Majorana Fermions on a Lattice and a Matrix Problem

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We investigate matrix problems arising in the study of Majorana Fermions on a lattice.

Key words: Majorana Fermions; Eigenvalue Problem; Anticommutator.

Majorana Fermions on a lattice have been studied by many authors [1 – 9] in particular in connection with Fermionic quantum computing. Here the following matrix problem arises: Find a pair of $n\times n$ hermitian matrices A and B such that $A^2 = I_n$, $B^2 = I_n$, and $[A,B]_+ = 0_n$ (i.e. the anticommutator vanishes). Here I_n is the $n\times n$ identity matrix, and 0_n is the $n\times n$ zero matrix. An example of such a pair of matrices is given by

$$A = \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad B = -\sigma_2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix},$$

where σ_1, σ_2, σ_3 are the Pauli spin matrices. Note that the Pauli spin matrices satisfy $[\sigma_1,\sigma_2]_+ = 0_2$, $[\sigma_2,\sigma_3]_+ = 0_2$, $[\sigma_2,\sigma_1]_+ = 0_2$, and $\sigma_1^2 = \sigma_2^2 = \sigma_3^2 = I_2$.

Here we derive properties of such pairs of $n\times n$ matrices A and B and give some applications.

First we show that the conditions cannot be satisfied if n is odd. From $AB = -BA$ we obtain $\det(AB) = \det(-BA)$. Thus $\det(AB) = (-1)^n\det(AB)$. Since $\det(AB) \neq 0$ this implies that the condition $[A,B]_+ = 0_n$ cannot be satisfied if n is odd.

Thus in the following we assume that n is even and A, B satisfy the conditions $A^2 = I_n$, $B^2 = I_n$, and $AB + BA = 0_n$. From the conditions $A^2 = I_n$ and $B^2 = I_n$ it follows that $A = A^{-1}$ and $B = B^{-1}$. From the condition $AB + BA = 0_n$, we also find that $\mathrm{tr}(AB) = 0$ and therefore $\det(\mathrm{e}^{AB}) = 1$. It also follows that $(AB)^2 = (BA)^2 = -I_n$.

Next we show that half of the eigenvalues of A (and of B) must be $+1$ and the other half must be -1. Let $\mathbf{v}$ be an eigenvector of A corresponding to the eigenvalue 1. Then from

$$(AB+BA)\mathbf{v} = AB\mathbf{v} + B\mathbf{v} = (A+I_n)B\mathbf{v} = \mathbf{0}$$

and the fact that $\mathbf{v} \neq \mathbf{0}$, $B\mathbf{v} \neq \mathbf{0}$ it follows that $B\mathbf{v}$ is an eigenvector of A corresponding to the eigenvalue -1. Similarly if $\mathbf{v}$ is an eigenvector of A corresponding to the eigenvalue -1, then $B\mathbf{v}$ is an eigenvector of A corresponding to the eigenvalue 1. Since B is invertible it follows that the eigenspaces of A corresponding to the eigenvalues 1 and -1 have the same dimension.

The above argument is also constructive, i.e. given A and an orthonormal basis

$$\{\mathbf{v}_{-1,1}, \mathbf{v}_{-1,2}, \ldots, \mathbf{v}_{-1,n/2}, \mathbf{v}_{1,1}, \mathbf{v}_{1,2}, \ldots, \mathbf{v}_{1,n/2}\}$$

composed of eigenvectors $\mathbf{v}_{-1,j}$ corresponding to the eigenvalue -1 of A and eigenvectors $\mathbf{v}_{1,j}$ corresponding to the eigenvalue 1 of A, we can construct a B satisfying the properties above as

$$B = \sum_{j=1}^{n/2} (\mathbf{v}_{1,j}\mathbf{v}_{-1,j}^* + \mathbf{v}_{-1,j}\mathbf{v}_{1,j}^*).$$

All B's can be constructed in this way by an appropriate choice of an orthonormal basis.

For any $n\times n$ matrices X and Y over $\mathbb{C}$ we have the following expansion utilizing the anticommutator [10, 11]:

$$\mathrm{e}^X Y \mathrm{e}^X = Y + [X,Y]_+ + \frac{1}{2!}[X,[X,Y]_+]_+ + \frac{1}{3!}[X,[X,[X,Y]_+]_+]_+ + \cdots .$$

Consequently we find

$$\mathrm{e}^X Y \mathrm{e}^{-X} = \left(Y + [X,Y]_+ + \frac{1}{2!}[X,[X,Y]_+]_+ + \frac{1}{3!}[X,[X,[X,Y]_+]_+]_+ + \cdots \right) \mathrm{e}^{-2X},$$

$$e^X Y e^{-X} = e^{2X}\left(Y - [X,Y]_+ + \frac{1}{2!}[X,[X,Y]_+]_+ - \frac{1}{3!}[X,[X,[X,Y]_+]_+]_+ + \cdots\right).$$

Thus since $[A,B]_+ = 0_n$ for the matrices A and B, we have

$$e^A B e^{-A} = B e^{-2A} \quad \text{and} \quad e^A B e^{-A} = e^{2A} B.$$

Utilizing $B^2 = I_n$, we have $e^{-A} = B e^A B$. Note that ($z \in \mathbb{C}$)

$$e^{zA} = I_n \cosh(z) + A \sinh(z),$$
$$e^{zB} = I_n \cosh(z) + B \sinh(z),$$

and

$$e^{zAB} = I_n \cos(z) + AB \sin(z).$$

Let $\otimes$ be the Kronecker product [12–15]. Then we have

$$e^{z(A\otimes B)} = (I_n \otimes I_n)\cosh(z) + (A\otimes B)\sinh(z)$$

and for the anticommutators

$$[A\otimes I_n, B\otimes I_n]_+ = 0_{n^2},$$
$$[I_n\otimes A, I_n\otimes B]_+ = 0_{n^2},$$

where $(A\otimes I_n)^2 = I_n \otimes I_n$ and $(B\otimes I_n)^2 = I_n\otimes I_n$. Thus using the Kronecker product and the identity matrix I_n, we can construct matrices in higher dimensions which satisfy the conditions. This can be extended. Let C be an $n\times n$ matrix with $C^2 = I_n$. Then the pair $C\otimes A$ and $C\otimes B$ also satisfies the conditions $(C\otimes A)^2 = I_n\otimes I_n$, $(C\otimes B)^2 = I_n \otimes I_n$, $[C\otimes A, C\otimes B]_+ = 0_{n^2}$. An example for $n = 2$ would be $A = \sigma_3\otimes\sigma_1$, $B = -\sigma_3\otimes\sigma_2$. However note that

$$[A\otimes A, B\otimes B]_+ = 2((AB)\otimes(AB)),$$
$$[A\otimes B, A\otimes B]_+ = 2I_n\otimes I_n,$$
$$[A\otimes B, B\otimes A]_+ = -2((AB)\otimes(AB)).$$

Let $\oplus$ be the direct sum and A, B be a pair of hermitian matrices satisfying $A^2 = B^2 = I_n$ and $[A,B]_+ = 0_n$. Then $A\oplus A$ and $B\oplus B$ is also such a pair (in $2n$ dimensions). Let A, B be such a pair with $n = 2$. Then the 4×4 matrices

$$A\star A := \begin{pmatrix} a_{11} & 0 & 0 & a_{12} \\ 0 & a_{11} & a_{12} & 0 \\ 0 & a_{21} & a_{22} & 0 \\ a_{21} & 0 & 0 & a_{22} \end{pmatrix}$$

and $B\star B$ are also such a pair.

From the pair A, B, we can also form the $n\times n$ matrix $A + iB$ which is non-normal, i.e. $(A+iB)^*(A+iB) \neq (A+iB)(A+iB)^*$. Such matrices play a role for the study of non-hermitian Hamilton operators [16]. Since $(A+iB)^2 = 0_n$, we obtain

$$e^{z(A+iB)} = I_n + z(A+iB).$$

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