One Dimensional Model of Martensitic Transformation Solved by Homotopy Analysis Method

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The homotopy analysis method (HAM) is applied to solve a nonlinear ordinary differential equation describing certain phase transition problem in solids. Both bifurcation conditions and analytical solutions are obtained simultaneously for the Euler–Lagrange equation of the martensitic transformation. HAM is capable of providing an analytical expression for the bifurcation condition to judge the occurrence of the phase transition, while other numerical techniques have difficulties in bifurcation analysis. The convergence of the analytical solutions on the one hand can be adjusted by the auxiliary parameter and on the other hand is always obtainable for all relevant physical parameters satisfying the bifurcation condition.

Key words: Homotopy Analysis Method; Martensitic Phase Transformation; Double-Well Potential; Bifurcation.

1. Introduction

Many scholars [1 – 5] considered a classical one-dimensional problem of equilibrium for a bar like shape memory alloys undergoing stress-induced martensitic transformation in a hard device, which reduces to the minimization of the energy functional

$$F = \int_0^L [W(u_x) + \alpha u_{xx}^2 + \beta(u - \gamma u_{\mathrm{H}})^2]\,\mathrm{d}x \qquad (1)$$

with the boundary condition

$$u(0) = 0, \;\; u(L) = dL, \;\; u''(0) = u''(L) = 0. \quad (2)$$

$u(x)$ denotes the displacement field, $u_{\mathrm{H}} = (u(L) - u(0))x/L + u(0)$ the field of classical homogeneous displacement of the bar, and d the overall imposed strain. The potential energy density is a double-well nonconvex function permitting phase transition and stress instability to occur. The parameters E, α, and β are positive constants. $\gamma = 0$ or 1 corresponds to the rigid or elastic foundation models, respectively. $\gamma = 1$ will be considered in this paper.

The minimizers of the energy functional must satisfy the following Euler–Lagrange (EL) equation:

$$\begin{aligned}&\frac{\partial^2 W(u_x(x))}{\partial u_x^2} u''(x) \\ &- 2\alpha u''''(x) - 2\beta(u(x) - u_{\mathrm{H}}(x)) = 0.\end{aligned} \qquad (3)$$

One major difficulty in solving (3) is the strong nonlinearity due to the nonconvex quartic potential energy, the other the indeterminate positions of the multiple phase interfaces. Analytic forms were obtained in [2] for the solutions with piecewise parabolic $W(u_x)$. Local bifurcation analysis based on linearization of inhomogeneous strain was performed in [3] for a quartic $W(u_x)$, and numerical solutions were obtained by the finite difference method. A dynamical solution is worked out for the strain field by adding a nonlocal term which is quadratic in strains and has a negative definite interaction kernel [4]. All of these successful studies of the martensitic transformation have great contributions on this field of research and have meanwhile left some topics for future research [2 – 4].

As analytical solutions are beneficial in many aspects, we try to analytically solve the EL equation

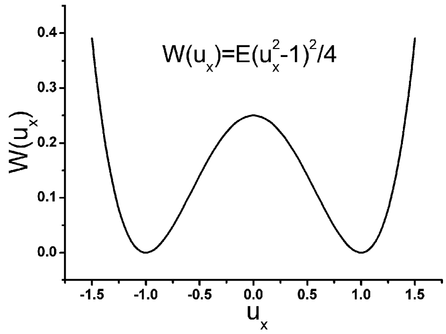

Fig. 1. Double-well potential energy density $W(u_x)$.

(3) with a quartic double-well potential $W(u_x) = E(u_x^2-1)^2/4$ (Fig. 1) in this paper. We adopt the homotopy analysis method (HAM) to give series solutions for the displacement and strain field during the martensitic transformation. HAM [6–8] is an analytic technique for nonlinear problems [9–18]. However, the HAM hasn't been applied to problems studying phase transitions. As for the application in phase transition to be presented here, the HAM serves as a good tool to implement bifurcation analysis. Furthermore, the multiple solutions are obtainable for all possible interface numbers permitting a phase transition.

The remaining sections of the paper are organized as follows. In Section 2, we implement the basic homotopy analysis method to get the series solution of the Euler–Lagrange equation (3). In Section 3, we present a way of finding a valid auxiliary parameter required to ensure the convergence of the series solutions. The HAM based bifurcation analysis is attained to enclose the confines of parameters that permit phase transition to occur. For different values of overall imposed strain, multiple solutions of strain fields with different interface numbers are presented. Also, we mention the assumption of global energy minima to be one of the ways of finding unique physical solutions. Conclusions are displayed in Section 4.

2. Application of HAM on Solving the One-Dimensional Model of the Martensitic Transformation

To nondimensionalize and homogenize the boundary condition of (3), we implement the following transformation:

$$\xi = \frac{x}{L}\pi, \ \ v(\xi) = \frac{u(x)-u_{\mathrm{H}}(x)}{L}. \tag{4}$$

For any α, β, d, (3) has the trivial solution $u(x) = u_{\mathrm{H}}(x)$ or $v(\xi) = 0$ corresponding to the classical homogeneous solution. But for certain values of α, β, d, there exist nontrivial solutions. Following the basic ideas of HAM [6] dealing with the bifurcation problem, we define

$$A = v\left(\frac{\pi}{2k}\right), \ \ z(\xi) = \frac{v(\xi)}{A}, \tag{5}$$

where the nonzero A, the wave amplitude, permits bifurcation to occur. Then (3) reduces to

$$\begin{aligned} &z^{(4)}(\xi) + cz''(\xi) + az(\xi) - bA^2[z'(\xi)]^2 z''(\xi) \\ &-\frac{2bd}{\pi}Az'(\xi)z''(\xi) = 0, \\ &z(0) = z(\pi) = z''(0) = z''(\pi) = 0, \end{aligned} \tag{6}$$

where

$$a = \frac{\bar{\beta}}{\pi^4\bar{\alpha}}, \ \ b = \frac{3}{2\bar{\alpha}}, \ \ c = \frac{1-3d^2}{2\pi^2\bar{\alpha}}. \tag{7}$$

Here $\bar{\alpha} = \alpha/EL^2$ and $\bar{\beta} = \beta L^2/E$ are the dimensionless interfacial and inhomogeneous energy coefficient, respectively, and primes denote the derivative with respect to ξ. Studies like numerics based dynamical model [4] and mathematical proof dealing with minimizers of functionals like (1) [19] imply that the strain is nearly periodic when equilibrium is attained. So according to the boundary condition in (2), we express $z(\xi)$ as

$$z(\xi) = \sum_{n=0}^{\infty} C_n \sin[(n+1)k\xi], \tag{8}$$

where k is used to quantify the number of phase interfaces. The initial guess is

$$z_0(\xi) = \sin k\xi. \tag{9}$$

Define a linear operator according to the solution expression (8)

$$L[\Phi(\xi;q)] = \frac{\partial^2\Phi(\xi;q)}{\partial\xi^2} + k^2\Phi(\xi;q) \tag{10}$$

with the property

$$L[C_1\sin(k\xi) + C_2\cos(k\xi)] = 0, \tag{11}$$

where C_1 and C_2 are arbitrary constants.

According to the basic ideas of HAM [6–8], we construct a so-called 0th-order deformation equation

$$\begin{aligned}&(1-q)L[\Phi(\xi;q)-z_0(\xi)]\\&\quad=\hbar qH(\xi)N[\Phi(\xi;q),\alpha(q)],\end{aligned}\tag{12}$$

subject to the boundary condition

$$\begin{aligned}\Phi(0;q)&=\Phi(\pi;q)=\Phi''(0;q)\\&=\Phi''(\pi;q)=0.\end{aligned}\tag{13}$$

Here, $q\in[0,1]$ is an embedding parameter, $\Phi(\xi;q)$ an unknown function depending on ξ and q, $\alpha(q)$ an unknown function of q, $\hbar\neq 0$ an auxiliary parameter used to control the convergence and to adjust the rate of it, and $H(\xi)\neq 0$ is an auxiliary function. N denotes a nonlinear operator of the same form of (6)

$$\begin{aligned}&N[\Phi(\xi;q),\alpha(q)]=\frac{\partial^4\Phi}{\partial\xi^4}+a\Phi+c\frac{\partial^2\Phi}{\partial\xi^2}\\&-b\alpha^2(q)\left(\frac{\partial\Phi}{\partial\xi}\right)^2\frac{\partial^2\Phi}{\partial\xi^2}-2\frac{bd}{\pi}\alpha(q)\frac{\partial\Phi}{\partial\xi}\frac{\partial^2\Phi}{\partial\xi^2}.\end{aligned}\tag{14}$$

Apparently, when $q=0$ and $q=1$, it holds

$$\Phi(\xi;0)=z_0(\xi),\ \ \Phi(\xi;1)=z(\xi),\ \ \alpha(1)=A.\tag{15}$$

Thus as q increases from 0 to 1, the solution $\Phi(\xi;q)$, $\alpha(q)$ varies from the initial guess $z_0(\xi),A_0$ to the accurate solution $z(\xi),A$, respectively.

The following is something noticeable. As for $0<q<1$, we do not mean that all $\hbar$ and H are suitable for the solution of the homotopy equation (12). Nor can we ensure or prove rigorously that for any H, we can find a corresponding $\hbar$ to give a convergent solution to (12). Generally, the HAM provides a method to solve a nonlinear partial differential equation (PDE) admissible for certain types of H. If there exist some forms (no matter what) of H except zero to make (12) solvable and give convergent solutions, we are successful in applying the HAM to solve our own nonlinear problems. For detail references see [6–13].

By Taylor's expansion, it holds

$$\Phi(\xi;q)=z_0(\xi)+\sum_{m=1}^{+\infty}z_m(\xi)q^m,\tag{16}$$

$$\alpha(q)=A_0+\sum_{m=1}^{+\infty}A_mq^m,\tag{17}$$

where

$$\begin{aligned}z_m(\xi)&=\frac{1}{m!}\frac{\partial^m\Phi(t;q)}{\partial q^m}\bigg|_{q=0},\\A_m&=\frac{1}{m!}\frac{\mathrm{d}^m\alpha(q)}{\mathrm{d}q^m}\bigg|_{q=0}.\end{aligned}\tag{18}$$

Differentiating m times the 0th-order deformation equation (12) and (13) with respect to q and then dividing them by $m!$ and finally setting $q=0$, the mth-order deformation equations are obtained:

$$\begin{aligned}&L[z_m(\xi)-\chi_mz_{m-1}(\xi)]=\hbar H(\xi)R_m(\mathbf{z}_{m-1},\mathbf{A}_{m-1}),\\&m>0,\end{aligned}\tag{19}$$

subject to the boundary conditions

$$z_m(0)=z_m(\pi)=0,\ \ z_m''(0)=z_m''(\pi)=0,\tag{20}$$

where

$$\chi_m=\begin{cases}0, & m\leq 1,\\1, & \text{other}.\end{cases}\tag{21}$$

The function $R_m(\mathbf{z}_{m-1},\mathbf{A}_{m-1})$ depends on the vectors $\mathbf{z}_{m-1}=\{z_0(\xi),z_1(\xi),z_2(\xi),\ldots,z_{m-1}(\xi)\}$ and $\mathbf{A}_{m-1}=\{A_0,A_1,A_2,\ldots,A_{m-1}\}$, and its expression is given by

$$\begin{aligned}&R_m(\mathbf{z}_{m-1},\mathbf{A}_{m-1})=\frac{1}{(m-1)!}\\&\cdot\frac{\partial^{m-1}N[\Phi(\xi;q),\alpha(q)]}{\partial q^{m-1}}\bigg|_{q=0}\\&=z_{m-1}^{(4)}+az_{m-1}+cz_{m-1}''\\&-\frac{2bd}{\pi}\left[\sum_{r=0}^{m-1}\left(\sum_{k=0}^{r}A_kz_{r-k}'\right)z_{m-1-r}''\right]\\&-b\sum_{j=0}^{m-1}\left[\left(\sum_{i=0}^{j}A_iA_{j-i}\right)\right.\\&\left.\cdot\left(\sum_{r=0}^{m-1-j}\left[\sum_{k=0}^{r}z_k'z_{r-k}'\right]z_{m-1-j-r}''\right)\right].\end{aligned}\tag{22}$$

The solution based on (19) and (20) with two unknown variables $z_m(\xi)$ and A_{m-1} is still an open problem, hence an extra equation is needed. If $H(\xi)\neq 0$, the right hand side of (19) can be written as $\sum_{n=0}^{\mu_m}b_{m,n}(\mathbf{A}_{m-1})\sin[(n+1)k\xi]$, where $b_{m,n}(\mathbf{A}_{m-1})$ is a coefficient. If $b_{m,0}(\mathbf{A}_{m-1})\neq 0$, $\sin(k\xi)$ will give rise to the existence of a secular term $\xi\cos(k\xi)$ in the solution of the linear ordinary differential equation (ODE) (19), which will violate the basic form of the solution expression (8). To avoid this, we must have

$$b_{m,0}(\mathbf{A}_{m-1}) = 0\,. \tag{23}$$

An alternative to get this extra algebraic equation is to operate L^{-1} at both sides of (19), and with the property (11) it holds

$$z_m(\xi) - \chi_m z_{m-1}(\xi) = \hbar H(\xi) L^{-1}[R_m(z_{m-1}, A_{m-1})] + C_1 \sin k\xi + C_2 \cos k\xi\,, \tag{24}$$

where

$$L^{-1}[\sin m\xi] = \frac{\sin m\xi}{k^2 - m^2} \tag{25}$$

is the inverse operator of L, satisfying $m \neq k$ and $m \geq 1$. Apparently, $\sin(k\xi)$ is supposed to vanish at the right hand side of (19), thus requiring (23). According to the *rule of solution expression* [6], we force $C_2 = 0$. As (8) leads to $z(\pi/2k) = 1$, we have

$$z_m\left(\frac{\pi}{2k}\right) = 0\,. \tag{26}$$

Then the solution of (19) and (20) is given by

$$z_m(\xi) = \chi_m z_{m-1}(\xi) + \sum_{n=1}^{2m} \frac{b_{m,n}}{[1-(n+1)^2]k^2} \sin[(n+1)k\xi] + C_1 \sin k\xi\,, \tag{27}$$

where C_1 is determined by (26).

Assuming that $\hbar$ and $H(\xi)$ are so properly selected that series (16) and (17) are convergent at $q = 1$, the Mth-order approximation is given by

$$z(\xi) = z_0(\xi) + \sum_{m=1}^{M} z_m(\xi)\,, \tag{28}$$

$$A = A_0 + \sum_{m=1}^{M} A_m\,. \tag{29}$$

The series solutions (28) and (29) are easily obtainable by symbol calculation software like Mathematica or Maple.

3. Results and Discussions

3.1. Examination of the Convergence of the HAM-Based Solutions

Generally from the experience in applying the HAM [6–8], we may start by choosing a constant auxiliary function as

$$H(\xi) = 1 \tag{30}$$

to test whether we can find some convergent solutions by using a proper nonzero auxiliary parameter $\hbar$. If this is the case, we can use this pair of auxiliary function and parameter to work out the HAM solutions. Otherwise, we may need to test some other auxiliary functions and parameters until some convergent solutions are obtained.

Note that the solution series (28) and (29) for given values of $\overline{\alpha}, \overline{\beta}, d, k$ contain the parameter $\hbar$ that enables us to adjust and control the convergence by means of the so-called $\hbar$ curve method. As pointed out by [6], the valid region of $\hbar$ should be a nearly horizontal line segment as in Figure 2a. In Figure 2a, the region $\hbar \in [1 \cdot 10^{-4}, 3 \cdot 10^{-4}]$ can be regarded as such a valid region in which the series A remains almost unchanged with the increasing HAM order and thus is supposed to converge. So if we choose any value of $\hbar$ from this region, the series A is convergent. See also how the sum of the convergent series A of increasing order of approximation increases at an ever-decreasing rate listed

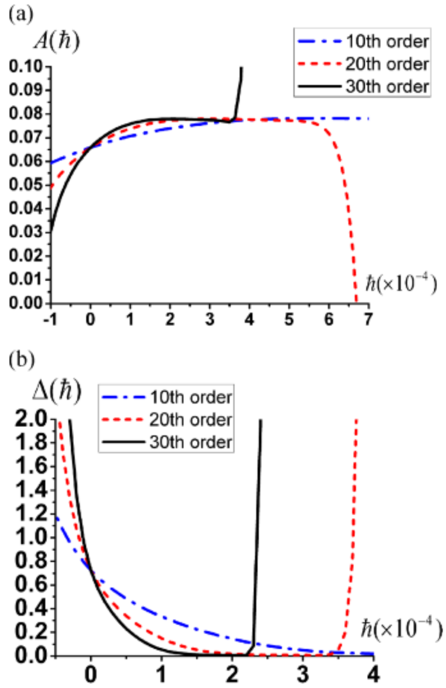

Fig. 2 (colour online). Two criteria to judge convergence ($\bar{\alpha} = 0.00015$, $\bar{\beta} = 15$, $k = 5$, $d = 0$); (a) $A(\hbar)$ of 10th, 20th, 30th-order of approximation, (b) residual integral $\Delta(\hbar)$ of 10th, 20th, 30th-order of approximation.

in Table 1 for two sets of parameters. Moreover, it has been observed that $\hbar$ has a closer relationship with $\overline{\alpha}$ than with $\overline{\beta}$ or as if $\hbar \sim \overline{\alpha}$ satisfies as a rough guess in many cases. Granted, this $\hbar$ curve-method is a straightforward way to show clearly a valid region of the auxiliary parameter for a convergent solution, but it's always a good thing if there's an alternative to convince us. Another approach [20] in 2009 is to substitute the solutions given by HAM into the governing equation (6) and to integrate the residual error

$$\Delta(\hbar) = \frac{1}{b^2}\int_0^{\pi} N^2[z(\xi,\hbar),A(\hbar)]\,\mathrm{d}\xi\,, \qquad (31)$$

which we call residual integral, where b and operator N are defined in (7) and (14), respectively. $\Delta(\hbar)$ for $\bar{\alpha} = 0.00015$, $\bar{\beta} = 15$, $k = 5$, $d = 0$ is plotted in Figure 2b, where $\hbar \in [1\cdot 10^{-4}, 2\cdot 10^{-4}]$ can be regarded as the optimum region to minimize the residual error. In this region, likewise, the residual error almost remains unchanged with the increasing HAM order and is thought to converge. Combining these two criteria into practice, we should choose $\hbar \in [1\cdot 10^{-4}, 2\cdot 10^{-4}]$ to get a more accurate series solution A.

There's some important experience offering us a shortcut to find a convergent solution for series $z(\xi)$. So long as series A in (29) is convergent, series $z(\xi)$ in (28) given by the same auxiliary parameter $\hbar$ should be convergent as well for $\forall \xi \in [0,\pi]$ [6]. These two series are bound to be a solution of the governing equation (6) as proved in [6]. In this case, we need only to find an appropriate auxiliary parameter $\hbar$ to ensure the convergence of A. One can get a solution of the displacement $u(x)$ or the strain $u'(x)$ under the transformations of (4) and (5), once a valid $\hbar$ is found like in Figure 2a and Figure 2b. The first ten terms of the 40th-order approximated inhomogeneous displacement for $\bar{\alpha} = 0.00015, \bar{\beta} = 15, k = 5, d = 0$ while $\hbar = 0.00015$ is given by

$$\begin{aligned} u(x) &= 0.0714491\sin(5x) - 0.00532939\sin(15x) \\ &+ 0.000923094\sin(25x) - 0.000181045\sin(35x) \\ &+ 0.0000362481\sin(45x) - 7.09452\cdot 10^{-6}\sin(55x) \end{aligned}$$

$\bar{\alpha} = 1.5\cdot 10^{-4}$, $\bar{\beta} = 15$, $d = 0$, $k = 5$, $\hbar = 1.5\cdot 10^{-4}$		$\bar{\alpha} = 0.001$, $\bar{\beta} = 50$, $d = 0.1$, $k = 4$, $\hbar = 0.001$	
order	A	order	A
13	0.073807	7	0.013309
14	0.074192	8	0.013311
15	0.074551	9	0.013313
16	0.074886	10	0.013315
17	0.075197	11	0.013316
18	0.075486	12	0.013317
19	0.075754	13	0.013319
20	0.076002	14	0.013320
21	0.076229	15	0.013321
22	0.076439	16	0.013322
23	0.076630	17	0.013322
24	0.076804	18	0.013323
25	0.076963	19	0.013323
26	0.077106	20	0.013324
27	0.077234	21	0.013324
28	0.077350	22	0.013325
29	0.077452	23	0.013325
30	0.077542	24	0.013326
31	0.077621	25	0.013326
32	0.077689	26	0.013326
33	0.077748	27	0.013326
34	0.077797	28	0.013326
35	0.077838	29	0.013327
36	0.077870	30	0.013327
37	0.077896	31	0.013327
38	0.077915	32	0.013327
39	0.077927	33	0.013327
40	0.077935	34	0.013327
41	0.077935	35	0.013327
42	0.077935	36	0.013327

Table 1. Convergence of A for two sets of parameters: $\bar{\alpha} = 0.00015$, $\bar{\beta} = 15$, $d = 0$, $k = 5$, and $\bar{\alpha} = 0.001$, $\bar{\beta} = 50$, $d = 0.1$, $k = 4$.

$$+1.32475\cdot 10^{-6}\sin(65x)-2.33153\cdot 10^{-7}\sin(75x)$$
$$+3.84677\cdot 10^{-8}\sin(85x)-5.93761\cdot 10^{-9}\sin(95x)$$
$$+8.57026\cdot 10^{-10}\sin(105x)\,. \tag{32}$$

3.2. Bifurcation Analysis and Inhomogeneous Strain Solutions

Next, bifurcation analysis will be done. From (14) and (22), one is able to get

$$\hbar R_1(z_0,A_0) = \hbar\left\{\left(k^4-ck^2+a+\frac{1}{4}bA_0^2k^4\right)\sin k\xi+\dots\right\}. \tag{33}$$

To satisfy (23), the coefficient of $\sin k\xi$ in the above formula should vanish. That leads to

$$A_0=\pm\frac{2}{k^2}\sqrt{\frac{1}{b}(ck^2-a-k^4)}\,. \tag{34}$$

$A_0\neq 0$ is the bifurcation condition [6]. With (7), we obtain

$$\frac{1}{2\pi^2}(1-3d^2)>\bar{\alpha}k^2+\frac{\bar{\beta}}{\pi^4}\frac{1}{k^2}\,. \tag{35}$$

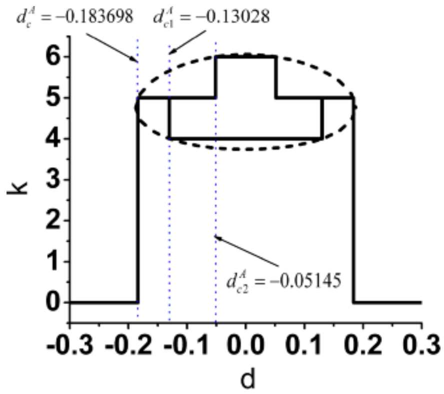

Fig. 3 (colour online). k–d bifurcation diagram of parameter set A: $\bar{\alpha}=0.001$, $\bar{\beta}=50$.

It should be emphasized that the above bifurcation condition is the same as that derived by a local bifurcation analysis in [3]. Thus, in addition to providing explicit series solutions, HAM proves to be a powerful tool for bifurcation analysis in solving phase transition problems as well.

The bifurcation condition (35) reveals a relationship between the overall strain d and the number of interfaces k. In Figure 3, the dashed curve represents the

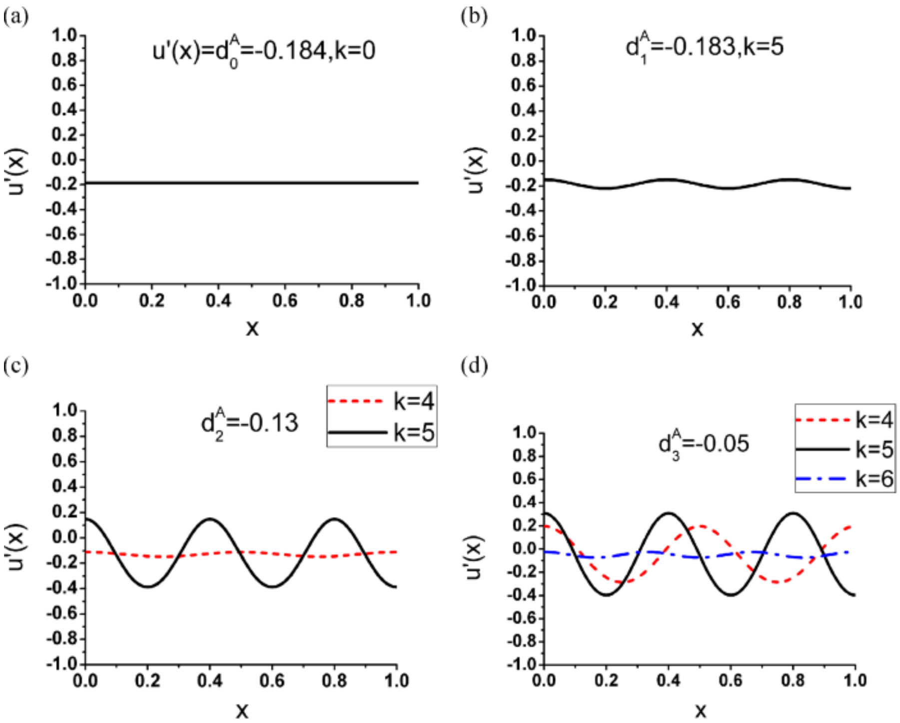

Fig. 4 (colour online). Solution(s) for the strain $u'(x)$ at different values of load d of parameter set A: $\bar{\alpha}=0.001$, $\bar{\beta}=50$. (a) Classical homogeneous solution ($k=0$) at $d_0^{\mathrm{A}}=-0.184<d_c^{\mathrm{A}}=-0.183698$ with no phase transition. (b) Emergence of the only inhomogeneous solution ($k=5$) at $d_1^{\mathrm{A}}=-0.183>d_c^{\mathrm{A}}$, hence the martensitic transformation occurs. (c) Emergence of the double inhomogeneous solutions ($k=4,5$) at $d_2^{\mathrm{A}}=-0.13>d_{c1}^{\mathrm{A}}=-0.13028$. (d) Emergence of the triple inhomogeneous solutions ($k=4,5,6$) at $d_3^{\mathrm{A}}=-0.05>d_{c2}^{\mathrm{A}}=-0.05145$.

critical curve for k satisfying the inequality (35)

$$k_{\pm}(d) = \sqrt{\frac{(1-3d^2) \pm \sqrt{(1-3d^2)^2 - 16\bar{\alpha}\bar{\beta}}}{4\bar{\alpha}\pi^2}}\,. \quad (36)$$

The sawtooth segments represent the exact confine of k and d permitting bifurcation or phase transition. The critical load d_c for phase transition satisfies

$$d_\mathrm{c}^2 = -\frac{2}{3}\pi^2\left(\bar{\alpha}k_\mathrm{c}^2 + \frac{\bar{\beta}}{\pi^4 k_\mathrm{c}^2}\right) + \frac{1}{3}\,, \quad (37)$$

where

$$k_\mathrm{c} = \left[\frac{1}{\pi}\left(\frac{\bar{\beta}}{\bar{\alpha}}\right)^{1/4}\right] \text{ or } \left[\frac{1}{\pi}\left(\frac{\bar{\beta}}{\bar{\alpha}}\right)^{1/4}\right] + 1\,. \quad (38)$$

Here $[x]$ denotes the maximum integer smaller than x.

When d is out of the range $[-|d_\mathrm{c}|, |d_\mathrm{c}|]$, (6) only has a trivial solution, thus (3) has a unique solution $u(x) = u_\mathrm{H}(x) = dx$, the inhomogeneous displacement and the phase transition will not occur. As $d \in [-|d_\mathrm{c}|, |d_\mathrm{c}|]$, (6) will have solutions for A_k and $z_k(\xi)$ with certain $k \in [k_-(d), k_+(d)]$ values. Equation (3) will then have inhomogeneous solutions $u_k(x) = u_\mathrm{H}(x) + LA_k z_k(\pi x/L)$. Such inhomogeneous solutions represent the martensitic transformation in certain alloys and the k value corresponds to the number of phase interfaces. Apparently from (35) for $\forall\overline{\alpha}$, $\overline{\beta}$, $|d_\mathrm{c}| < 1/\sqrt{3}$ is a necessity. To get a full appreciation of this point, one has only to turn to that double-well potential $W(u_x) = E(u_x^2 - 1)^2/4$, in that the second derivative is negative for $u_x^2 < 1/3$ where the energy is unstable hence phase transition may take place. As far as the effects of $\overline{\alpha}$ and $\overline{\beta}$ upon the k–d bifurcation diagram are concerned, one can refer to [3] for details. After all, this bifurcation analysis by means of the HAM is physically reasonable.

Without loss of generality, we firstly choose the parameter set A: $\bar{\alpha} = 0.001$ and $\bar{\beta} = 50$ used in Figure 3 as an example, where $d_\mathrm{c}^\mathrm{A} = -0.183698$. When $d < d_\mathrm{c}^\mathrm{A}$, say $d = d_0^\mathrm{A} = -0.184$ (here only $d < 0$ is considered and $d > 0$ is its symmetric case), only a trivial solution satisfies the governing equation (6) as in Figure 4a. As d increases to a value larger than d_c^A, say $d_1^\mathrm{A} = -0.183$, the martensitic transformation occurs meanwhile $k = 5$ is the only possible interface number for a nontrivial solution plotted in Figure 4b. It's noteworthy that in the tiny neighbourhood of the critical load d_c^A, the HAM results are good. As d increases to, say $d_2^\mathrm{A} = -0.13 > d_\mathrm{c1}^\mathrm{A} = -0.13028$, two interface numbers $k = 4,5$ are possible with the corresponding strain in Figure 4c. As d increases to, say $d_3^\mathrm{A} = -0.05 > d_\mathrm{c2}^\mathrm{A} = -0.05145$, there exist triple inhomogeneous solutions $k = 4,5,6$ for strain in Figure 4d. With the increasing $d (d \le 0)$, there exist more possible nontrivial solutions. The more complicated bifurcation diagram of another example, parameter set B: $\bar{\alpha} = 0.00015$ and $\bar{\beta} = 15$ is shown in Figure 5a and Figure 5b, where $d_\mathrm{c}^\mathrm{B} = -0.519286$. Similar to the example A, any value of d, say $d_0^\mathrm{B} = -0.5193 < d_\mathrm{c}^\mathrm{B}$ will result in the only trivial solution corresponding to the classical homogeneous solution as in Figure 6a. When d takes the value of $d_1^\mathrm{B} = -0.518 < d_\mathrm{c1}^\mathrm{B}$ and $d_2^\mathrm{B} = -0.5178 > d_\mathrm{c1}^\mathrm{B}$, there exist respectively the only inhomogeneous solution ($k = 6$) in Figure 6b and double solutions ($k = 5,6$) in Figure 6c. But when $d = d_3^\mathrm{B} = 0$, the interface number can be any value between 2 and 18 and three of the solutions with very different pattern formations are shown in Figure 6d. In a nutshell, HAM

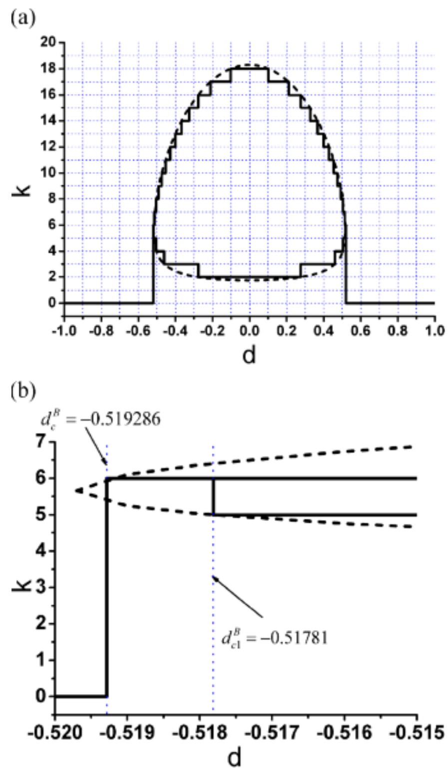

Fig. 5 (colour online). k–d bifurcation diagram of parameter set B: $\bar{\alpha} = 0.00015$, $\bar{\beta} = 15$; (a) whole view, (b) local view near bifurcation.

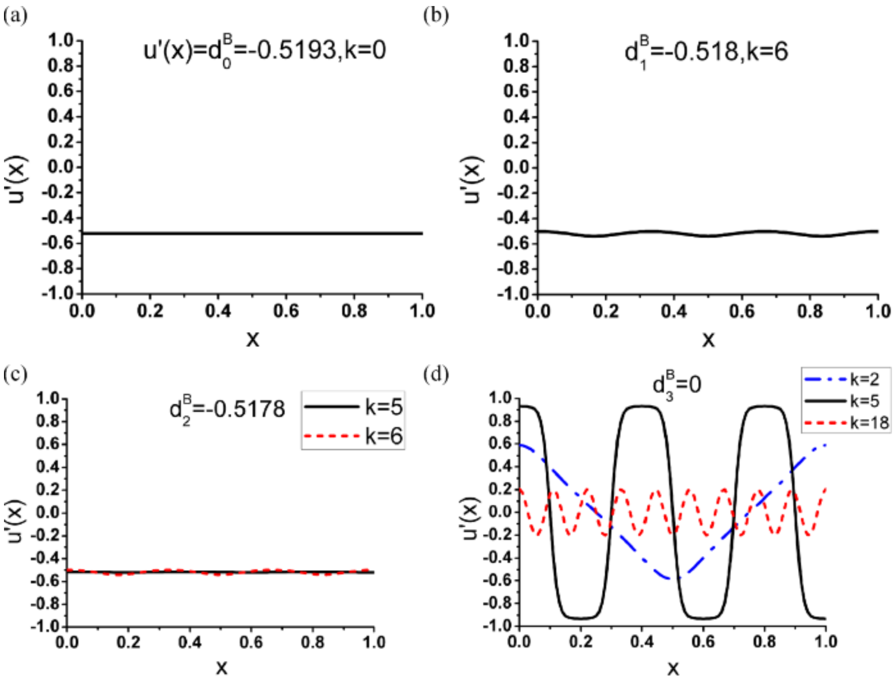

Fig. 6 (colour online). Solution(s) for the strain $u'(x)$ at different values of load d of parameter set B: $\bar{\alpha} = 0.00015$, $\bar{\beta} = 15$. (a) Classical homogeneous solution ($k = 0$) at $d_0^B = -0.5193 < d_c^B = -0.519286$ with no phase transition. (b) Only inhomogeneous solution ($k = 6$) at $d_1^B = -0.518 < d_{c1}^B = -0.51781$. (c) Emergence of the double solutions ($k = 5,6$) at $d_2^B = -0.5178 > d_{c1}^B = -0.51781$. (d) Three of the many solutions ($k = 2 \sim 18$) at $d_3^B = 0$.

is capable of giving convergent analytical solutions for any admissible interface number.

It should be pointed out that, although few cases of parameters are illustrated in this paper due to the limitation of length, our analytic solutions and conclusions are valid for most parameters. With all of these factors considered, HAM is no doubt applicable in the one-dimensional model of the martensitic transformation.

We are able to obtain quite a few series solutions of the Euler–Lagrange equation (3) only when the interface number k is prescribed to be known. One of the possible ways to pick up from all mathematically possible solutions a unique physically real one with a definite k is the assumption of the real physical solution as the global energy minimizer [2, 4]. With this assumption, one can obtain such results as the evolution of strain or pattern formations during the whole process of martensitic transformation. Limited by the capacity of this paper, the results and analysis obtained from the solutions of the Euler–Lagrange equation are reserved for future papers.

4. Conclusions

In this paper, we implement the HAM to solve the Euler–Lagrange equation describing the one-dimensional model of the martensitic transformation. The convergence of the analytical solutions can be ensured by an auxiliary parameter by means of the $\hbar$ curve method and the residual integral method, and the rate of convergence can also be adjusted by it. On the one hand, HAM is able to obtain nonlinear solutions for bifurcation while the perturbation method is only valid for linear or infinitesimal nontrivial solutions; on the other hand, HAM can provide an analytical expression for the bifurcation condition while other numerical techniques have difficulties in bifurcation analysis. Also the displacement and strain field converges fast and is obtainable for all parameters allowed for bifurcation. With all of the above considered, it's reasonable to say that HAM is applicable in the one-dimensional model of the martensitic phase transformation.

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