D_e r_e^2 (E_{n\kappa} + M - C_s)}, \quad (32)$$

where  $n = 0, 1, 2, \dots$ , and the wave function is same as (23) with

$$\gamma = \sqrt{(\kappa + T + 1/2)^2 + D_e r_e^2 (E_{n\kappa} + M - C_s)}, \quad (33)$$

$$\varepsilon = \sqrt{(M + E_{n\kappa} - C_s)(M - E_{n\kappa} + D_e)}.$$

### 3.4. Nonrelativistic Limit

By applying the following approximate mapping  $E_{n\kappa} - Mc^2 \rightarrow E_{n\ell}$ ,  $C_s = 0$ ,  $T = 0$ , and  $1/\hbar^2 c^2 \cdot (Mc^2 + E_{n\ell}) \rightarrow 2\mu/\hbar^2$  to (25), we obtain the energy levels of the Schrödinger equation for any arbitrary orbital quantum number  $\ell$  as

$$E_{n\ell} = c - \frac{2\mu b^2}{\hbar^2} \cdot \left[ 1 + 2n + 2\sqrt{\left(\ell + \frac{1}{2}\right)^2 + \frac{2\mu a}{\hbar^2}} \right]^{-2}, \quad (34)$$

which is identical to the ones obtained previously by using the Nikiforov-Uvarov method [43, 44] and is also identical to the one obtained by using the asymptotic iteration method for the special case  $c = 0$  [10]. Some numerical results of (34) are given in Table 2.

Table 2. Ro-vibrational energy spectral (in eV) for various  $n$  and  $\ell$  quantum numbers for two diatomic molecules, where  $\hbar c = 197.329$ ,  $a = D_e r_e^2$ ,  $b = 2D_e r_e$ , and  $c = 0, D_e$ .

$n$	$\ell$	$c$	CO (Present)	CO ([44])	NO (Present)	NO ([44])	Non-Relativistic CO	Non-Relativistic NO
0	0	$D_e$	0.0508296037	0.050823	0.4112332403	0.0411118	5.139293772771225E-7	4.1587977150925326E-7
	0		-10.79431511	-10.79431532	-8.002659244	-8.002659417	-10.845144196070622	-8.043782152120228
	0	$D_e$	0.1513054257	0.151287	0.1223262450	0.122311	1.541788005710032E-6	1.247639211499063E-6
	1		0.1517741263	0.151755	0.1227392593	0.122724	1.5417880572243803E-6	1.2476392541316272E-6
1	0		-10.69383928	-10.69383992	-7.921456323	-7.921456839	-10.845143168211994	-8.043781320360788
	1		-10.69337058	-10.69337123	-7.921043309	-7.921043834	-10.845143168211942	-8.043781320360745
	0	$D_e$	0.2503848730	0.250354	.2022994594	0.202274	2.5696464884816805E-6	2.079398520038467E-6
	1		0.2508470750	0.250816	0.2027062350	0.202681	2.569646538219672E-6	2.079398562671031E-6
2	2		0.2517713576	0.251744	0.2035196592	0.203494	2.569646635919298E-6	2.079398649712516E-6
	0		-10.59475984	-10.59476089	-7.841483109	-7.841483956	-10.845142140353511	-8.04378048860148
	1		-10.59429764	-10.59429869	-7.841076333	-7.841077188	-10.845142140353461	-8.043780488601437
	2		-10.59337335	-10.59337441	-7.840262909	-7.840263771	-10.845142140353364	-8.04378048860135
	0	$D_e$	0.3480937013	0.348051	.2810676721	0.281033	3.5975048255920683E-6	2.911157700680178E-6
	1		0.3485495242	0.348507	0.2814683340	0.281434	3.59750487533006E-6	2.9111577433127422E-6
	2		0.3494610508	0.349418	0.2822695334	0.282235	3.597504973029686E-6	2.9111578303542274E-6
3	3		0.3508280427	0.350785	0.2834710209	0.283436	3.59750511691459E-6	2.9111579618046335E-6
	0		-10.49705101	-10.49705246	-7.762714896	-7.762716066	-10.845141112495174	-8.043779656842299
	1		-10.49659519	-10.49659664	-7.762314234	-7.762315413	-10.845141112495124	-8.043779656842256
	2		-10.49568366	-10.49568512	-7.761513035	-7.761514218	-10.845141112495027	-8.043779656842169
	3		-10.49431667	-10.49431814	-7.760311547	-7.760312744	-10.845141112494883	-8.043779656842037
	0	$D_e$	0.4444570753	0.444403	.3586549706	0.358611	4.625363018817552E-6	3.74291675164784E-6
	1		0.4449066360	0.444852	0.3590496408	0.359006	4.62536306855544E-6	3.742916796056761E-6
	2		0.4458056402	0.445751	0.3598388590	0.359795	4.625363162702456E-6	3.7429168813218894E-6
	3		0.4471538535	0.447099	0.3610223806	0.360978	4.625363308363717E-6	3.7429170109959387E-6
4	4		0.4489509248	0.448895	0.3625998392	0.362555	4.625363503762969E-6	3.742917183302552E-6
	0		-10.40068763	-10.40068947	-7.685127597	-7.685129079	-10.84514008463698	-8.043778825083248
	1		-10.40023807	-10.40023992	-7.684732927	-7.684734417	-10.845140084636931	-8.043778825083203
	2		-10.39933907	-10.39934092	-7.683943709	-7.683945203	-10.845140084636837	-8.043778825083118
	3		-10.39799086	-10.39799272	-7.682760187	-7.682761696	-10.845140084636691	-8.043778825082988
	4		-10.39619379	-10.39619567	-7.681182729	-7.681184246	-10.845140084636496	-8.043778825082816
	0	$D_e$	0.5394995845	0.539434	.4350848437	0.435032	5.6532210610527045E-6	4.57467567471781E-6
	1		0.5399429972	0.539877	0.4354736410	0.435421	5.653221112567053E-6	4.574675717350374E-6
	2		0.5408297073	0.540764	0.4362511158	0.436198	5.653221208490322E-6	4.574675804391859E-6
	3		0.5421594846	0.542093	0.4374170279	0.437364	5.653221354151583E-6	4.574675934065908E-6
	4		0.5439319837	0.543865	0.4389710177	0.438917	5.653221549550835E-6	4.574676104596165E-6
5	5		0.5461467446	0.546082	0.4409126059	0.440858	5.653221792911722E-6	4.574676321311699E-6
	0		-10.30564513	-10.30564735	-7.608697724	-7.608699509	-10.845139056778939	-8.043777993324325
	1		-10.30520171	-10.30520394	-7.608308927	-7.608310719	-10.845139056778887	-8.043777993324282
	2		-10.30431500	-10.30431723	-7.607531452	-7.607533248	-10.845139056778791	-8.043777993324195
	3		-10.30298523	-10.30298747	-7.606365540	-7.606367349	-10.845139056778645	-8.043777993324065
	4		-10.30121273	-10.30121499	-7.604811550	-7.604813368	-10.84513905677845	-8.043777993323895
	5		-10.29899797	-10.29900024	-7.602869962	-7.602871795	-10.845139056778207	-8.043777993323678

#### 4. Numerical Results

The following parameters  $M = 5.0 \text{ fm}^{-1}$ ,  $T = 0, 10, 15$ ,  $C_s = 0$ ,  $r_e = 0.35 \text{ fm}$ , and  $D_e = 1.25 \text{ fm}^{-1}$  have been used in our numerical calculations to obtain energies of Dirac valence states. The numerical results are summarized in Table 1.

In Table 1, we see that energies of bound states such as:  $(np_{1/2}, np_{3/2})$ ,  $(nd_{3/2}, nd_{5/2})$ ,  $(nf_{5/2}, nf_{7/2})$ ,  $(ng_{7/2}, ng_{9/2}), \dots$  (where each pair is considered as a spin doublet) in the absence of the tensor potential are degenerate but in the presence of the tensor potential, these degeneracies are removed. Further, we investigated the effects of the parameters

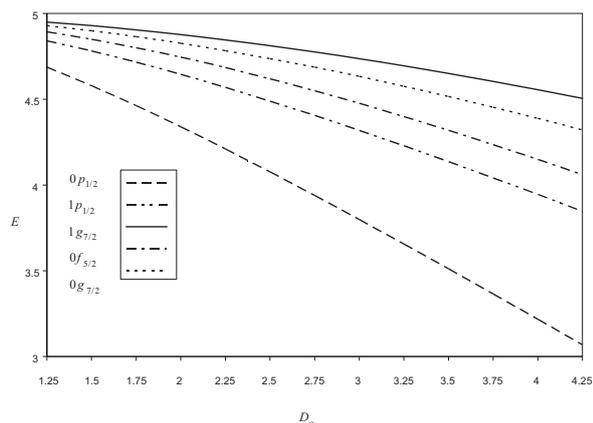


Fig. 1. Valence bound states of the Kratzer-Fues potential for several states under the exact spin symmetry with  $D_e$  ( $\text{fm}^{-1}$ ) for  $M = 5.0 \text{ fm}^{-1}$ ,  $T = 0$ , and  $r_e = 0.35 \text{ fm}$ .

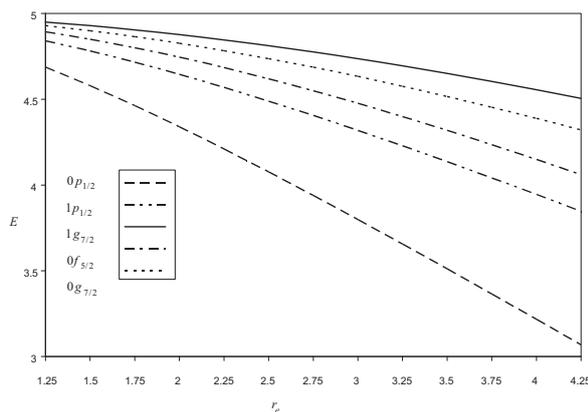


Fig. 2. Valence bound states of the Kratzer-Fues potential for several states under the exact spin symmetry with  $r_e$  (fm) for  $M = 5.0 \text{ fm}^{-1}$ ,  $T = 0$ , and  $D_e = 1.25 \text{ fm}^{-1}$ .

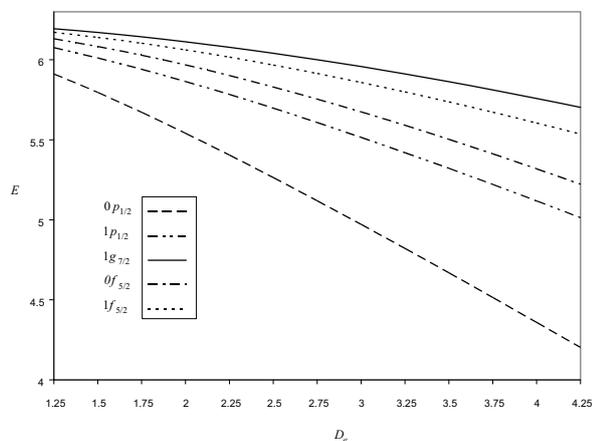


Fig. 3. Valence bound states of the modified Kratzer potential for several states under the exact spin symmetry with  $D_e$  ( $\text{fm}^{-1}$ ) for  $M = 5.0 \text{ fm}^{-1}$ ,  $T = 0$ , and  $r_e = 0.35 \text{ fm}$ .

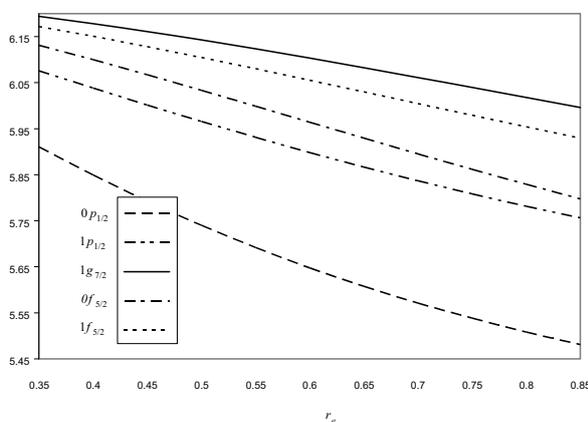


Fig. 4. Valence bound states of the modified Kratzer potential for several states under the exact spin symmetry with  $r_e$  (fm) for  $M = 5.0 \text{ fm}^{-1}$ ,  $T = 0$ , and  $D_e = 1.25 \text{ fm}^{-1}$ .

$D_e$  and  $r_e$  on the bound states under the condition of the spin symmetry limits for  $T = 0$ . It is clearly seen that if  $D_e$  and  $r_e$  increase, the valence energy states of the Kratzer-Fues potential decrease for several states. Now, if  $D_e$  increasing, the bound state energy eigenvalues of the modified Kratzer potential increases, and with  $r_e$  increases, the valence energy state of this potential decreases for several states.

For a given value of  $n$  and  $\kappa$  (or  $l$ ) the energy equation in spin symmetry provides two distinct positive and negative energy spectra related with  $E_{n\kappa}^+$  or  $E_{n\kappa}^-$ , respectively. However, the positive energy solution is

valid for the spin symmetry limit since  $E \neq -M$  in the denominator of the lower-spinor component of the wave function. Otherwise the wave function becomes not integrable or normalized in such a case [41–43].

We also extended our numerical computation to nonrelativistic limits. Using the parameters  $D_e = 10.84514471$ ,  $r_e = 1.1282$ , and  $\mu = 6.86058600$  for diatomic molecule CO and  $D_e = 8.043782568$ ,  $r_e = 1.1508$ , and  $\mu = 7.46844100$  for diatomic molecule NO, the ro-vibrational energies of the modified Kratzer potential and Kratzer-Fues potential for these two selected diatomic molecules are calculated by means of (34). The numerical results are given in Table 2. Obvi-

ously, our results are in excellent agreement with the ones obtained previously via other methods.

## 5. Conclusion

In this paper, we have exactly solved the Dirac equation under the condition of spin symmetry for the Mie-type potential in the presence of the tensor interaction by using the LTA. The bound state energy equation and the corresponding normalized eigenfunctions of the Dirac equation are obtained explicitly, which are likely of much interest in different fields of physics. Some numerical results for the energy eigenvalues of the Kratzer–Fues and the modified Kratzer potentials are summarized in Tables 1 and 2, respectively. By using the LTA, for the first time, our results in Table 1 when the tensor interaction is vanishing (i.e.,  $T = 0$ ) are in high agreement with those obtained by Aydogdu and Sever [1] and Hamzavi et al. [14] in the framework of the asymptotic iteration method and the Nikiforov–Uvarov method, respectively. Our energy levels are accurate to the sixth digit after the decimal point if compared with Tables 1 and 2 of [1]. We also calculated these states in the presence of tensor interaction for the case  $T \neq 0$ . Overmore, our energy eigenvalues in our Table 1 for  $\kappa > 0$  and  $\kappa < 0$  are in high agreement (nearly the same) with those in Tables 2 and 4 of [14] in the presence of tensor interaction, i.e., when  $T = 0, 10, 15$  ( $\alpha = 0, 10, 15$  in notations of [19]) and with [1] in the absence of tensor interaction, i.e.,  $T = 0$  ( $\alpha = 0$  in notation of [1]). Also our results in Table 2 are in exact agreement with those in Tables 6 and 8 of [14] when  $T = 0, 10, 15$  and [1] for  $T = 0$ . It is found that the tensor interaction removes the degeneracy between two states in spin doublets. Using the LTA, we applied the nonrelativistic model to generate the rotational-vibrational energy eigenvalues of the Kratzer and modified Kratzer potentials for some diatomic molecules like CO and NO. As shown in Tables 4 and 5, the comparison of our nonrelativistic results with those obtained by Berkdemir et al. [44] using the Nikiforov–Uvarov method shows a good agreement. This reveals the high accuracy and efficiency of the present method which reaches the seventh digit after the decimal point. The physical significance of the results demonstrate that it can be applied to various physical models in relativistic as well as nonrelativistic cases. We also find that this treatment is quite reliable, and further analytical calcula-

tions with the Laplace transform method would be useful.

## Appendix

### Normalization Constant of the Radial Wave Function

Unlike the nonrelativistic case, the normalization condition for the Dirac spinor combines the two individual normalization constants  $N_{n\kappa}$  in one single integral [35]. The radial wave functions are normalized according to the formula  $\int_0^\infty \psi_{n\kappa}^\dagger(r) \psi_{n\kappa}(r) r^2 dr = 1$  which explicitly implies for the two-spinor components  $f_{n\kappa}(r)$  and  $g_{n\kappa}(r)$  in (4) that [35]

$$\int_0^\infty (f_{n\kappa}^2(r) + g_{n\kappa}^2(r)) r^2 dr = \int_0^\infty (F_{n\kappa}^2(r) + G_{n\kappa}^2(r)) dr = 1. \quad (\text{A1})$$

The upper full radial wave function is

$$F_{n\kappa}(r) = Nr^{\gamma+1/2} e^{-\varepsilon r} {}_1F_1(-n; 2\gamma+1; 2\varepsilon r), \quad (\text{A2})$$

where  $\gamma = \sqrt{(\kappa + T + 1/2)^2 + a(M + E_{n\kappa} - C_s)}$ ,  $\varepsilon = \sqrt{(M + E_{n\kappa} - C_s)(M - E_{n\kappa} + c)}$ , respectively. After making transformations  $E_{n\kappa} \rightarrow -E_{n\kappa}$ ,  $V(r) \rightarrow -V(r)$ ,  $\kappa \rightarrow \kappa - 1$ ,  $F_{n\kappa}(r) \rightarrow G_{n\kappa}(r)$ , and  $C_s \rightarrow -C_{ps}$ , we obtain the lower radial wave function as

$$G_{n\kappa}(r) = Nr^{\omega+1/2} e^{-\varepsilon' r} {}_1F_1(-n; 2\omega+1; 2\varepsilon' r), \quad (\text{A3})$$

where  $\omega = \sqrt{(\kappa + T - 1/2)^2 + a(E_{n\kappa} - M - C_{ps})}$ ,  $\varepsilon' = \sqrt{(M - E_{n\kappa} + C_{ps})(M + E_{n\kappa} - c)}$ , respectively. Therefore relation (A1) with (A2) and (A3) becomes

$$N^2 \left[ \left( \frac{n! \Gamma(2\gamma+1)}{\Gamma(n+2\gamma+1)} \right)^2 \int_0^\infty r^{2\gamma+1} e^{-2\varepsilon r} [L_n^{2\gamma}(2\varepsilon r)]^2 dr + \left( \frac{n! \Gamma(2\omega+1)}{\Gamma(n+2\omega+1)} \right)^2 \int_0^\infty r^{2\omega+1} e^{-2\varepsilon' r} [L_n^{2\omega}(2\varepsilon' r)]^2 dr \right] = 1, \quad (\text{A4})$$

where we have used  $L_n^p(x) = \frac{\Gamma(n+p+1)}{n! \Gamma(p+1)} {}_1F_1(-n; p+1; x)$  [40]. Now using

$$\int_0^\infty x^{k+1} e^{-x} [L_n^k(x)]^2 dx = \frac{\Gamma(n+k+1)}{n!} (2n+k+1), \quad (\text{A5})$$

we finally get

$$N^2 \left[ \left( \frac{\Gamma(2\gamma+1)}{(2\varepsilon)^{\gamma+1}} \right)^2 \frac{n!(2n+2\gamma+1)}{\Gamma(n+2\gamma+1)} + \left( \frac{\Gamma(2\omega+1)}{(2\varepsilon')^{\omega+1}} \right)^2 \frac{n!(2n+2\omega+1)}{\Gamma(n+2\omega+1)} \right] = 1. \quad (\text{A6})$$

In the nonrelativistic limit, the normalization constant becomes

$$C_{nl} = \sqrt{\frac{n!}{\Gamma(n+2l+2)(2n+2l+2)}}$$

$$\cdot \left( 2\sqrt{-2\mu E_{nl}} \right)^{l+3/2}. \quad (\text{A7})$$

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