A Variety of Exact Periodic Wave and Solitary Wave Solutions for the Coupled Higgs Equation

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In this work, the coupled Higgs field equation is studied. The extended Jacobi elliptic function expansion methods are efficiently employed to construct the exact periodic solutions of this model. As a result, many exact travelling wave solutions are obtained which include new shock wave solutions or kink-shaped soliton solutions, solitary wave solutions or bell-shaped soliton solutions, and combined solitary wave solutions are formally obtained.

Key words: Coupled Higgs Field Equation; Jacobi Elliptic Function Method; Solitary Wave Solutions.

1. Introduction

In recent years, there has been an increasing interest in the investigation of exact solutions for nonlinear partial differential equations (NLPDEs) which are widely used as models to describe the wave dynamics in various fields of nonlinear science. Solving nonlinear equations may guide authors to know the described process deeply and sometimes leads them to know some facts which are not simply understood through common observations [1].

Recently, many powerful methods have been established and developed to carry out the integrations of NLPDEs of all kinds, such as the subsidiary ordinary differential equation method (sub-ODE method for short) [2 – 4], solitary wave ansatz method [5, 6], sine-cosine method [7, 8], Hirota bilinear method [9, 10], F-expansion method [11], the Jacobi elliptic functions method [12], and so on.

Liu et al. proposed Jacobi elliptic sine (or cosine or the third-kind Jacobi elliptic) function expansion methods [13, 14], and obtained some exact periodic solutions of some nonlinear evolution equations. El-Wakil et al. [15] used the extended Jacobi elliptic function expansion method to solve some nonlinear evolution equations including the generalized Zakharov equations, the $(2+1)$-dimensional Davey–Stewartson equation, the higher-order nonlinear Schrödinger equation, and the $(2+1)$-dimensional Broer–Kaup–Kupershmidt system to get some new exact periodic solutions.

In present paper, we consider the following coupled Higgs field equation:

$$u_{tt} - u_{xx} - \beta u + \gamma |u|^2 u - 2uv = 0\,, \tag{1}$$

$$v_{tt} + v_{xx} - \gamma \left(|u|^2\right)_{xx} = 0\,, \tag{2}$$

which describes a system of conserved scalar nucleons interacting with neutral scalar mesons in particle physics. Here, the function $v = v(x,t)$ represents a real scalar meson field and $u = u(x,t)$ a complex scalar nucleon field. Equation (1) – (2) is the coupled nonlinear Klein–Gordon equation for $\beta < 0, \gamma < 0$ and the Higgs equation for $\beta > 0, \gamma > 0$. This model has important applications in various fields, such as particle physics, field theory and electromagnetic waves.

In this work, we are interested in constructing the periodic wave solutions of (1) and (2) and discussing their conditions of existence. For such a goal, we use the Jacobi elliptic sine, cosine, and combined sine-cosine functions expansion methods for deriving a variety of periodic solutions of the model under discussion.

2. Basic Idea of the Developed Jacobi Elliptic Function Expansion Method

The basic definitions and fundamental operations of the developed Jacobi elliptic function expansion method are defined as follows [15]: Consider a nonlinear evolution equation in the form

$$N\left(u,|u|,u_x,u_t,u_{xt},u_{tt},\ldots\right)=0\,, \tag{3}$$

where $u_x=\frac{\mathrm{d}u}{\mathrm{d}x}$, $u_{xx}=\frac{\mathrm{d}^2u}{\mathrm{d}x^2}, u_{xt}=\frac{\mathrm{d}^2u}{\mathrm{d}x\mathrm{d}t},\ldots$ and $|u|$ denotes the modulus of u, the same hereafter.

We can seek their envelope periodic solutions of the form

$$u(x,t)=\phi(\xi)\,\mathrm{e}^{\mathrm{i}(kx-\omega t)},\ \ \xi=x-\lambda t\,, \tag{4}$$

where ϕ is a real valued function, λ is a constant parameter, and k and ω denote the wave number and the circular frequency, respectively. Substituting (4) into (3) leads to an ordinary differential equation (ODE)

$$N\left(\phi,\frac{\mathrm{d}\phi}{\mathrm{d}\xi},\frac{\mathrm{d}^2\phi}{\mathrm{d}\xi^2},\frac{\mathrm{d}^3\phi}{\mathrm{d}\xi^3},\ldots\right)=0\,, \tag{5}$$

where $\phi(\xi)$ is expressed as a finite series of Jacobi elliptic sine functions $\mathrm{sn}(\xi)$ (similarly, Jacobi elliptic cosine function $\mathrm{cn}(\xi)$, or Jacobi elliptic functions of the third kind $\mathrm{dn}(\xi)$), i.e., the ansatz

$$\phi(\xi)=\sum_{i=1}^{n}a_i\mathrm{sn}^i(\xi)\,, \tag{6}$$

where a_i are constants to be determined later, and n is a positive integer that will be determined by using the balance method. Substituting (6) into the reduced ODE (5) and equating to zero the coefficients of all power of $\mathrm{sn}(\xi)$, $\mathrm{cn}(\xi)$, $\mathrm{dn}(\xi)$ yields a set of algebraic equations for a_i. Finally by inserting each solution of this set of algebraic equations into (6), then into (4), the exact periodic solutions of (1) and (2) are obtained.

3. Periodic Wave and Solitary Wave Solutions

To begin with, let us consider the following gauge transformation [16]:

$$u(x,t)=\mathrm{e}^{\mathrm{i}(kx+\omega t+\xi_0)}\psi(x,t)\,, \tag{7}$$

where $\psi(x,t)$ is a real-valued function, k, ω are two real constants to be determined, ξ_0 is an arbitrary constant.

Substituting (7) into (1) and (2), separating the real and imaginary parts of (1) leads to

$$\psi_{tt}-\psi_{xx}+\left(k^2-\omega^2-\beta\right)\psi+\gamma\psi^3-2\psi v=0\,, \tag{8}$$

$$\omega\psi_t-k\psi_x=0\,, \tag{9}$$

$$v_{tt}+v_{xx}-\gamma\left(\psi^2\right)_{xx}=0\,. \tag{10}$$

In view of (9), we assume that the travelling wave solutions of (8) and (10) is of the form

$$\psi(x,t)=\psi(\xi)\,,\ \ \xi=\omega x+kt+\xi_1\,, \tag{11}$$

$$v(x,t)=v(\xi)\,, \tag{12}$$

where ξ_1 is an arbitrary constant.

Substituting (11) and (12) into (8) and (10), yields an ordinary differential equations for $\psi(\xi)$ and $v(\xi)$:

$$\begin{aligned}&\left(k^2-\omega^2\right)\psi''+\left(k^2-\omega^2-\beta\right)\psi\\&+\gamma\psi^3-2\psi v=0\,,\end{aligned} \tag{13}$$

$$\left(\omega^2+k^2\right)v''-\gamma\omega^2\left(\psi^2\right)''=0\,, \tag{14}$$

where the prime denotes the derivative with respect to ξ.

Integrating (14) once and taking integration constant to zero, and integrating it again with respect to the variable ξ gives

$$v(\xi)=\frac{\gamma\omega^2}{\omega^2+k^2}\psi^2+C\,, \tag{15}$$

where C is an integration constant.

Inserting (15) into the ODE (13), we have

$$\psi''+\frac{k^2-\omega^2-\beta-2C}{k^2-\omega^2}\psi+\frac{\gamma}{\omega^2+k^2}\psi^3=0\,, \tag{16}$$

which can be rewritten as

$$\psi''+A\psi+B\psi^3=0\,, \tag{17}$$

where

$$A=\frac{k^2-\omega^2-\beta-2C}{k^2-\omega^2}\,,\ \ B=\frac{\gamma}{\omega^2+k^2}\,. \tag{18}$$

3.1. Using the Jacobi Elliptic Sine Function Expansion Method

To seek the envelope periodic solutions of (17), we first employ the Jacobi elliptic sine function expansion method which admits the use of the ansatz [15]

$$\psi(\xi)=\sum_{i=1}^{n}a_i\mathrm{sn}^i(\xi)\,, \tag{19}$$

where $\mathrm{sn}(\xi)$ is the Jacobi elliptic sine function, a_i are constants to be determined later, and n is a parameter which can be found by balancing the highest-order linear term with the nonlinear terms. Note that the Jacobi elliptic functions possess properties of triangular functions [15]:

$$\begin{aligned}
&\mathrm{sn}^2(\xi)+\mathrm{cn}^2(\xi)=1\\
&\mathrm{dn}^2(\xi)+m^2\mathrm{sn}^2(\xi)=1\,,\\
&(\mathrm{sn}(\xi))'=\mathrm{cn}(\xi)\,\mathrm{dn}(\xi)\,,\\
&(\mathrm{cn}(\xi))'=-\mathrm{sn}(\xi)\,\mathrm{dn}(\xi)\,,\\
&(\mathrm{dn}(\xi))'=-m^2\mathrm{sn}(\xi)\,\mathrm{cn}(\xi)\,,
\end{aligned}\tag{20}$$

where $\mathrm{sn}(\xi)=\mathrm{sn}(\xi,m)$ and $\mathrm{cn}(\xi)=\mathrm{cn}(\xi,m)$ are, respectively, the Jacobi elliptic sine and cosine functions. m is the modulus of the elliptic function, and $\mathrm{dn}(\xi)$ is the Jacobi elliptic function of the third kind.

By balancing the highest linear term ψ'' with the nonlinear ψ^3 term in (17), we get $n+2=3n$ so that $n=1$ in (19). Accordingly, we assume that

$$\psi(\xi)=a_0+a_1\mathrm{sn}(\xi)\,.\tag{21}$$

Substituting (21) and making use of (20) into (17) and equating the coefficients of all powers of $\mathrm{sn}^i(\xi)$ $(i=0,1,2,3)$, yields a set of algebraic equations for a_0, a_1:

$$Aa_0+Ba_0^3=0\,,\tag{22}$$

$$-a_1\left(m^2+1\right)+Aa_1+3Ba_0^2a_1=0\,,\tag{23}$$

$$3Ba_0a_1^2=0\,,\tag{24}$$

$$2a_1m^2+Ba_1^3=0\,.\tag{25}$$

Solving the above equations, we can determine the coefficients as

$$a_0=0\,,\;\; a_1=\pm\sqrt{-\frac{2}{B}}m\,,\;\; A=m^2+1\,.\tag{26}$$

Thus using (18), (26) gives

$$\begin{aligned}
&a_0=0\,,\;\; a_1=\pm\sqrt{-\frac{2\left(\omega^2+k^2\right)}{\gamma}}m\,,\\
&C=\frac{m^2\left(\omega^2-k^2\right)-\beta}{2}\,.
\end{aligned}\tag{27}$$

By combining (7), (11), (12), (15), (21), (27), we get final solutions in the form

$$\begin{aligned}
v(x,t)=&-2m^2\omega^2\mathrm{sn}^2\left(\omega x+kt+\xi_1\right)\\
&+\frac{m^2\left(\omega^2-k^2\right)-\beta}{2}\,,
\end{aligned}\tag{28}$$

$$\begin{aligned}
u(x,t)=&\pm\sqrt{-\frac{2\left(\omega^2+k^2\right)}{\gamma}}m\\
&\cdot\mathrm{sn}\left(\omega x+kt+\xi_1\right)\mathrm{e}^{\mathrm{i}(kx+\omega t+\xi_0)}\,,
\end{aligned}\tag{29}$$

which are the exact periodic solutions of the model (1) and (2). Remarkably, the solution $u(x,t)$ in (29) exists provided that $\gamma<0$.

As long as $m\to 1$, then $\mathrm{sn}(\xi)=\tanh(\xi)$. Thus, the solitary wave solutions (28) and (29) are degenerated as the following form:

$$\begin{aligned}
v(x,t)=&-2\omega^2\tanh^2\left(\omega x+kt+\xi_1\right)\\
&+\frac{\omega^2-k^2-\beta}{2}\,,
\end{aligned}\tag{30}$$

$$\begin{aligned}
u(x,t)=&\pm\sqrt{-\frac{2\left(\omega^2+k^2\right)}{\gamma}}\\
&\cdot\tanh\left(\omega x+kt+\xi_1\right)\mathrm{e}^{\mathrm{i}(kx+\omega t+\xi_0)}\,,
\end{aligned}\tag{31}$$

which are the envelope shock wave solution or kink-shaped soliton solutions of the considered (1) and (2). Note that the solution $u(x,t)$ in (31) exists provided that $\gamma<0$.

3.2. Using the Jacobi Elliptic Cosine Function Expansion Method

Similarly, using Jacobi elliptic cosine function expansion method, the ansatz solution is

$$\psi(\xi)=\sum_{i=1}^{n}a_i\mathrm{cn}^i(\xi)\,,\tag{32}$$

where $\mathrm{cn}(\xi)$ is the Jacobi elliptic cosine function. By balancing the highest linear term ψ'' with the nonlinear ψ^3 term in (17), we get $n=1$ in (32) so that

$$\psi(\xi)=a_0+a_1\mathrm{cn}(\xi)\,.\tag{33}$$

Substituting (33) and making use of (20) into (17) and equating the coefficients of all powers of $\mathrm{cn}^i(\xi)$

$(i=0,1,2,3)$, yields a set of algebraic equations

$$Aa_0+Ba_0^3=0\,, \tag{34}$$
$$a_1\left(2m^2-1\right)+Aa_1+3Ba_0^2a_1=0\,, \tag{35}$$
$$3Ba_0a_1^2=0\,, \tag{36}$$
$$-2a_1m^2+Ba_1^3=0\,. \tag{37}$$

Solving (34)–(37), we get

$$a_0=0\,,\;\; a_1=\pm\sqrt{\frac{2}{B}}m\,,\;\; A=1-2m^2\,. \tag{38}$$

Using (18), (38) gives

$$\begin{aligned} &a_0=0\,,\;\; a_1=\pm\sqrt{\frac{2\left(\omega^2+k^2\right)}{\gamma}}m\,,\\ &C=\frac{2m^2\left(k^2-\omega^2\right)-\beta}{2}\,. \end{aligned} \tag{39}$$

By combining (7), (11), (12), (15), (33), (39), we get the final solutions in the form

$$\begin{aligned} v(x,t)=&\,2m^2\omega^2\mathrm{cn}^2\left(\omega x+kt+\xi_1\right)\\ &+\frac{2m^2\left(k^2-\omega^2\right)-\beta}{2}\,, \end{aligned} \tag{40}$$

$$\begin{aligned} u(x,t)=&\pm\sqrt{\frac{2\left(\omega^2+k^2\right)}{\gamma}}m\\ &\cdot\mathrm{cn}\left(\omega x+kt+\xi_1\right)\mathrm{e}^{\mathrm{i}(kx+\omega t+\xi_0)}\,, \end{aligned} \tag{41}$$

which are another exact periodic solutions of the Higgs equation (1) and (2). It should be noted that the solution $u(x,t)$ exist provided that $\gamma>0$. When $m\to 1$, then $\mathrm{cn}(\xi)=\sec h(\xi)$, thus the solitary wave solutions (40) and (41) are obtained as follows:

$$\begin{aligned} v(x,t)=&\,2\omega^2\,\mathrm{sech}^2\left(\omega x+kt+\xi_1\right)\\ &+\frac{\omega^2-k^2-\beta}{2}\,, \end{aligned} \tag{42}$$

$$\begin{aligned} u(x,t)=&\pm\sqrt{\frac{2\left(\omega^2+k^2\right)}{\gamma}}\\ &\cdot\mathrm{sech}\left(\omega x+kt+\xi_1\right)\mathrm{e}^{\mathrm{i}(kx+\omega t+\xi_0)}\,, \end{aligned} \tag{43}$$

which are envelope solitary wave solutions of bell-shape for the Higgs equation (1) and (2). Note that these solutions exist provided that $\gamma>0$.

3.3. Using the Jacobi Elliptic Sine-Cosine Function Expansion Method

Let us now assume an ansatz solution for (17) in the form [17]

$$\begin{aligned} \psi(\xi)=a_0+&\sum_{i=1}^{l}\mathrm{sn}^{i-1}(\xi)\\ &\cdot\left[a_{2i-1}\mathrm{sn}(\xi)+a_{2i}\mathrm{cn}(\xi)\right]. \end{aligned} \tag{44}$$

Balancing the highest-order derivative ψ'' with the nonlinear ψ^3 term in (17), we get $l=1$ in (44). Accordingly, (44) has the following formal solution:

$$\psi(\xi)=a_0+a_1\mathrm{sn}(\xi)+a_2\mathrm{cn}(\xi)\,, \tag{45}$$

where a_0, a_1, and a_2 are coefficients to be determined later.

In the limit $a_0=a_1=0$, we obtain periodic solutions with the cn-type shape, but when $a_2=0$ the solution given in (45) exactly transforms to periodic solutions of sn-type function. The presence of the parameters a_0, a_1, and a_2 permits the ansatz (45) to describe the features of periodic solutions with combined $\mathrm{sn}(\xi)-\mathrm{cn}(\xi)$ shape.

Substituting (45) and using (20) into (17), expanding cn terms to sn terms, and equating the coefficient terms containing independent combinations of cn and sn functions to zero, we obtain the following seven independent parametric equations:

$$Aa_0+B{a_0}^3+3B{a_2}^2a_0=0\,, \tag{46}$$

$$\begin{aligned} &-a_1\left(1+m^2\right)\\ &+Aa_1+3B{a_0}^2a_1+3B{a_2}^2a_1=0\,, \end{aligned} \tag{47}$$

$$3B{a_1}^2a_0-3B{a_2}^2a_0=0\,, \tag{48}$$

$$2a_1m^2+B{a_1}^3-3B{a_2}^2a_1=0\,, \tag{49}$$

$$2a_2m^2-B{a_2}^3+3B{a_1}^2a_2=0\,, \tag{50}$$

$$-a_2+Aa_2+B{a_2}^3+3B{a_0}^2a_2=0\,, \tag{51}$$

$$6Ba_0a_1a_2=0\,. \tag{52}$$

Solving the above equations yields

$$\begin{aligned} &a_0=0\,,\;\; a_1=\pm\mathrm{i}\frac{m}{\sqrt{2B}}\,,\\ &a_2=\pm\frac{m}{\sqrt{2B}}\,,\;\; A=1-\frac{m^2}{2}\,. \end{aligned} \tag{53}$$

Using (17), (53) gives

$$a_0 = 0\,,\ \ a_1 = \pm \mathrm{i}\sqrt{\frac{\omega^2+k^2}{2\gamma}}m\,,$$
$$a_2 = \pm\sqrt{\frac{\omega^2+k^2}{2\gamma}}m\,, \tag{54}$$
$$C = \frac{2m^2\left(k^2-\omega^2\right)-\beta}{2}\,,$$

where $\mathrm{i} = \sqrt{-1}$ is the imaginary unit.

By combining (7), (11), (12), (15), (45), (54), we get a final solutions in the form

$$v(x,t) = \frac{\gamma\omega^2}{\omega^2+k^2}[\pm \mathrm{i}\lambda_1 \mathrm{sn}(\omega x+kt+\xi_1) \pm \lambda_1 \mathrm{cn}(\omega x+kt+\xi_1)]^2 + \frac{2m^2\left(k^2-\omega^2\right)-\beta}{2}, \tag{55}$$

$$u(x,t) = [\pm \mathrm{i}\lambda_1 \mathrm{sn}(\omega x+kt+\xi_1) \pm \lambda_1 \mathrm{cn}(\omega x+kt+\xi_1)]\,\mathrm{e}^{\mathrm{i}(kx+\omega t+\xi_0)}, \tag{56}$$

where $\lambda_1 = m\sqrt{(\omega^2+k^2)/2\gamma}$.

As long as $m \to 1$, then $\mathrm{sn}(\xi) = \tanh(\xi)$ and $\mathrm{cn}(\xi) = \mathrm{sech}(\xi)$. Thus, the solitary wave solutions (55) and (56) take the forms:

$$v(x,t) = \frac{\gamma\omega^2}{\omega^2+k^2}[\pm \mathrm{i}\lambda_2 \tanh(\omega x+kt+\xi_1) \pm \lambda_2 \sec h(\omega x+kt+\xi_1)]^2 + \frac{2\left(k^2-\omega^2\right)-\beta}{2}, \tag{57}$$

$$u(x,t) = [\pm \mathrm{i}\lambda_2 \tanh(\omega x+kt+\xi_1) \pm \lambda_2 \mathrm{sech}(\omega x+kt+\xi_1)]\,\mathrm{e}^{\mathrm{i}(kx+\omega t+\xi_0)}, \tag{58}$$

where $\lambda_2 = \sqrt{(\omega^2+k^2)/2\gamma}$. These solutions represent the combined solitary wave solutions for the Higgs equation (1) and (2) that can describe the simultaneous propagation of bright and dark solitary waves in a combined form in nonlinear media. We should point out that the existence of combined solitary wave solutions has been proven for the first time for the higher-order nonlinear Schrödinger equation with constant coefficients [18], and variable coefficients [19].

4. Conclusion

The coupled Higgs field equation has been investigated. The developed Jacobi elliptic sine and cosine functions expansion methods were used to construct the new exact periodic solutions of the considered model. By adopting an amplitude ansatz in terms of combined $\mathrm{sn}(\xi) - \mathrm{cn}(\xi)$ functions, we have derived a new periodic wave solution for the coupled Higgs field equation model which shows that the coupled Higgs field equation model can support many exact solitary wave solutions including the new bright–dark solitons in the limit $m \to 1$. The work reveals the power of the adaptive method in handling two coupled NLPDEs. The applied method will be used in further works to establish more entirely new exact solutions for other kinds of three and multi-component system of NLPDEs.

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