Breathing Azimuthons in Nonlocal Nonlinear Media

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Two-dimensional self-similar azimuthons are introduced and investigated analytically and numerically in nonlocal nonlinear media with space-dependent diffractive, gain (attenuation) coefficient based on the similarity transformation and variational approach. We demonstrate that the azimuthons of critical power in the strongly nonlocal limit are more stable than the ones with lower nonlocality. Remarkably, these self-similar azimuthons have the azimuthal angle modulated by the distributed diffractive coefficient, apart from the beam width and intensity changing self-similarly.

Key words: Nonlocal Nonlinear Media; Self-Similar; Azimuthons.
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1. Introduction

The nonlocal nonlinearity exists in many physical systems such as plasmas, Bose–Einstein condensates, and some optical materials. Recently, there are a number of experimental observations and theoretical treatments of self-trapping effects and spatial solitary waves in different types of nonlocal nonlinear media [1 – 22]. Nonlocal nonlinear response has showed profound consequences on the formation of localized structures and wave propagation. In the field of optics, the propagation of beams in nonlocal nonlinear media is governed by the nonlocal nonlinear Schrödinger equation (NNLSE). In general, the nonlinear term has the nonlocal form associated with a symmetric and real-valued response kernel. Snyder and Mitchell [16] simplified the NNLSE to a linear model (called the standard Snyder–Mitchell mode, SMM) in the strongly nonlocal case. The propagation of optical beams in strongly nonlocal media have attracted extensive interest and been widely investigated in recent years [4 – 18]. And there are many novel features of the nonlocal nonlinearity for the evolution of waves, such as vortex solitons, Gaussian solitons, soliton cluster, ellipticons, and the rotating nonlinear wave solutions, the so called azimuthons. Some of them are as follows: In [11], the existence and the stability of Whittaker solitons have been introduced and analyzed numerically. These higher-order solitons are obtained as a generalization of the Whittaker linear modes in the case of the Gaussian response function. Reference [12] studied Bessel solitary wave (BSW) solutions to a two-dimensional strongly nonlocal nonlinear Schrodinger equation with distributed coefficients and compared the features of BSW with that of Hermite solitary waves and Laguerre solitary waves. Recently, a variety of dynamics both for vortex–vortex and vortex–antivortex pairs in nonlocal nonlinear media have been demonstrated in [13]. The nonlocal nonlinear response suppressed azimuthal instability and elliptical instability and formed quasi-stable rotating or breathing states for vortex pairs depending on their circulations. The azimuthons [19 – 22] represent a generic type of singular localized beams, which can be described as a vortex soliton continuously modulated along the ring with topological charge and azimuthal modulation. Importantly, the azimuthon family includes the known soliton cluster and vortex soliton as particular members, with zero and maximum modulation depths, respectively. The presence of azimuthal intensity and phase modulation leads to spatial rotation of the ring, or spiralling, characterized by the angular velocity. Reference [22] reported on the first observation of optical azimuthons employing rubidium vapours as an isotropic

self-focusing nonlinear medium. This observation indicates that the presence of modulation stabilises the anomalously rotating azimuthons.

Self-similar dynamical effects have saturated in various fields such as quantum field theory, hydrodynamics, and nonlinear optics. Several remarkable results have been obtained in the context of nonlinear optics, such as the evolutions of self-written wave guides, the self-similar regime of collapse for spiral laser beams in nonlinear media, nonlinear compression of chirped solitary waves, and the nonlinear propagation of pulses with parabolic intensity, etc. Recently, exact optical self-similar waves, such as the exact soliton solutions and quasi-soliton solutions have been investigated in many papers [23–33]. The governing equation of these optical similar waves are various kinds of non-autonomous and inhomogeneous nonlinear Schrödinger equations.

In this paper, self-similar technique and variational approach allow us to derive breathing azimuthons in nonlocal nonlinear media with space-dependent diffractive, gain (attenuation) coefficient. These higher-order azimuthons are obtained in the case of the Gaussian response function. Further on, we analyze the propagation of these azimuthons numerically. The stabilization of the solutions can be improved by moving deeper into the strongly nonlocal regime.

2. Self-Similar Azimuthons

The two-dimensional generalized nonlocal nonlinear equation (GNNE) which governing the propagation of a two-dimensional light beam is [27]

$$\mathrm{i}\psi_z + \frac{\beta(z)}{2}\nabla^2\psi + s(z)N(I,\vec{\rho})\psi = \mathrm{i}\frac{g(z)}{2}\psi, \quad (1)$$

where z and $\vec{\rho} = (x,y)$ stand for the propagation and transverse coordinates, respectively. The nonlinearity depends on the light intensity $I \equiv |\psi|^2$ via the following phenomenological nonlocal relation:

$$N(I,\vec{\rho}) = \int \mathfrak{R}(|\vec{\rho} - \vec{\rho}'|)I(\vec{\rho}',z)\,\mathrm{d}\vec{\rho}'. \quad (2)$$

The actual form of the response function $\mathfrak{R}(\rho)$ is determined by the details of a particular physical system. Here, we will consider a Gaussian model of nonlocal response,

$$\mathfrak{R}(\rho) = (\pi\sigma^2)^{-1}\exp(-\rho^2/\sigma^2), \quad (3)$$

where σ measures the degree of nonlocality. The last three terms of (1) represent diffraction, nonlinearity, and gain (attenuation), respectively.

In the case where the nonlocality is weak, the characteristic length of the response function $\mathfrak{R}(\rho)$ is narrow compared to the width of the beam, then the intensity I can be expanded by means of Taylor's expansion. Here, expanding $I(\vec{\rho}',z)$ in (2) around the point $\vec{\rho}' = \vec{\rho}$ to the second order and considering $\int \mathfrak{R}(\vec{\rho})\,\mathrm{d}\vec{\rho} = 1$, we obtain

$$N(I,\vec{\rho}) = I + \frac{1}{4}\nabla^2 I \int \rho'^2 \mathfrak{R}(\vec{\rho}')\,\mathrm{d}\vec{\rho}'. \quad (4)$$

Especially, when the response length $\sigma \to 0$, (1) is reduced to the nonlinear Schrödinger equation with Kerr nonlinearity. And as $\sigma \to \infty$, it corresponds to a strongly nonlocal regime. Now taking the Taylor expansion of $\mathfrak{R}(\vec{\rho})$ at $\rho = 0$ to the second order. For convenience, we simply adopt

$$\begin{aligned} N(I,\vec{\rho}) = \mathfrak{R}(0)E_0 - \frac{1}{2}\gamma E_0 \rho^2 \\ - \frac{1}{2}\gamma \int \rho'^2 I(\vec{\rho}',z)\,\mathrm{d}\vec{\rho}', \end{aligned} \quad (5)$$

where the beam power $E_0 = \int I(\vec{\rho},z)\,\mathrm{d}\vec{\rho}$, and the material constant

$$\gamma = -\frac{\partial^2 \mathfrak{R}(\vec{\rho}')}{\partial x_i'^2}\Big|_{\vec{\rho}'=0}.$$

The last term of the Taylor series of $N(I,\vec{\rho})$ above is negligible in the strongly nonlocal regime. In this case, we assume (1) to take the following approximate form:

$$\begin{aligned} &\mathrm{i}\psi_z + \frac{\beta(z)}{2}\nabla^2\psi + s(z)\left[\mathfrak{R}(0)E_0 - \frac{1}{2}\gamma E_0(x^2+y^2)\right]\psi \\ &= \mathrm{i}\frac{g(z)}{2}\psi. \end{aligned} \quad (6)$$

Reference [8] studied Laguerre-Gaussian spatial solitary waves for (6) with $\beta(z) = 1, s(z) = s_0 =$ constant. In [10], author discussed (6) with $\beta(z) = s(z) = 1$, $g(z) = 0$, and gain the rotating azimuthon, which can be reduced to the radially symmetric optical vortex soliton under certain conditions. Further on, [12] investigated (6) with space-dependent diffractive and gain coefficient for the first time and gain the non-rotating Bessel solitary wave. We introduce a set of self-similar transformation as follows:

$$\begin{aligned} \Psi(x,y,z) = \frac{G(z)\Phi(X(x,z),Y(y,z),Z(z))}{H(z)} \\ \cdot \exp[\mathrm{i}\eta(x,y,z)] \end{aligned} \quad (7)$$

with

$$X = \frac{x}{H(z)}\,,\quad Y = \frac{y}{H(z)}\,,$$
$$\eta = \Re(0)E_0 + \Lambda Z(z) + k(z)(x^2+y^2)\,, \tag{8}$$

where $G(z) = \exp[\int_0^z \frac{g(z')}{2}\,\mathrm{d}z']$ is the change of the energy of the beam. Λ is the propagation constant. The beam width $H(z)$, the effective propagation distance $Z(z)$, and the variable $k(z)$ are to be determined. Inserting (8) into (6), subsequently, we obtain the following equation in the cylindrical coordinate system:

$$\frac{1}{2}\frac{\partial^2\Phi}{\partial r^2} + \frac{1}{2r}\frac{\partial\Phi}{\partial r} + \frac{1}{2r^2}\frac{\partial^2\Phi}{\partial\theta^2}$$
$$- I\Omega\frac{\partial\Phi}{\partial\theta} - \Lambda\Phi - \frac{1}{2}\gamma E_0 r^2\Phi = 0 \tag{9}$$

with

$$r = \sqrt{X^2+Y^2}\,,\quad \theta = \varphi - \Omega Z(z)\,,$$
$$\varphi = \arccos\left(\frac{X}{\sqrt{X^2+Y^2}}\right),\quad Z(z) = Z_0 + \int\frac{\beta(z')}{H(z')^2}\,\mathrm{d}z'\,,$$
$$s(z) = -\frac{\mathrm{d}k(z)}{\mathrm{d}z} - 2\beta(z)k(z)^2 + \frac{\beta(z)}{2H(z)^4}\,, \tag{10}$$
$$H(z) = H_0\exp\left(\int 2\beta(z')k(z')\,\mathrm{d}z'\right),$$

where $\beta(z)$ and $k(z)$ are arbitrary functions of z, H_0, and Z_0 are the arbitrary constants, and Ω is the angular frequency.

Equation (9) is the associated Euler–Lagrange equation [34] of the following calculus of variations:

$$\delta\int_0^\infty\int_0^{2\pi}\left[\frac{I}{2}\Omega r\left(\Phi\frac{\partial\Phi^*}{\partial\theta} - \Phi^*\frac{\partial\Phi}{\partial\theta}\right) - \frac{1}{2r}\left|\frac{\partial\Phi}{\partial\theta}\right|^2\right.$$
$$\left. - \frac{1}{2}r\left|\frac{\partial\Phi}{\partial r}\right|^2 - \frac{1}{2}\gamma E_0 r^3|\Phi|^2 - \Lambda r|\Phi|^2\right]\mathrm{d}\theta\,\mathrm{d}r = 0\,. \tag{11}$$

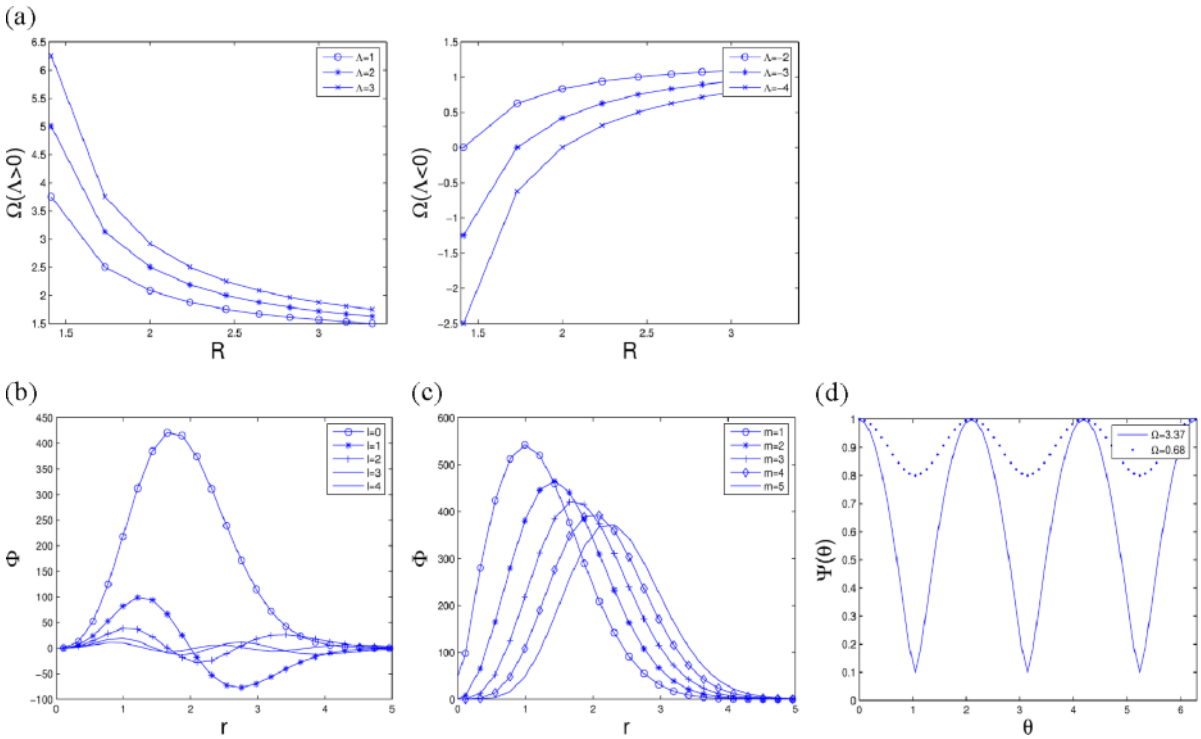

Fig. 1 (colour online). (a) Angular frequency versus radius of azimuthon with $l = 0$, $R(r) = \int|\Psi|^2 r^2\,\mathrm{d}r$. (b) Amplitude comparison of radial distribution of azimuthon for different l with $m = 3$. (c) Amplitude comparison of radial distribution of azimuthon for different m with $l = 0$. (d) Azimuthal envelopes of solution, for example, with different angular frequency Ω. The other parameters in common are $q = 0.5, \sigma = 10\sqrt{10}$.

The separation of variables $\Phi(r,\theta) = \Phi_0 V(r)$ $[\cos(m\theta) + \mathrm{i}q\sin(m\theta)]$ in (11) leads to an average calculus of variations, which equals to

$$\delta \int_0^\infty -\frac{\Phi_0^2\pi}{2r}\left\{V^2(1+q^2)(\gamma E_0 r^4 + m^2) + \left[-4\Omega qmV^2 + (1+q^2)\left(2\Lambda V^2 + \left(\frac{\mathrm{d}V}{\mathrm{d}r}\right)^2\right)\right]r^2\right\}\mathrm{d}r = 0\,. \tag{12}$$

To meet the requirements of (12), we find

$$V(r) = r^m L_l^m(r^2)\exp\left(-\frac{r^2}{2}\right) \tag{13}$$

with

$$l = -\frac{m+1+\Lambda}{2} + \frac{\Omega qm}{1+q^2} \tag{14}$$

and

$$E_0 = 1/\gamma,\ \ \Phi_0 = \sqrt{\frac{2l!}{\gamma(m+l)!}}\,, \tag{15}$$

where $L_l^m(r^2)$ is the generalized Laguerre polynomial, the parameters Φ_0 is the normalization constant, $q \in [0,1]$ determines the modulation depth of the beam intensity, m is a real constant called the topological charge, and l is a non-negative integer.

Finally, we obtain the families of the modulated self-similar azimuthons to (6):

$$\begin{aligned}\Psi_{l,m} &= \exp\left[\int_0^z \frac{g(z')}{2}\mathrm{d}z'\right]\Phi_0 H(z)^{-1} r^m L_l^m(r^2)\\ &\cdot\exp\left(-\frac{r^2}{2}\right)[\cos(m\theta) + \mathrm{i}q\sin(m\theta)]\\ &\cdot\exp[\mathrm{i}\Re(0)E_0 + \mathrm{i}\Lambda Z(z) + \mathrm{i}k(z)(x^2+y^2)]\end{aligned} \tag{16}$$

with (8), (10), (15), which are characterized by the azimuthal quantum numbers m and l and parametrized by the angular frequency Ω and the propagation constant Λ, see Figure 1. For $\Omega = 0$, azimuthons become ordinary (non-rotating) solitons. It should be stressed that there exist three kinds of non-rotating stationary solutions in the form of (16): (i) Gaussian solitons ($m = 0$); (ii) Vortex solitons ($q = 1$ and $m \neq 0$); (iii) Soliton cluster ($q = 0$).

3. Discussion

To find stationary soliton solutions of (1), we resort to a variational numerical procedure. Choosing initial conditions that are consistent with the solution of linear (6), $\Psi = \Psi(0,r,\theta)$, which is obtained above, and also for $H(z) = H_0 = 1, g(z) = 0, \Lambda = -m-1-2l$. Note that such a choice of the initial optical field allows the possibility of having propagating fields with fractional topological charges m, provided the parameter q is chosen accordingly. Such a possibility has been dis-

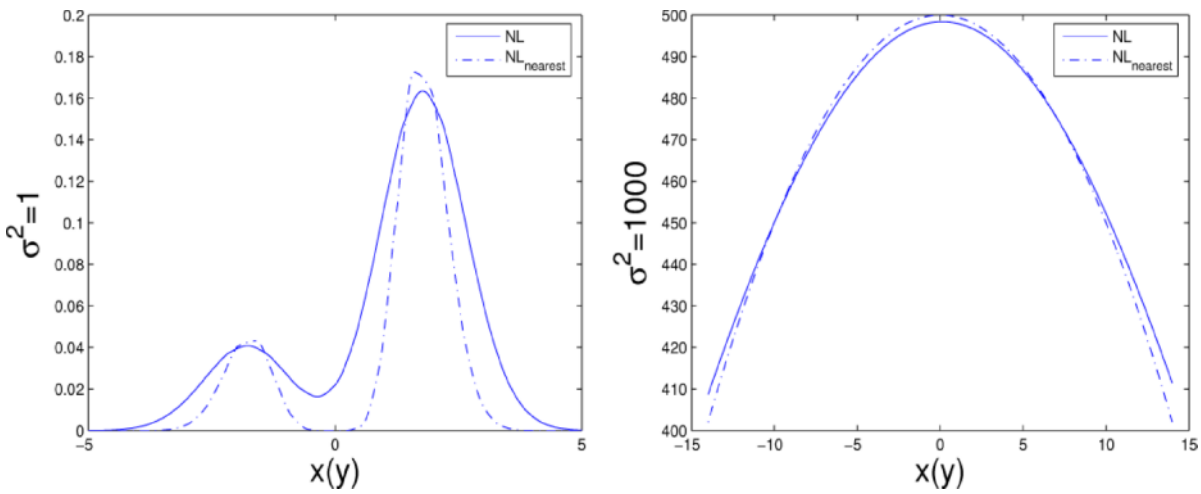

Fig. 2 (colour online). Comparison between the numerical nonlinear term (2) and the nearest one on the assumption that the response of the material is of the Gaussian-type with different parameter σ.

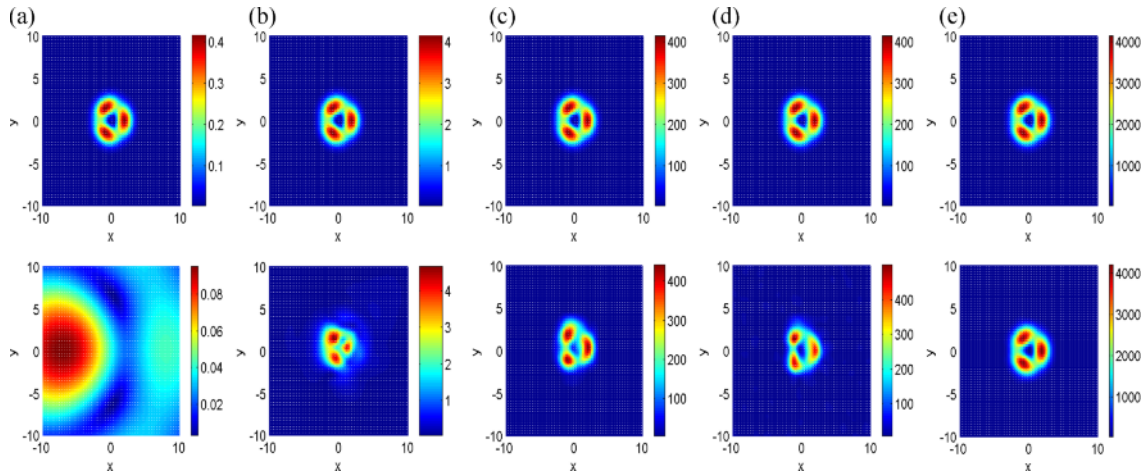

Fig. 3 (colour online). Initial intensity distribution of an azimuthon (top) and dynamics of it (bottom) with different σ from left to right at the same propagation distance $z = 10$; the other parameters are: $H = H_0 = 1$, $g = 0$, $m = 3$, $l = 0$, $q = 0.5$, $\Lambda = -4$. (a) $\sigma = 1$; (b) $\sigma = \sqrt{10}$; (c) $\sigma = 10\sqrt{10}$; (d) Parameters are same as in (c), except that a white noise of variance $v^2 = 0.02$ is added; (e) $\sigma = 100$ with a white noise of variance $v^2 = 0.02$.

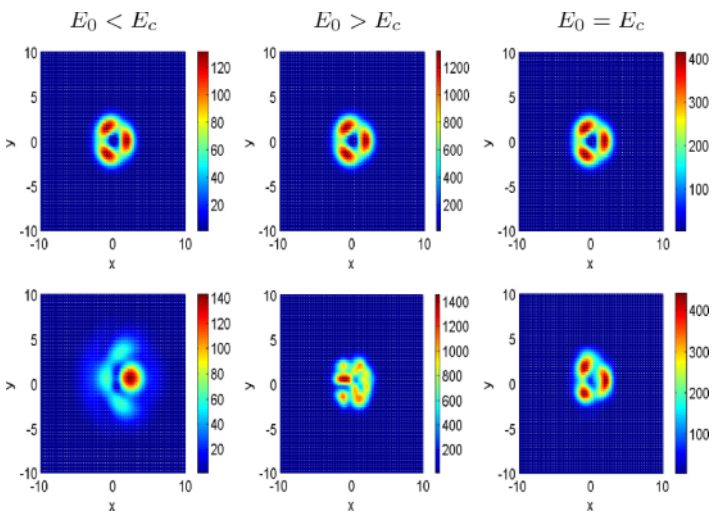

Fig. 4 (colour online). Initial intensity distribution of an azimuthon (top) and dynamics of it (bottom) with $E_0 < E_c$, $E_0 > E_c$, $E_0 = E_c$ from left to right at the same propagation distance $z = 10$.

cussed theoretically [35, 36] and demonstrated experimentally [37 – 39]. Figure 2 displays the comparison between the numerical nonlinear term (2) and the corresponding nearest one. Figure 3 depicts the numerical simulation at the same propagation distance with different σ. The distance z is given in units of the diffraction length. The idea is to demonstrate that the solution so obtained (in the strongly nonlocal limit) is more stable than the solutions with lower values of σ. The Gaussian response function $\mathfrak{R}(\vec{\rho})$ with $\sigma = 100$ is chosen in (3), so as to be in the strongly nonlocal region. In general, the evolution of the azimuthon is considered stable, although the beams exhibit slow radical expansion as they propagate. In the presented example, the deformation is even negligible over the distances considered for propagation. When the degree of nonlocality becomes weaker, the differences between the initial form and the evolutive form are changing more and more for the worse, as shown in Figure 3. To confirm the stabilization of the azimuthon, we simulated the propagation of the azimuthon with a white noise. It is shown that the noise does not result in a collapse of the case $\sigma = 100$, and no apparent difference is observed. Furthermore, propagating of the beam in

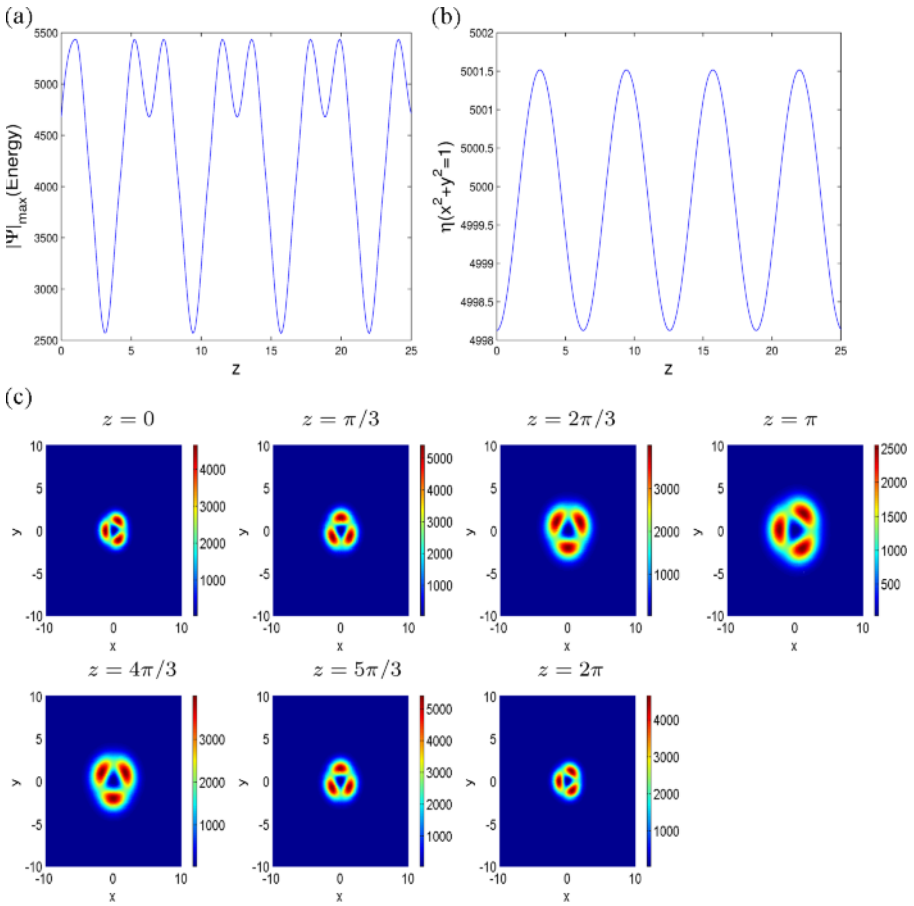

Fig. 5 (colour online). (a) Profile of the breathing azimuthon energy along the distance. (b) Corresponding phase of the azimuthon at $x^2+y^2=1$ along the distance. (c) Propagation of the azimuthon with $l=0$, $m=3$, $\sigma=100$.

weakly nonlocal nonlinear media is, however, distinct from the strongly nonlocal case. When a little perturbation is performed, the beam collapses in the case of $\sigma=10\sqrt{10}$ at a distance $z=10$, which indicates that the beam is instable, see Figure 3d.

As we known, $E_c=1/\gamma$ is the critical power for the soliton propagation. In order to explore the influence on the stationary solution of E_c, we set the other parameters same as that in Figure 3. When beam power $E_0<E_c$, the beam diffraction overcomes the beam-induced refraction and the beam initially broadens, whereas the reverse happens, and the beam initially narrows for $E_0>E_c$, respectively. When beam power $E_0=E_c$, the beam diffraction initially equals to the beam-induced refraction, and the pattern of the beam varies periodically, as seen in Figure 4, where $\sigma=10\sqrt{10}$.

In particular, let us have a closer look at this case of strong nonlocality with the critical power E_c. Our numerical results demonstrate that the self-similar azimuthons mentioned above rotate in the transverse (x,y)-plane as they evolve. As shown in Figure 5, these breathing azimuthons rotate counter-clockwise around the vertical axis. The parameters are $m=3$, $l=0$, $q=0.5$, $\sigma=100$, $\Lambda=-1.6$, $a=1$, $Z_0=0$, $H_0=1$, and

$$g(z)=4\sin(10z)\,,\ \ \beta(z)=-\sin(z)\exp[-0.6\cos(z)]\,,$$
$$k(z)=-0.15\exp[0.6\cos(z)]\,,$$

therefore we have

$$H(z)=\exp[-0.3\cos(z)]\,,\ \ G(z)=\exp[-0.2\cos(10z)]\,,$$
$$Z(z)=\cos(z)\,,\ \ s(z)=-0.545\sin(z)\exp[0.6\cos(z)]\,.$$

Further on, we find that the sense of rotation depends on the sign of the topological of m,l, and the distributed diffractive coefficient $\beta(z)$. When $\Omega Z(z)>$

0, the wave's rotation is counter-clockwise; when $\Omega Z(z) < 0$, it is clockwise. These are circular breathers whose widths vibrate or whose patterns vary periodically as they travel in the straight path along the z-axis, as well as the maximum of optical intensity.

4. Conclusion

In summary, with the help of a universal self-similarity technique and variational approach, we have presented a systematic analysis of the GNNE. We first simplified the GNNE to an approximate form (6) and obtained the self-similar azimuthons (16), and then analyzed numerically the existence and the stabilization of these azimuthons (16) in the GNNE. We have a large freedom to choose functions in (16) to obtain physical meaningful solutions. In addition, the self-similar azimuthons include the self-similar soliton cluster and the self-similar vortex soliton as particular members, with zero and maximum modulation depths, respectively. Our major attentions have been paid on a periodic distributed amplification system with the distributed diffractive coefficient $\beta(z)$, the gain (attenuation) coefficient $g(z)$, and the beam width $M(z)$. Under the conditions of critical power E_c and strong nonlocality, the self-similar azimuthons in the GNNE propagate stably, and this structure will not be destroyed through the large-scale variation of the initial distance of the beam center. Remarkably, these self-similar azimuthons have the azimuthal angle modulated by the distributed diffractive coefficient $\beta(z)$, apart from the beam width and intensity changing self-similarly as the breather.

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