Approximate Solution of the Spin-0 Particle Subject to the Trigonometric Pöschl–Teller Potential with Centrifugal Barrier

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The trigonometric Pöschl–Teller (PT) potential describes the diatomic molecular vibration. In this paper, we study the approximate solutions of the radial Klein–Gordon (KG) equation for the rotating trigonometric PT potential using the Nikiforov–Uvarov (NU) method. The energy eigenvalues and their corresponding eigenfunctions are calculated for arbitrary \( l \)-states in closed form. We obtain the non-relativistic limit and present some numerical results for both relativistic and non-relativistic cases.

Key words: Klein–Gordon Equation; Trigonometric Pöschl–Teller Potential; Nikiforov–Uvarov Method.

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1. Introduction

Relativistic wave equations and particularly the Dirac and the Klein–Gordon (KG) equations have been of interest for theoretical physicists in many branches of physics [1, 2]. In recent years, there has been an increased interest in finding exact solutions to relativistic spinless KG particles with various vector and scalar potentials [3–8]. The most commonly used techniques to explore these wave equations are the Nikiforov–Uvarov (NU) method [9, 10], the supersymmetric quantum mechanics method [11, 12], the point canonical transformation [13], the iteration method [14–16], the exact quantization rule [17], the shifted \( 1/N \) expansion (SE) technique [18], and the ansatz approach [19].

The trigonometric Pöschl–Teller (PT) potential proposed for the first time by Pöschl and Teller [20] in 1933 was to describe the diatomic molecular vibration. Chen [21] and Zhang and Wang [22] have studied the relativistic bound state solutions for the trigonometric PT potential and hyperbolical PT (second PT) potential, respectively. Liu et al. [23] studied the trigonometric PT potential within the framework of the Dirac theory. Very recently, Hamzavi and Rajabi studied the exact \( s \)-wave solution \((l = 0)\) of the Schrödinger equation for the vibrational trigonometric PT potential [24].

This potential takes the following form:

\[ V(r) = \frac{V_1}{\sin^2(\alpha r)} + \frac{V_2}{\cos^2(\alpha r)}, \]

where the parameters \( V_1 \) and \( V_2 \) describe the property of the potential well while the parameter \( \alpha \) is related to the range of this potential [23, 25, 26]. In Figure 1, we draw the trigonometric PT potential (1) for parameter values \( V_1 = 5.0 \text{ fm}^{-1}, V_2 = 3.0 \text{ fm}^{-1}, \) and \( \alpha = 0.8 \text{ fm}^{-1}. \)

The aim of the present work is to extend our previous work [24] to the relativistic limit and the case of \( l \neq 0 \) (rotational case). We introduce a convenient approximation scheme to deal with the strong singular centrifugal term. The ansatz of this approximation possesses the same form of the potential and is singular as the centrifugal term \( r^{-2} \). Thus, the KG equation with the trigonometric PT potential is solved approximately for its energy eigenvalues and corresponding wave functions with an arbitrary rotation–vibration \((n, l)\) state.

This work is arranged as follows: In Section 2, we solve the KG equation for the given equally mixed scalar–vector trigonometric PT potential and obtain its energy eigenvalues and the corresponding wave functions. Some numerical results are obtained for any arbitrary vibration–rotation quantum numbers \( n \) and \( l \). The
non-relativistic limit is discussed in this section too. Finally, the relevant conclusion is given in Section 3.

2. KG Solution of Equally Mixed Scalar–Vector Trigonometric PT Potential

In relativistic quantum mechanics, we usually use the KG equation for describing a scalar particle, i.e., the spin-0 particle dynamics. The discussion of the relativistic behaviour of spin-0 particles requires understanding the single particle spectrum and the exact solutions to the KG equation which are constructed by using the four-vector potential \(A_\lambda\) \((\lambda = 0, 1, 2, 3)\) and the scalar potential \(S(r)\). In order to simplify the analytical solution of the KG equation, the four-vector potential is represented by a vector potential \(V(r)\), i.e., \(A_0 = V(r)\). In this case, the motion of a relativistic spin-0 particle in a potential is always described by the KG equation with the potentials \(V(r)\) and \(S(r)\). For equally mixed scalar and vector potentials, \(S(r) = \pm V(r)\) cases, the \((3+1)\)-dimensional KG equation is reduced to a Schrödinger-type equation and thereby the bound state solutions are easily obtained by using the well-known methods developed in non-relativistic quantum mechanics [27, 28].

Let us now consider the \((3+1)\)-dimensional time-independent KG equation describing a scalar particle (spin-0 particle) with Lorentz scalar \(S(r)\) and Lorentz vector \(V(r)\) potentials which takes the form [29]

\[
\left\{ \nabla^2 + \frac{1}{\hbar^2 c^2} \left[ \left( E_R - \frac{1}{2} V(r) \right)^2 - \left( mc^2 + \frac{1}{2} S(r) \right)^2 \right] \right\} \psi_{KG}(\vec{r}) = 0,
\]

where \(m\) and \(E_R\) denote the reduced mass and relativistic binding energy of two interacting particles, respectively, with \(\vec{P}_{op} = -i\hbar \vec{\nabla}\) is the momentum operator. It would be natural to scale the potential terms in (2) so that in the non-relativistic limit the interaction potential becomes \(V(r)\), not \(2V(r)\). We will follow Alhaidari et al. [28] to reduce the above equation to the form

\[
\left\{ \nabla^2 + \frac{1}{\hbar^2 c^2} \left[ \left( E_R - \frac{1}{2} V(r) \right)^2 - \left( mc^2 + \frac{1}{2} S(r) \right)^2 \right] \right\} \psi_{KG}(\vec{r}) = 0.
\]

Thus, after making use of the equal scalar and vector Yukawa-type functions \(S(r) = \pm V(r)\), (2) recasts to

\[
\left\{ \nabla^2 - \frac{1}{\hbar^2 c^2} \left[ \left( mc^2 + E_R \right)^2 \right] \right\} \psi_{KG}(\vec{r}) = 0,
\]

\[(4a)\]
where \( l \) is the centrifugal potential and the boundary conditions \( g(0) = 0 \) and \( g(\infty) \to 0 \) as we are dealing with bound state solutions. Where the radial wave function \( R_{\alpha l}(r) \) has to satisfy the required boundary conditions, namely, \( R_{\alpha l}(0) = 0 \) and \( R_{\alpha l}(\pi/2) = 0 \) on the edges. Since the SE with the trigonometric PT potential has no analytical solution for \( l \neq 0 \) states, we resort to use our approximate approximation scheme to deal with the centrifugal potential term as

\[
\frac{1}{r^2} \xrightarrow{\alpha \to 0} \alpha^2 \left( d_0 + \frac{1}{\sin^2(\alpha r)} \right), \quad 0 < \alpha r < \pi/2,
\]

where \( d_0 = 1/12 \) is a dimensionless shifting parameter and \( \alpha \ll 1 \). The approximation (7) is done on the basis that \( \sin(z) = z - z^3/3! + z^5/5! - z^7/7! + \cdots \), and in the limit when \( z \to 0 \), \( \sin(z) \approx z \). To show the validity and accuracy of our choice to the approximation scheme (7), we plot the centrifugal potential term \( 1/r^2 \) and its approximations, \( \alpha^2 / \sin^2(\alpha r) \) and \( \alpha^2 \left( d_0 + 1/\sin^2(\alpha r) \right) \), in Figure 2. As illustrated, the three curves coincide together and show how accurate this is replacement.

2.1. Equal Mixture of \( S(r) = +V(r) \) Case

As for the KG equation, for equally mixed scalar \( S(r) \) and vector \( V(r) \) potentials, i.e., \( S(r) = +V(r) \), (6) becomes

\[
\frac{d^2g(s)}{ds^2} + \left( \frac{1}{2} - s \right) \frac{dg(s)}{ds} + \frac{1}{s(1-s)} g(s) = 0, \quad (9a)
\]

\[
A = -\frac{\beta^2}{4\alpha^2h^2c^2} \sum_{l} \left( l(l+1)\hbar^2c^2 + \frac{l(l+1)}{4} \right), \quad (9b)
\]

\[
B = -\frac{\beta}{4\alpha^2h^2c^2} \sum_{l} \left( \beta^2(V_1 - V_2) \right), \quad (9c)
\]

\[
C = -\frac{\beta^2}{4\alpha^2h^2c^2} \sum_{l} \left( l(l+1) + 1 \right), \quad (9d)
\]

Comparing (9a) with (A.2), we get

\[
c_1 = \frac{1}{2}, \quad c_2 = 1, \quad c_3 = 1, \quad (10)
\]

and by the use of (A.5), we can obtain

\[
c_4 = \frac{1}{4}, \quad c_5 = -\frac{1}{2}, \quad c_6 = \frac{1}{4}(1+4A), \quad c_7 = -\frac{1}{4}(1+4B), \quad (11)
\]

\[
c_8 = \frac{1}{16}(1+16C), \quad c_9 = A - B + C + \frac{1}{16}, \quad (12)
\]

\[
c_{10} = \frac{1}{4}(1+\sqrt{1+16C}), \quad c_{11} = \frac{1}{2}(A - B + C + \frac{1}{16}), \quad (13)
\]

The energy equation can be obtained by using (A.10), (10), and (11) to get

\[
\left( n + \frac{1}{2} \right)^2 \left( A - B + C + \frac{1}{16} + C + \frac{1}{16} \right)^2 = A \quad (14)
\]

or it can be explicitly expressed in terms of the energy as (in units \( \hbar = c = 1 \))
where
\[ A_n \]
\[ \rho(s) = s^{\left( l + \frac{1}{2} \right)^2 + \frac{E_{nl}+mV_1}{4\alpha^2} - 1 - s} \sqrt{\frac{\left| E_{nl}+mV_1 \right|}{a^2}} + \frac{1}{2} \right], \]
\[ \phi(s) = s^{\left( l + \frac{1}{2} \right)^2 + \frac{E_{nl}+mV_1}{4\alpha^2} - 1 - s} \sqrt{\frac{\left| E_{nl}+mV_1 \right|}{a^2}} \cdot (1-s)^{\left( l + \frac{1}{2} \right)^2 + \frac{E_{nl}+mV_1}{4\alpha^2} - 1 - s} \right] \]}

Hence, (A.13) with the help of the weight function \( \rho(s) \) in (14) gives
\[ n + \frac{1}{2} + \sqrt{\frac{(E_{nl}+m)V_1}{4\alpha^2}} + \frac{1}{2} + \frac{1}{16} \]
\[ - \frac{1}{16} \]
\[ \frac{1}{4} \]
\[ + \frac{1}{16} \]
\[ (1 + l + 1) \frac{d^2 g(r)}{dr} - \frac{1}{h^2 c^2} \left[ \beta^2 \left( \beta^2 - V(r) \right) \right] + \frac{l(l+1)h^2 c^2}{r^2} \]
\[ g(r) = 0. \]

To avoid repetition in the solution of (17), it is necessary to apply the following appropriate transformations:
\[ \beta^2 \rightarrow \beta^2, \quad \beta^2 \rightarrow \beta^2 \quad \text{i.e.,} \quad E \rightarrow -E, \]
and
\[ V(r) \rightarrow -V(r) \]
on (13) and (16) and obtain the energy equation and wave function. We just write the final forms for energy equation as
\[ n + \frac{1}{2} + \sqrt{\frac{(E_{nl}+m)V_1}{4\alpha^2}} + \frac{1}{4} \]
\[ + \frac{1}{16} \]
\[ \frac{1}{4} \]
\[ + \frac{1}{16} \]
\[ (1 + l + 1) \frac{d^2 g(r)}{dr} - \frac{1}{h^2 c^2} \left[ \beta^2 \left( \beta^2 - V(r) \right) \right] + \frac{l(l+1)h^2 c^2}{r^2} \]
\[ g(r) = 0. \]

Further, the wave functions become
\[ R_{nl}(r) = A_{nl}(\sin(\alpha r)) \left( \frac{1}{l+\frac{1}{2}} + \frac{1}{l+\frac{1}{2}} \right) \]
\[ \cdot (1 - \sin^2(\alpha r))^{\frac{1}{2}} \sqrt{\frac{|E_{nl}+mV_1|}{a^2}} + \frac{1}{2} \right] \right) \]
\[ \cdot \left( \cos(2\alpha r) \right) \]}
\[ \left( \sqrt{\frac{(l+1)^2 + \frac{E_{nl}+mV_1}{4\alpha^2}}{a^2}} + \frac{1}{2} \right) \]}

Some numerical results of (13) and (19) are given in Tables 1 and 2, respectively, where we used the parameters values as \( m = 10 \text{ fm}^{-1}, V_1 = 5.0 \text{ fm}^{-1}, \) \( V_2 = 3.0 \text{ fm}^{-1}, \) and \( \alpha = 1.2, 0.8, 0.4, 0.2, 0.02, 0.002 \) [23]. Further, when potential range parameter \( \alpha \) approaches zero, the energy eigenvalues approaches a constant. When \( S(r) = -V(r) \), from (13) we find that this constant is \( M + V_1 + V_2 + 2\sqrt{V_1V_2} \), i.e. \( \lim_{\alpha \to 0} E_{nl} = M + V_1 + V_2 + 2\sqrt{V_1V_2} \), what can be seen from Table 1. When \( S(r) = V(r) \), from (19) we find that this constant is \( M + V_1 + V_2 - 2\sqrt{V_1V_2} \), i.e. \( \lim_{\alpha \to 0} E_{nl} = M + V_1 + V_2 - 2\sqrt{V_1V_2} \), what can be seen from Table 2.
KG equation can be reduced to the solution of the \( \beta_1 \) equation, which produces into a constant value [24]:

Then (21) becomes

\[ E_{nl} = \frac{2\hbar^2 \alpha^2}{\mu} \left[ n + 1 + \frac{1}{4} \sqrt{(2l + 1)^2 + \frac{8\mu V_1}{\hbar^2 \alpha^2}} \right]^2 \]

where \( n = 0, 1, 2, \ldots \) and \( l = 0, 1, 2, \ldots \) are the vibration and rotation quantum numbers, respectively.

Finally, we get the non-relativistic radial wave functions as

\[ E_{nl}^{\text{non-rel}}(s) = \frac{\sqrt{2}}{s} \left[ (l + 1) + \frac{8\mu V_1}{\hbar^2 \alpha^2} \right] \left( 1 + s^2 \right)^{-1} \cdot \int \right. \]

or, inserting \( s = \sin^2(\alpha r) \) in the above equation, we get

\[ E_{nl}^{\text{non-rel}}(r) = N_{nl} (\sin(\alpha r))^{1 + \eta_r/2} \left( \cos(\alpha r))^{1 + \delta/2} \right)^{1/2} \left( \cos(2\alpha r) \right)^{1/2} \]

where \( N_{nl} \) is a normalization factor to be calculated from the normalization conditions.
3. Final Remarks and Conclusion

In this work, we have obtained the approximate bound state solutions of the KG equation with the trigonometric Pöschl–Teller potential for arbitrary l-state in the framework of a new approximation for the centrifugal term r^2. We employed a shortcut of the NU method in finding the energy eigenvalues and corresponding wave functions. Some numerical results are given in Tables 1 and 2, and it is found that in the limit when the potential range parameter \( \alpha \to 0 \), the energy levels approach to a constant value \( M + V_1 + V_2 \pm 2\sqrt{V_1V_2} \) when \( S(r) = \pm V(r) \). Finally, we discussed the non-relativistic limit, and it is found that in the limit when the potential range parameter \( \alpha \to 0 \), the energy levels approach to a constant value \( (\sqrt{V_1} + \sqrt{V_2})^2 \).

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Appendix: A Shortcut of the NU Method

The NU method is used to solve second-order differential equations with an appropriate coordinate transformation \( s = s(r) \) [30]

\[
\frac{d^2 \psi_n}{ds^2} + \frac{\tilde{\tau}(s)}{\sigma(s)} \frac{d \psi_n}{ds} + \frac{\tilde{\sigma}(s)}{\sigma^2(s)} \psi_n(s) = 0, \quad (A.1)
\]

where \( \sigma(s) \) and \( \tilde{\sigma}(s) \) are polynomials, at most of second degree, and \( \tilde{\tau}(s) \) is a first-degree polynomial. To make the application of the NU method simpler and direct without need to check the validity of solution, we present a shortcut for the method. So, at first we write the general form of the Schrödinger-like equation (A.1) in a more general form applicable to any potential as follows [31–33]:

\[
\frac{d^2 \psi_n}{ds^2} + \left( \frac{c_1 - c_2 s}{s(1 - c_3 s)} \right) \frac{d \psi_n}{ds} + \left( \frac{-A s^2 + B s - C}{s^2(1 - c_3 s)^2} \right) \psi_n(s) = 0,
\]

(A.2)

satisfying the wave functions

\[
\psi_n(s) = \varphi(s) \gamma_n(s),
\]

(A.3)

Comparing (A.2) with its counterpart (A.1), we obtain the following identifications:

\[
\tilde{\tau}(s) = c_1 - c_2 s, \quad \sigma(s) = s(1 - c_3 s), \quad \tilde{\sigma}(s) = -\xi_1 s^2 + \xi_2 s - \xi_3.
\]

(A.4)

Following the NU method [30], we obtain the following shortcut procedure [31–33]:

(i) The relevant constants:

\[
c_4 = \frac{1}{2} \left( 1 - c_1 \right), \quad c_5 = \frac{1}{2} \left( c_2 - 2 c_3 \right), \quad c_6 = c_3^2 + A, \quad c_7 = 2 c_4 c_5 - B, \quad c_8 = c_4^2 + C, \quad c_9 = c_3 \left( c_7 + c_3 c_6 \right) + c_6, \quad c_{10} = c_1 + 2 c_4 + 2 \sqrt{c_8} - 1 > -1, \quad (A.5)
\]

\[
c_{11} = 1 - c_1 - 2 c_4 + \frac{2}{c_3} \sqrt{c_9} > -1, \quad c_3 \neq 0,
\]

\[
c_{12} = c_4 + \sqrt{c_8} > 0,
\]

\[
c_{13} = -c_4 + \frac{1}{c_3} \left( \sqrt{c_3} - c_3 \right) > 0, \quad c_3 \neq 0.
\]

(ii) The essential polynomial functions:

\[
\pi(s) = c_4 + c_5 s - \left[ \left( \sqrt{c_9} + c_3 \sqrt{c_6} \right) s - \sqrt{c_8} \right], \quad (A.6)
\]

\[
k = - \left( c_7 + 2 c_3 c_8 \right) - 2 \sqrt{c_9 c_6}, \quad (A.7)
\]

\[
\tau(s) = c_1 + 2 c_4 - (c_2 - 2 c_3) s - 2 \left( \sqrt{c_9} + c_3 \sqrt{c_6} \right) s - \sqrt{c_8}, \quad (A.8)
\]

\[
\tau(s) = -2c_3 - 2 \left( \sqrt{c_9} + c_3 \sqrt{c_6} \right) < 0. \quad (A.9)
\]

(iii) The energy equation:

\[
c_2 n - (2 n + 1) c_5 + (2 n + 1) \left( \sqrt{c_9} + c_3 \sqrt{c_6} \right) + n(n - 1) c_3 + c_7 + 2 c_3 c_8 + 2 \sqrt{c_9 c_6} = 0. \quad (A.10)
\]

(iv) The wave functions:

\[
\rho(s) = s^{c_{10}} \left( 1 - c_3 s \right)^{c_{11}}, \quad (A.11)
\]

\[
\varphi(s) = s^{c_{12}} \left( 1 - c_3 s \right)^{c_{13}}, \quad c_{12} > 0, \quad c_{13} > 0, \quad (A.12)
\]

\[
\gamma_n(s) = P_n^{(c_{10}, c_{11})}(1 - 2 c_3 s), \quad c_{10} > -1, \quad c_{11} > 1, \quad (A.13)
\]

\[
\psi_n(s) = N_n s^{c_{12}} \left( 1 - c_3 s \right)^{c_{13}} P_n^{(c_{10}, c_{11})}(1 - 2 c_3 s), \quad (A.14)
\]

where \( P_n^{(\mu, \nu)}(x) \), \( \mu < -1, \nu < -1, \) and \( x \in [-1, 1] \) are Jacobi polynomials with
\[ P_n^{(\alpha, \beta)}(1 - 2s) = \frac{\Gamma(1 + n)}{n!} \cdot _2F_1(-n, 1 + \alpha + \beta + n; \alpha + 1; s), \]  

(A.15)

\[ \psi_{nl}(s) = N_{nl}s^{c_{12}}(1 - c_3s)^{c_{13}} \cdot _2F_1(-n, 1 + c_{10} + c_{11} + n; c_{10} + 1; c_3s), \]  

(A.16)

where \( c_{12} > 0, c_{13} > 0, \) and \( s \in [0, 1/c_3], \ c_3 \neq 0. \)

References: