Approximate Solution of the Spin-0 Particle Subject to the Trigonometric Pöschl–Teller Potential with Centrifugal Barrier

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The trigonometric Pöschl–Teller (PT) potential describes the diatomic molecular vibration. In this paper, we study the approximate solutions of the radial Klein–Gordon (KG) equation for the rotating trigonometric PT potential using the Nikiforov–Uvarov (NU) method. The energy eigenvalues and their corresponding eigenfunctions are calculated for arbitrary l-states in closed form. We obtain the non-relativistic limit and present some numerical results for both relativistic and non-relativistic cases.

Key words: Klein–Gordon Equation; Trigonometric Pöschl–Teller Potential; Nikiforov–Uvarov Method.
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1. Introduction

Relativistic wave equations and particularly the Dirac and the Klein–Gordon (KG) equations have been of interest for theoretical physicists in many branches of physics [1, 2]. In recent years, there has been an increased interest in finding exact solutions to relativistic spinless KG particles with various vector and scalar potentials [3 – 8]. The most commonly used techniques to explore these wave equations are the Nikiforov–Uvarov (NU) method [9, 10], the supersymmetric quantum mechanics method [11, 12], the point canonical transformation [13], the iteration method [14 – 16], the exact quantization rule [17], the shifted $1/N$ expansion (SE) technique [18], and the ansatz approach [19].

The trigonometric Pöschl–Teller (PT) potential proposed for the first time by Pöschl and Teller [20] in 1933 was to describe the diatomic molecular vibration. Chen [21] and Zhang and Wang [22] have studied the relativistic bound state solutions for the trigonometric PT potential and hyperbolical PT (second PT) potential, respectively. Liu et al. [23] studied the trigonometric PT potential within the framework of the Dirac theory. Very recently, Hamzavi and Rajabi studied the exact s-wave solution $(l = 0)$ of the Schrödinger equation for the vibrational trigonometric PT potential [24]. This potential takes the following form:

$$V(r) = \frac{V_1}{\sin^2(\alpha r)} + \frac{V_2}{\cos^2(\alpha r)}\,, \tag{1}$$

where the parameters V_1 and V_2 describe the property of the potential well while the parameter α is related to the range of this potential [23, 25, 26]. In Figure 1, we draw the trigonometric PT potential (1) for parameter values $V_1 = 5.0\,\mathrm{fm}^{-1}$, $V_2 = 3.0\,\mathrm{fm}^{-1}$, and $\alpha = 0.8\,\mathrm{fm}^{-1}$.

The aim of the present work is to extend our previous work [24] to the relativistic limit and the case of $l \neq 0$ (rotational case). We introduce a convenient approximation scheme to deal with the strong singular centrifugal term. The ansatz of this approximation possesses the same form of the potential and is singular as the centrifugal term r^{-2}. Thus, the KG equation with the trigonometric PT potential is solved approximately for its energy eigenvalues and corresponding wave functions with an arbitrary rotation–vibration (n,l) state.

This work is arranged as follows: In Section 2, we solve the KG equation for the given equally mixed scalar–vector trigonometric PT potential and obtain its energy eigenvalues and the corresponding wave functions. Some numerical results are obtained for any arbitrary vibration–rotation quantum numbers n and l. The

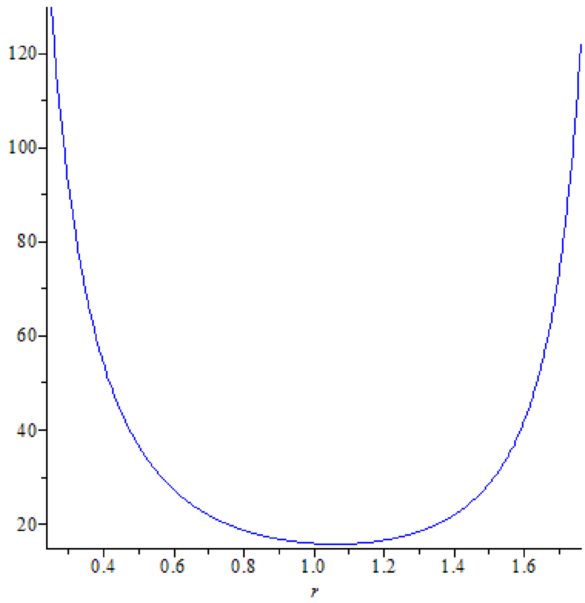

Fig. 1 (colour online). Plot of the trigonometric PT potential for $\alpha = 0.8\,\mathrm{fm}^{-1}$.

non-relativistic limit is discussed in this section too. Finally, the relevant conclusion is given in Section 3.

2. KG Solution of Equally Mixed Scalar–Vector Trigonometric PT Potential

In relativistic quantum mechanics, we usually use the KG equation for describing a scalar particle, i. e., the spin-0 particle dynamics. The discussion of the relativistic behaviour of spin-0 particles requires understanding the single particle spectrum and the exact solutions to the KG equation which are constructed by using the four-vector potential A_λ $(\lambda = 0,1,2,3)$ and the scalar potential $S(r)$. In order to simplify the analytical solution of the KG equation, the four-vector potential can be written as $A_\lambda = (A_0, 0, 0, 0)$. The first component of the four-vector potential is represented by a vector potential $V(r)$, i. e., $A_0 = V(r)$. In this case, the motion of a relativistic spin-0 particle in a potential is always described by the KG equation with the potentials $V(r)$ and $S(r)$. For equally mixed scalar and vector potentials, $S(r) = \pm V(r)$ cases, the (3+1)-dimensional KG equation is reduced to a Schrödinger-type equation and thereby the bound state solutions are easily obtained by using the well-known methods developed in non-relativistic quantum mechanics [27, 28].

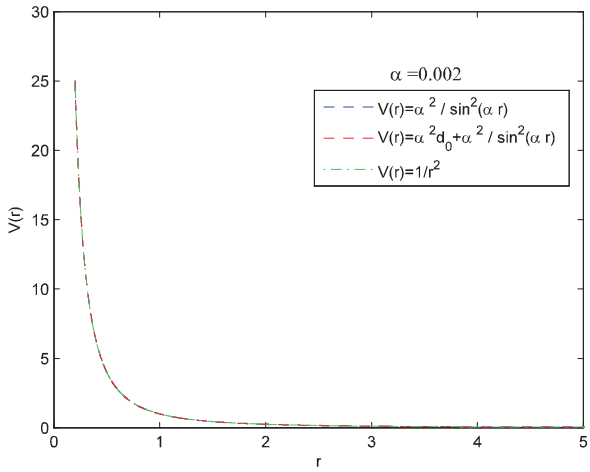

Fig. 2 (colour online). Centrifugal term $1/r^2$ (green line) and its approximations (6).

Let us now consider the (3+1)-dimensional time-independent KG equation describing a scalar particle (spin-0 particle) with Lorentz scalar $S(r)$ and Lorentz vector $V(r)$ potentials which takes the form [29]

$$\left[c^2 P_{\mathrm{op}}^2 - (V(r) - E_{\mathrm{R}})^2 + \left(S(r) + mc^2\right)^2\right]\psi_{\mathrm{KG}}(\vec{r}) = 0\,, \tag{2}$$

where m and E_{R} denote the reduced mass and relativistic binding energy of two interacting particles, respectively, with $\vec{P}_{\mathrm{op}} = -\mathrm{i}\hbar\vec{\nabla}$ is the momentum operator. It would be natural to scale the potential terms in (2) so that in the non-relativistic limit the interaction potential becomes $V(r)$, not $2V(r)$. We will follow Alhaidari et al. [28] to reduce the above equation to the form

$$\left\{\nabla^2 + \frac{1}{\hbar^2 c^2}\left[\left(E_{\mathrm{R}} - \frac{1}{2}V(r)\right)^2 - \left(mc^2 + \frac{1}{2}S(r)\right)^2\right]\right\}\psi_{\mathrm{KG}}(\vec{r}) = 0\,. \tag{3}$$

Thus, after making use of the equal scalar and vector Yukawa-type functions $S(r) = \pm V(r)$, (2) recasts to

$$\left\{\nabla^2 - \frac{1}{\hbar^2 c^2}\left[\left(m^2c^4 - E_{\mathrm{R}}^2\right) \pm V(r)\left(mc^2 \pm E_{\mathrm{R}}\right)\right]\right\}\psi_{\mathrm{KG}}(\vec{r}) = 0\,, \tag{4a}$$

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right], \quad r^2 = \sum_{j=1}^{3} x_j^2 . \tag{4b}$$

In addition, we take the interaction potential as in (1) and decompose the total wave function $\psi_{\mathrm{KG}}(\vec{r})$, with a given angular momentum l, as a product of a radial wave function $R_l(r) = g(r)/r$, and the angular dependent spherical harmonic functions $Y_l^m(\hat{r})$ [29]

$$\psi_{\mathrm{KG}}(\vec{r}) = \frac{g(r)}{r} Y_l^m(\hat{r}) , \tag{5}$$

with angular momentum quantum numbers being l and m. This reduces (4a) into the form

$$\frac{\mathrm{d}^2 g(r)}{\mathrm{d}r^2} - \frac{1}{\hbar^2 c^2}\left[\left(m^2c^4 - E_{\mathrm{R}}^2\right) \pm V(r) \cdot \left(mc^2 \pm E_{\mathrm{R}}\right) + \frac{l(l+1)\hbar^2 c^2}{r^2}\right] g(r) = 0 , \tag{6}$$

where $l(l+1)r^{-2}$ is the centrifugal potential and the boundary conditions $g(0) = 0$ and $g(\infty) \to 0$ as we are dealing with bound state solutions. Where the radial wave function $R_{n,l}(r)$ has to satisfy the required boundary conditions, namely, $R_{n,l}(0) = 0$ and $R_{n,l}(\pi/2) = 0$ on the edges. Since the SE with the trigonometric PT potential has no analytical solution for $l \neq 0$ states, we resort to use an appropriate approximation scheme to deal with the centrifugal potential term as

$$\frac{1}{r^2} = \lim_{\alpha\to 0} \alpha^2 \left(d_0 + \frac{1}{\sin^2(\alpha r)}\right) , \quad 0 < \alpha r < \pi/2 , \tag{7}$$

where $d_0 = 1/12$ is a dimensionless shifting parameter and $\alpha r \ll 1$. The approximation (7) is done on the basis that $\sin(z) = z - z^3/3! + z^5/5! - z^7/7! + \cdots$, and in the limit when $z \to 0$, $\sin(z) \approx z$. To show the validity and accuracy of our choice to the approximation scheme (7), we plot the centrifugal potential term $1/r^2$ and its approximations, $\alpha^2/\sin^2(\alpha r)$ and $\alpha^2\left(d_0 + 1/\sin^2(\alpha r)\right)$, in Figure 2. As illustrated, the three curves coincide together and show how accurate is this replacement.

2.1. Equal Mixture of $S(r) = +V(r)$ Case

As for the KG equation, for equally mixed scalar $S(r)$ and vector $V(r)$ potentials, i. e., $S(r) = +V(r)$, (6) becomes

$$\frac{\mathrm{d}^2 g(r)}{\mathrm{d}r^2} - \frac{1}{\hbar^2 c^2}\left[\beta_2^2\left(\beta_1^2 + \frac{V_1}{\sin^2(\alpha r)} + \frac{V_2}{\cos^2(\alpha r)}\right) + l(l+1)\hbar^2 c^2 \alpha^2 \left(d_0 + \frac{1}{\sin^2(\alpha r)}\right)\right] g(r) = 0 , \tag{8}$$

$$\beta_1^2 = mc^2 - E_{\mathrm{R}} , \quad \beta_2^2 = mc^2 + E_{\mathrm{R}} ,$$

and using the transformation $s(r) = \sin^2(\alpha r)$, we can rewrite it as follows:

$$\frac{\mathrm{d}^2 g(s)}{\mathrm{d}s^2} + \frac{\left(\frac{1}{2} - s\right)}{s(1-s)}\frac{\mathrm{d}g(s)}{\mathrm{d}s} + \frac{1}{s^2(1-s)^2}\left(-As^2 + Bs - C\right) g(s) = 0 , \tag{9a}$$

$$A = -\frac{\beta_1^2\beta_2^2}{4\alpha^2\hbar^2c^2} - \frac{l(l+1)d_0}{4} , \tag{9b}$$

$$B = -\frac{\beta_1^2\beta_2^2}{4\alpha^2\hbar^2c^2} + \frac{\beta_2^2(V_1 - V_2)}{4\alpha^2\hbar^2c^2} + \frac{l(l+1)}{4}(1 - d_0) , \tag{9c}$$

$$C = \frac{\beta_2^2 V_1}{4\alpha^2\hbar^2c^2} + \frac{l(l+1)}{4} . \tag{9d}$$

Comparing (9a) with (A.2), we get

$$c_1 = \frac{1}{2} , \quad c_2 = 1 , \quad c_3 = 1 , \tag{10}$$

and by the use of (A.5), we can obtain

$$\begin{aligned} c_4 &= \frac{1}{4} , \quad c_5 = -\frac{1}{2} , \\ c_6 &= \frac{1}{4}(1+4A) , \quad c_7 = -\frac{1}{4}(1+4B) , \\ c_8 &= \frac{1}{16}(1+16C) , \quad c_9 = A - B + C + \frac{1}{16} , \\ c_{10} &= \frac{1}{2}\sqrt{1+16C} , \quad c_{11} = 2\sqrt{A - B + C + \frac{1}{16}} , \\ c_{12} &= \frac{1}{4}\left(1 + \sqrt{1+16C}\right) , \\ c_{13} &= \frac{1}{4} + \sqrt{A - B + C + \frac{1}{16}} . \end{aligned} \tag{11}$$

The energy equation can be obtained by using (A.10), (10), and (11) to get

$$\left(n + \frac{1}{2} + \sqrt{A - B + C + \frac{1}{16}} + \sqrt{C + \frac{1}{16}}\right)^2 = A \tag{12}$$

or it can be explicitly expressed in terms of the energy as (in units $\hbar = c = 1$)

$$\left(n+\frac{1}{2}+\sqrt{\frac{(E_{nl}+m)V_1}{4\alpha^2}+\frac{1}{16}}\right.$$
$$\left.+\sqrt{\frac{(E_{nl}+m)V_1}{4\alpha^2}+\frac{l(l+1)}{4}+\frac{1}{16}}\right)^2 \tag{13}$$
$$=\frac{(E_{nl}-m)(E_{nl}+m)}{4\alpha^2}-\frac{l(l+1)d_0}{4}\,.$$

On the other hand, to find the corresponding wave functions, referring to (11), (A.11), and (A.12), we find the functions

$$\rho(s)=s^{\sqrt{\left(l+\frac{1}{2}\right)^2+\frac{(E_{nl}+m)V_1}{\alpha^2\hbar^2c^2}}}(1-s)^{\sqrt{\frac{(E_{nl}+m)V_2}{\alpha^2\hbar^2c^2}+\frac{1}{4}}}\,,$$
$$\phi(s)=s^{\left(\frac{1}{4}+\frac{1}{2}\sqrt{\left(l+\frac{1}{2}\right)^2+\frac{(E_{nl}+m)V_1}{\alpha^2\hbar^2c^2}}\right)} \tag{14}$$
$$\cdot(1-s)^{\frac{1}{4}+\sqrt{\frac{(E_{nl}+m)V_2}{\alpha^2\hbar^2c^2}+\frac{1}{4}}}\,.$$

Hence, (A.13) with the help of the weight function $\rho(s)$ in (14) gives

$$y_n(s)=P_n^{\left(\sqrt{\left(l+\frac{1}{2}\right)^2+\frac{(E_{nl}+m)V_1}{\alpha^2\hbar^2c^2}},\sqrt{\frac{(E_{nl}+m)V_2}{\alpha^2\hbar^2c^2}+\frac{1}{4}}\right)} \tag{15}$$
$$\cdot(1-2s)\,,$$

Further, using $R_{nl}(s)=\varphi(s)y_n(s)$, we get the wave function for the spinless Klein-Gordon particle as

$$R_{nl}(s)=A_{nl}s^{\left(\frac{1}{4}+\frac{1}{2}\sqrt{\left(l+\frac{1}{2}\right)^2+\frac{(E_{nl}+m)V_1}{\alpha^2\hbar^2c^2}}\right)}$$
$$\cdot(1-s)^{\frac{1}{4}+\sqrt{\frac{(E_{nl}+m)V_2}{\alpha^2\hbar^2c^2}+\frac{1}{4}}} \tag{16a}$$
$$\cdot P_n^{\left(\sqrt{\left(1+\frac{1}{2}\right)^2+\frac{(E_{nl}+m)V_1}{\alpha^2\hbar^2c^2}},\sqrt{\frac{(E_{nl}+m)V_2}{\alpha^2\hbar^2c^2}+\frac{1}{4}}\right)}(1-2s)\,,$$

or equivalently,

$$R_{nl}(r)=A_{nl}(\sin(\alpha r))^{\left(\frac{1}{2}+\sqrt{\left(l+\frac{1}{2}\right)^2+\frac{(E_{nl}+m)V_1}{\alpha^2\hbar^2c^2}}\right)}$$
$$\cdot(1-\sin^2(\alpha r))^{\frac{1}{4}+\sqrt{\frac{(E_{nl}+m)V_2}{\alpha^2\hbar^2c^2}+\frac{1}{4}}} \tag{16b}$$
$$\cdot P_n^{\left(\sqrt{\left(l+\frac{1}{2}\right)^2+\frac{(E_{nl}+m)V_1}{\alpha^2\hbar^2c^2}},\sqrt{\frac{(E_{nl}+m)V_2}{\alpha^2\hbar^2c^2}+\frac{1}{4}}\right)}$$
$$\cdot(\cos(2\alpha r))\,,$$

where A_{nl} is the normalization constant.

2.2. Equal Mixture of $S(r)=-V(r)$ Case

Here, we want to solve the following KG equation:

$$\frac{\mathrm{d}^2g(r)}{\mathrm{d}r^2}-\frac{1}{\hbar^2c^2}\left[\beta_1^2(\beta_2^2-V(r))\right.$$
$$\left.+\frac{l(l+1)\hbar^2c^2}{r^2}\right]g(r)=0\,. \tag{17}$$

To avoid repetition in the solution of (17), it is necessary to apply the following appropriate transformations:

$$\beta_1^2\to\beta_2^2\,,\ \ \beta_2^2\to\beta_1^2\ \ (\text{i.e., } E\to -E),$$
$$\text{and } V(r)\to -V(r) \tag{18}$$

on (13) and (16) and obtain the energy equation and wave function. We just write the final forms for energy equation as

$$\left(n+\frac{1}{2}+\sqrt{\frac{(E_{nl}-m)V_1}{4\alpha^2}+\frac{1}{16}}\right.$$
$$\left.+\sqrt{\frac{(E_{nl}-m)V_1}{4\alpha^2}+\frac{l(l+1)}{4}+\frac{1}{16}}\right)^2 \tag{19}$$
$$=\frac{(E_{nl}-m)(E_{nl}+m)}{4\alpha^2}-\frac{l(l+1)d_0}{4}\,.$$

Further, the wave functions become

$$R_{nl}(r)=A_{nl}(\sin(\alpha r))^{\left(\frac{1}{2}+\sqrt{\left(l+\frac{1}{2}\right)^2+\frac{(E_{nl}-m)V_1}{\alpha^2\hbar^2c^2}}\right)}$$
$$\cdot(1-\sin^2(\alpha r))^{\frac{1}{4}+\sqrt{\frac{(E_{nl}-m)V_2}{\alpha^2\hbar^2c^2}+\frac{1}{4}}} \tag{20}$$
$$\cdot P_n^{\left(\sqrt{\left(l+\frac{1}{2}\right)^2+\frac{(E_{nl}-m)V_1}{\alpha^2\hbar^2c^2}},\sqrt{\frac{(E_{nl}-m)V_2}{\alpha^2\hbar^2c^2}+\frac{1}{4}}\right)}(\cos(2\alpha r))\,.$$

Some numerical results of (13) and (19) are given in Tables 1 and 2, respectively, where we used the parameters values as $m=10\,\mathrm{fm}^{-1}$, $V_1=5.0\,\mathrm{fm}^{-1}$, $V_2=3.0\,\mathrm{fm}^{-1}$, and $\alpha=1.2, 0.8, 0.4, 0.2, 0.02, 0.002$ [23]. Further, when potential range parameter α approaches zero, the energy eigenvalues approaches a constant. When $S(r)=+V(r)$, from (13) we find that this constant is $M+V_1+V_2+2\sqrt{V_1V_2}$, i.e. $\lim_{\alpha\to 0}E_{nl}=M+V_1+V_2+2\sqrt{V_1V_2}$, what can be seen from Table 1. When $S(r)=-V(r)$, from (19) we find that this constant is $M+V_1+V_2-2\sqrt{V_1V_2}$, i.e. $\lim_{\alpha\to 0}E_{nl}=M+V_1+V_2-2\sqrt{V_1V_2}$, what can be seen from Table 2.

Table 1. Relativistic limit bound state energy levels (in units fm^{-1}) of the trigonometric PT potential when $S(r) = +V(r)$.

(n,l)	$\alpha = 1.2$	$\alpha = 0.8$	$\alpha = 0.4$	$\alpha = 0.2$	$\alpha = 0.02$	$\alpha = 0.002$
1,0	30.59082676	28.96349389	26.54464562	26.54464562	25.82563383	25.75393131
2,0	33.80353672	31.10492368	27.07742562	27.07742562	25.87875072	25.75924112
2,1	33.93794296	31.16696458	27.08151720	27.08151720	25.87879224	25.75924153
3,0	37.01055585	33.24743192	27.61092000	27.61092000	25.93187788	25.76455103
3,1	37.13968528	33.30779391	27.61498476	27.61498476	25.93191937	25.76455144
3,2	37.39457430	33.42773881	27.62311046	27.62311046	25.93200235	25.76455227
4,0	40.20605385	35.38789895	28.14501074	28.14501074	25.98501514	25.76986104
4,1	40.33029451	35.44664484	28.14904849	28.14904849	25.98505661	25.76986146
4,2	40.57575711	35.56341597	28.15712027	28.15712027	25.98513954	25.76986229
4,3	40.93675721	35.73681451	28.16921860	28.16921860	25.98526393	25.76986354

Table 2. Relativistic limit bound state energy levels (in units fm^{-1}) of the trigonometric PT potential when $S(r) = -V(r)$.

(n,l)	$\alpha = 1.2$	$\alpha = 0.8$	$\alpha = 0.4$	$\alpha = 0.2$	$\alpha = 0.02$	$\alpha = 0.002$
1,0	17.84511999	15.28076597	12.46883921	10.97773969	10.01428383	10.00014371
2,0	22.36673714	18.56400017	14.33675956	11.95610175	10.03889589	10.00039559
2,1	22.69698416	18.75782335	14.41589079	11.98844474	10.03961054	10.00040293
3,0	26.53900347	21.58758153	16.08093493	12.92684330	10.07485741	10.00077330
3,1	26.80872662	21.74417864	16.14392285	12.95261128	10.07555859	10.00078066
3,2	27.32146041	22.04554086	16.26674812	13.00322346	10.07694850	10.00079526
4,0	30.48880611	24.43978348	17.72815988	13.86430089	10.12116080	10.00127672
4,1	30.72096299	24.57355996	17.78133514	13.88593231	10.12184320	10.00128409
4,2	31.16768759	24.83355766	17.88577583	13.92865902	10.12320023	10.00129875
4,3	31.79970143	25.20661639	18.03799063	13.99147546	10.12521711	10.00132056

2.3. The Non-Relativistic Limit

Note that the solution of the (3+1)-dimensional KG equation can be reduced to the solution of the Schrödinger equation with the following appropriate choice of parameters: $\beta_1^2 \to -E_{\mathrm{NR}}$ and $\beta_2^2/\hbar^2c^2 \to 2\mu/\hbar^2$, where $\mu = m_1m_2/(m_1+m_2)$ is the reduced atomic mass for the two particle system.

In the non-relativistic limit, the Schrödinger solution can be obtained from the $S(r) = V(r)$ case by means of (13). Applying the transformations $E_{nl} + M \approx 2\mu/\hbar^2$, $E_{nl} - M \approx E_{nl}$, one obtains the energy formula

$$E_{nl} = \frac{\hbar^2\alpha^2 l(l+1)d_0}{2\mu} + \frac{2\hbar^2\alpha^2}{\mu}\left[n + \frac{1}{2} + \frac{1}{4} \cdot \left(\sqrt{(2l+1)^2 + \frac{8\mu V_1}{\hbar^2\alpha^2}} + \sqrt{1 + \frac{8\mu V_2}{\hbar^2\alpha^2}}\right)\right]^2 . \tag{21}$$

In the limit when $\alpha \to 0$, the energy formula (21) reduces into a constant value [24]:

$$\lim_{\alpha\to 0} E_{nl} = \left(\sqrt{V_1} + \sqrt{V_2}\right)^2 . \tag{22}$$

Further, there is no less of generality if $d_0 = 0$, then (21) becomes

$$E_{nl} = \frac{2\hbar^2\alpha^2}{\mu}\left[n + \frac{1}{2} + \frac{1}{4}\left(\sqrt{(2l+1)^2 + \frac{8\mu V_1}{\hbar^2\alpha^2}} + \sqrt{1 + \frac{8\mu V_2}{\hbar^2\alpha^2}}\right)\right]^2 , \tag{23}$$

where $n = 0,1,2,\ldots$ and $l = 0,1,2,\ldots$ are the vibration and rotation quantum numbers, respectively.

Finally, we get the non-relativistic radial wave functions as

$$R_{n,l}^{\text{non-rel.}}(s) = s^{\frac{1}{4}\left[1+\sqrt{(2l+1)^2+\frac{8\mu V_1}{\hbar^2\alpha^2}}\right]}(1-s)^{\frac{1}{4}\left(1+\sqrt{1+\frac{8\mu V_2}{\hbar^2\alpha^2}}\right)} \cdot P_n^{\left(\frac{1}{2}\sqrt{(2l+1)^2+\frac{8\mu V_1}{\hbar^2\alpha^2}},\frac{1}{2}\sqrt{1+\frac{8\mu V_2}{\hbar^2\alpha^2}}\right)}(1-2s) \tag{24}$$

or, inserting $s = \sin^2(\alpha r)$ in the above equation, we get

$$R_{n,l}^{\text{non-rel.}}(r) = N_{nl}(\sin(\alpha r))^{(1+\eta_l)/2} (\cos(\alpha r))^{(1+\delta)/2} P_n^{(\eta_l/2,\delta/2)}(\cos(2\alpha r)) , \tag{25a}$$

$$\eta_l = \sqrt{(2l+1)^2 + \frac{8\mu V_1}{\hbar^2\alpha^2}} , \quad \delta = \sqrt{1 + \frac{8\mu V_2}{\hbar^2\alpha^2}} , \tag{25b}$$

where N_{nl} is a normalization factor to be calculated from the normalization conditions.

3. Final Remarks and Conclusion

In this work, we have obtained the approximate bound state solutions of the KG equation with the trigonometric Pöschl–Teller potential for arbitrary l-state in the framework of a new approximation for the centrifugal term r^{-2}. We employed a shortcut of the NU method in finding the energy eigenvalues and corresponding wave functions. Some numerical results are given in Tables 1 and 2, and it is found that in the limit when the potential range parameter $\alpha \to 0$, the energy levels approach to a constant value $M+V_1+V_2 \pm 2\sqrt{V_1V_2}$ when $S(r) = \pm V(r)$. Finally, we discussed the non-relativistic limit, and it is found that in the limit when the potential range parameter $\alpha \to 0$, the energy levels approach to a constant value $\left(\sqrt{V_1}+\sqrt{V_2}\right)^2$.

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Appendix : A Shortcut of the NU Method

The NU method is used to solve second-order differential equations with an appropriate coordinate transformation $s = s(r)$ [30]

$$\psi_n''(s) + \frac{\tilde{\tau}(s)}{\sigma(s)}\psi_n'(s) + \frac{\tilde{\sigma}(s)}{\sigma^2(s)}\psi_n(s) = 0\,, \tag{A.1}$$

where $\sigma(s)$ and $\tilde{\sigma}(s)$ are polynomials, at most of second degree, and $\tilde{\tau}(s)$ is a first-degree polynomial. To make the application of the NU method simpler and direct without need to check the validity of solution, we present a shortcut for the method. So, at first we write the general form of the Schrödinger-like equation (A.1) in a more general form applicable to any potential as follows [31–33]:

$$\begin{aligned}&\psi_n''(s) + \left(\frac{c_1 - c_2 s}{s(1-c_3 s)}\right)\psi_n'(s)\\ &+ \left(\frac{-As^2 + Bs - C}{s^2(1-c_3 s)^2}\right)\psi_n(s) = 0\,,\end{aligned} \tag{A.2}$$

satisfying the wave functions

$$\psi_n(s) = \varphi(s) y_n(s)\,. \tag{A.3}$$

Comparing (A.2) with its counterpart (A.1), we obtain the following identifications:

$$\begin{aligned}&\tilde{\tau}(s) = c_1 - c_2 s\,,\ \ \sigma(s) = s(1-c_3 s)\,,\\ &\tilde{\sigma}(s) = -\xi_1 s^2 + \xi_2 s - \xi_3\,.\end{aligned} \tag{A.4}$$

Following the NU method [30], we obtain the following shortcut procedure [31–33]:

(i) The relevant constants:

$$\begin{aligned}&c_4 = \frac{1}{2}(1-c_1)\,,\ \ c_5 = \frac{1}{2}(c_2 - 2c_3)\,,\\ &c_6 = c_5^2 + A\,,\ \ c_7 = 2c_4c_5 - B\,,\\ &c_8 = c_4^2 + C\,,\ \ c_9 = c_3(c_7 + c_3c_8) + c_6\,,\\ &c_{10} = c_1 + 2c_4 + 2\sqrt{c_8} - 1 > -1\,,\\ &c_{11} = 1 - c_1 - 2c_4 + \frac{2}{c_3}\sqrt{c_9} > -1\,,\ \ c_3 \neq 0\,,\\ &c_{12} = c_4 + \sqrt{c_8} > 0\,,\\ &c_{13} = -c_4 + \frac{1}{c_3}\left(\sqrt{c_3} - c_5\right) > 0\,,\ \ c_3 \neq 0\,.\end{aligned} \tag{A.5}$$

(ii) The essential polynomial functions:

$$\pi(s) = c_4 + c_5 s - \left[\left(\sqrt{c_9} + c_3\sqrt{c_8}\right)s - \sqrt{c_8}\right]\,, \tag{A.6}$$

$$k = -\left(c_7 + 2c_3c_8\right) - 2\sqrt{c_8c_9}\,, \tag{A.7}$$

$$\begin{aligned}\tau(s) = {}& c_1 + 2c_4 - (c_2 - 2c_5)s\\ &- 2\left[\left(\sqrt{c_9} + c_3\sqrt{c_8}\right)s - \sqrt{c_8}\right]\,,\end{aligned} \tag{A.8}$$

$$\tau'(s) = -2c_3 - 2\left(\sqrt{c_9} + c_3\sqrt{c_8}\right) < 0\,. \tag{A.9}$$

(iii) The energy equation:

$$\begin{aligned}&c_2 n - (2n+1)c_5 + (2n+1)\left(\sqrt{c_9} + c_3\sqrt{c_8}\right)\\ &+ n(n-1)c_3 + c_7 + 2c_3c_8 + 2\sqrt{c_8c_9} = 0\,.\end{aligned} \tag{A.10}$$

(iv) The wave functions:

$$\rho(s) = s^{c_{10}}(1-c_3 s)^{c_{11}}\,, \tag{A.11}$$

$$\varphi(s) = s^{c_{12}}(1-c_3 s)^{c_{13}}\,,\ \ c_{12} > 0\,,\ \ c_{13} > 0\,, \tag{A.12}$$

$$\begin{aligned}&y_n(s) = P_n^{(c_{10},c_{11})}(1-2c_3 s)\,,\\ &\qquad c_{10} > -1\,,\ \ c_{11} > -1\,,\end{aligned} \tag{A.13}$$

$$\psi_{nl}(s) = N_{nl}s^{c_{12}}(1-c_3 s)^{c_{13}}P_n^{(c_{10},c_{11})}(1-2c_3 s)\,, \tag{A.14}$$

where $P_n^{(\mu,\nu)}(x)$, $\mu > -1$, $\nu > -1$, and $x \in [-1,1]$ are Jacobi polynomials with

$$P_n^{(\alpha,\beta)}(1-2s) = \frac{(\alpha+1)_n}{n!} \cdot {}_2F_1(-n, 1+\alpha+\beta+n; \alpha+1; s), \tag{A.15}$$

and N_{nl} is a normalization constant. Also, the above wave functions can be expressed in terms of the hypergeometric function as

$$\psi_{nl}(s) = N_{nl} s^{c_{12}} (1-c_3 s)^{c_{13}} \cdot {}_2F_1(-n, 1+c_{10}+c_{11}+n; c_{10}+1; c_3 s) \tag{A.16}$$

where $c_{12} > 0$, $c_{13} > 0$, and $s \in [0, 1/c_3]$, $c_3 \neq 0$.

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