Solution of the Nonlinear Fractional Diffusion Equation with Absorbent Term and External Force Using Optimal Homotopy-Analysis Method

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In this article, the optimal homotopy-analysis method (HAM) is used to obtain approximate analytic solutions of the time-fractional nonlinear diffusion equation in the presence of an external force and an absorbent term. The fractional derivatives are considered in the Caputo sense to avoid nonzero derivative of constants. Unlike usual HAM this method contains at the most three convergence control parameters which determine the fast convergence of the solution through different choices of convergence control parameters. Effects of proper choice of parameters on the convergence of the approximate series solution by minimizing the averaged residual error for different particular cases are depicted through tables and graphs.

Key words: Fractional Diffusion Equation; Nonlinearity; Optimal Homotopy-Analysis Method; Fractional Brownian Motion; Absorbent Term; Error Analysis.
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1. Introduction

Nonlinear diffusion equations which are an important class of parabolic equations arise from a variety of diffusion phenomena appearing in the modelling of various physical problems. The fractional diffusion equation is obtained from the classical diffusion equation in mathematical physics by replacing the first-order time derivative by a fractional derivative of order α where $0 < \alpha < 1$; of late this being a field of growing interest as evident from literature survey. Thus appearances of fractional-order derivatives make the study more involved and challenging. An important phenomenon of these evolution equations is that it generates the fractional Brownian motion (FBM) which is a generalization of the Brownian motion (BM). The fractional differential equations have gained much attention recently due to the fact that the fractional-order system response ultimately converges to the integer-order system response. Various definitions of fractional calculus are available in many books [1 – 3].

Recently, the nonlinear fractional diffusion equations have gained a lot of attention. In fact, they have been applied in several situations such as percolation of gases through porous media [4], thin saturated regions in porous media [5], modelling of non-Markovian dynamical processes in protein folding [6], and anomalous transport in disordered systems [7].

The general equation of the one-dimensional nonlinear diffusion equation with fractional time derivative is

$$\begin{aligned} \frac{\partial^\alpha u(x,t)}{\partial t^\alpha} &= \frac{\partial}{\partial x}\left((u(x,t))^n \frac{\partial u(x,t)}{\partial x}\right) \\ &\quad - \frac{\partial}{\partial x}\big(F(x)u(x,t)\big) + a(t)u(x,t), \\ &\quad 0 < \alpha \le 1,\ t > 0,\ x > 0, \end{aligned} \tag{1}$$

where $F(x)$ is an external force, $a(t) = a\frac{t^\beta}{\Gamma(\beta+1)}$, $0 < \beta < 1$, physically represents a source term if $a > 0$ and an absorbent term if $a < 0$, which may be related to a reaction diffusion process, and $\Gamma(.)$ is the well-known gamma function.

Most of the nonlinear problems do not have a precise analytical solution; especially it is hard to obtain one for the fractional nonlinear equations. So these types of equations should be solved by any approximate methods or numerical methods. Schot et al. [8] have given an approximate solution of (1) for the linear case (i.e., for $n = 0$) in terms of a Fox H-function. Zahran [9] has offered a closed form solution in terms of a Fox H-function of the generalized linear fractional reaction-diffusion equation subjected to an external linear force field, one that is used to describe the transport processes in disorder systems. Das and Gupta [10] have solved the similar type of linear fractional diffusion equation by the homotopy perturbation method (HPM). Recently, Das et al. [11] have solved the approximate analytical solution of the general nonlinear diffusion equation with fractional time derivative in the presence of different types of absorbent terms and a linear external force using HPM. In another recent article of Yao [12], it is seen that the fractal geometry theory is combined with seepage flow mechanism to establish the nonlinear diffusion equation of fluid flow in fractal reservoir. It is to be noted that some works on fractional diffusion equations have already been done by Li et al. [13], Ganji and Sadighi [14] etc., using various mathematical techniques. But to the best of authors' knowledge the convergence of the solution of the considered nonlinear fractional problem by the minimization of residual error has not yet been studied by any researcher.

The homotopy analysis method (HAM) proposed by Liao [15] is a mathematical tool to get the series solution of linear and nonlinear partial differential equations (PDEs). The difference to the other perturbation methods is that this method is independent of small/large physical parameters. Another important advantage as compared to the other existing perturbation and non-perturbation methods lies in the flexibility to choose proper base functions to get better approximate solutions of the problems. It also provides a simple way to ensure the convergence of a series solution. Recently, Liao [16] has claimed that the difference to the other analytical methods is that one can ensure the convergence of series solution by means of choosing a proper value of the convergence control parameter. Ganjiani [17] recently has applied the HAM to solve a set of nonlinear fractional differential equations and compared the results with the exact one. But there are a lot of restrictions of the method, e.g., in usual HAM one cannot predict which value of convergence control parameter c_0 gives better convergence even through the plotting of c_0-curves. To overcome these restrictions, the authors have used as new mathematical tool the optimal homotopy analysis method, also proposed by Liao [18] to find the approximate analytical solution of our considered problem where the rate of convergence of the series solution is faster. Recently, this method has been successfully applied by Wang [19]. The basic difference of the method from usual HAM is that here we have to consider at the most three parameters c_0, c_1, c_2 ($|c_1| \leq 1$, $|c_2| \leq 1$), which are known as convergence control parameters whereas in usual HAM there was only one parameter c_0. The present approach contains special deformation functions which are determined by two parameters c_1 and c_2. The salient feature of our article is the introduction of a new type of residual error which helps to find out the optimal values of these parameters for getting better convergence of the solution.

2. Basic Ideas of Fractional Calculus

In this section, we give some definitions and properties of the fractional calculus which is used further in this paper.

Definition 1. A real function $f(t)$, $t > 0$, is said to be in the space C_μ, $\mu \in \mathbb{R}$, if there exists a real number $p > \mu$, such that $f(t) = t^p f_1(t)$, where $f_1(t) \in C[0,\infty)$, and is said to be in the space C_μ^n if and only if $f^{(n)} \in C_\mu$, $n \in \mathbb{N}$.

Definition 2. The Riemann–Liouville fractional integral operator J_t^α of order $\alpha > 0$ of a function $f \in C_\mu$, $\mu \geq -1$, is defined as

$$J_t^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\xi)^{\alpha-1} f(\xi)\,\mathrm{d}\xi, \ \alpha > 0,\ t > 0,$$
$$J_t^0 f(t) = f(t). \tag{2}$$

Definition 3. The fractional derivative D_t^α of $f(t)$ in the Caputo sense is defined as

$$D_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\xi)^{n-\alpha-1} f^{(n)}(\xi)\,\mathrm{d}\xi, \tag{3}$$

for $n-1 < \alpha < n$, $n \in \mathbb{N}$, $t > 0$, $f \in C_{-1}^n$.

The followings are two basic properties of the Caputo fractional derivative:

(i) Let $f \in C_{-1}^n$, $n \in \mathbb{N}$, then $D_t^\alpha f$, $0 \leq \alpha \leq n$, is well defined and $D_t^\alpha f \in C_{-1}$.

(ii) Let $n-1 \leq \alpha \leq n$, $n \in \mathbb{N}$, and $f \in C_\mu^n$, $\mu \geq -1$, then

(a) $D_t^\alpha(c) = 0$, where c is a constant;
(b) $D_t^\alpha J_t^\alpha f(t) = f(t)$;
(c) $J_t^\alpha D_t^\alpha f(t) = f(t) - \sum_{k=0}^{n-1} f^{(k)}(0^+) \frac{t^k}{k!}$.

3. Solution of the Problem by Optimal Homotopy Analysis Method

The present article is concerned with solutions of the following one-dimensional nonlinear fractional diffusion equation with external force and absorbent term:

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} - \frac{\partial}{\partial x}\left(u(x,t)\frac{\partial u(x,t)}{\partial x}\right) - \frac{\partial}{\partial x}(xu(x,t)) - a(t)u(x,t) = 0,\ 0<\alpha\le 1,\ 0<t<1,\ 0<x<1, \tag{4}$$

subject to the initial condition

$$u(x,0) = x. \tag{5}$$

To solve (4) by the optimal homotopy analysis method, we choose the initial approximation

$$u_0(x,t) = x \tag{6}$$

and the linear auxiliary operator

$$L[\phi(x,t;p)] = \frac{\partial^\alpha \phi(x,t;p)}{\partial t^\alpha} \tag{7}$$

with the property

$$L[c] = 0,$$

where c is an integral constant and $\phi(x,t;p)$ an unknown function. Furthermore, in the view of (4), we have defined the nonlinear operator as

$$\begin{aligned} N[\phi(x,t;p)] = {} & \frac{\partial^\alpha \phi(x,t;p)}{\partial t^\alpha} \\ & - \frac{\partial}{\partial x}\left(\phi(x,t;p)\frac{\partial \phi(x,t;p)}{\partial x}\right) \\ & - \frac{\partial}{\partial x}(x\phi(x,t;p)) - a(t)\phi(x,t;p). \end{aligned} \tag{8}$$

By means of the optimal homotopy analysis method, Liao [18] constructs the so-called zeroth-order deformation equation as

$$\begin{aligned} &(1-B(p))L[\phi(x,t;p)-u_0(x,t)] \\ &= c_0 A(p) N[\phi(x,t;p)], \end{aligned} \tag{9}$$

where $p\in[0,1]$ is the embedding parameter, c_0 is convergence control parameter, $A(p)$ and $B(p)$ are so-called deformation functions satisfying

$$A(0) = B(0) = 0 \quad\text{and}\quad A(1) = B(1) = 1. \tag{10}$$

The Maclaurin series of these functions are given by

$$A(p) = \sum_{m=1}^{\infty} \mu_m p^m, \tag{11}$$

$$B(p) = \sum_{m=1}^{\infty} \sigma_m p^m, \tag{12}$$

which exist and convergent for $|p|\le 1$. As given by Liao [18], there exist a large number of deformation functions satisfying these properties, but for the sake of computer efficiency, we use here so called one parameter deformation functions which are given as

$$A(p;c_1) = \sum_{m=1}^{\infty} \mu_m(c_1) p^m, \tag{13}$$

$$B(p;c_2) = \sum_{m=1}^{\infty} \sigma_m(c_2) p^m, \tag{14}$$

where $|c_1|\le 1$ and $|c_2|\le 1$ are constants, called the convergence control parameters. One can define μ_m and σ_m as

$$\begin{aligned} &\mu_1(c_1) = (1-c_1), \\ &\mu_m(c_1) = (1-c_1)c_1^{m-1},\ m>1, \end{aligned} \tag{15}$$

$$\begin{aligned} &\sigma_1(c_2) = (1-c_2), \\ &\sigma_m(c_2) = (1-c_2)c_2^{m-1},\ m>1. \end{aligned} \tag{16}$$

Thus the zeroth-order deformation (9) becomes

$$\begin{aligned} &(1-B(p;c_2))L[\phi(x,t;p)-u_0(x,t)] \\ &= c_0 A(p;c_1) N[\phi(x,t;p)]. \end{aligned} \tag{17}$$

It is obvious that for the embedding parameter $p=0$ and $p=1$, (17) becomes

$$\begin{aligned} &\phi(x,t;0) = u_0(x,t), \\ &\phi(x,t;1) = u(x,t), \end{aligned}$$

respectively. Thus, as p increases from 0 to 1, the solution $\phi(x,t;p)$ varies from the initial guess $u_0(x,t)$ to the solution $u(x,t)$. Expanding $\phi(x,t;p)$ in Maclaurin's series with respect to p, one has

$$\phi(x,t;p) = u_0(x,t) + \sum_{k=1}^{\infty} p^k u_k(x,t), \tag{18}$$

where

$$u_k(x,t) = \frac{1}{k!}\frac{\partial^k \phi(x,t;p)}{\partial p^k}\Big|_{p=0}. \tag{19}$$

If the auxiliary linear operator, the initial guess, and the convergence control parameters are properly chosen, the series (18) converges at $p = 1$. In this case one has

$$\phi(x,t;1) = u_0(x,t) + \sum_{k=1}^{\infty} u_k(x,t), \tag{20}$$

which must be one of the solutions of the original equation, as proven by Liao [18].

Let G be a function of p $(0 \le p \le 1)$. Define the so-called mth-order homotopy derivative as

$$D_m[G] = \frac{1}{m!}\frac{\partial^m G}{\partial p^m}\bigg|_{p=0}. \tag{21}$$

Taking the above operation on both sides of zeroth-order (17), we have the so-called mth-order deformation equations

$$\begin{aligned} &L\left[u_m(x,t) - \chi_m \sum_{k=1}^{m-1} \sigma_{m-k}(c_2) u_m(x,t)\right] \\ &= c_0 \sum_{k=0}^{m-1} \mu_{m-k}(c_1) R_k(x,t) \end{aligned} \tag{22}$$

with the initial condition

$$u_m(x,0) = 0, \tag{23}$$

where

$$\chi_m = \begin{cases} 0, & m \le 1, \\ 1, & m > 1, \end{cases}$$

and

$$\begin{aligned} R_k(x,t) = D_k N[\phi(x,t;p)] = \frac{\partial^\alpha u_k(x,t)}{\partial t^\alpha} \\ - \sum_{i=0}^{k} u_i'(x,t) u_{k-i}'(x,t) - \sum_{i=0}^{k} u_i(x,t) u_{k-i}''(x,t) \\ - \frac{\partial}{\partial x}(x u_k(x,t)) - a(t) u_k(x,t), \end{aligned}$$

where

$$u'(x,t) = \frac{\partial u(x,t)}{\partial x} \quad \text{and} \quad u''(x,t) = \frac{\partial^2 u(x,t)}{\partial x^2}.$$

Applying the idea of optimal homotopy analysis method, we have

$$\begin{aligned} u_m(x,t) = &\chi_m \sum_{k=1}^{m-1} \sigma_{m-k}(c_2) u_m(x,t) \\ &+ c_0 \sum_{k=0}^{m-1} \mu_{m-k}(c_1) J_t^\alpha R_k(x,t) + c, \end{aligned} \tag{24}$$

where the integration constant c is determined by the initial condition (23).

It is clear from (24) that $u_m(x,t)$ contains at most three unknown convergence control parameters c_0, c_1, and c_2, which determine the convergence region and rate of the homotopy-series solution.

As given in [18], at the mth-order of approximation, one can define the exact square residual error as

$$\Delta_m = \int_0^1 \int_0^1 \left(N\left[\sum_{i=0}^{m} u_i(x,t)\right]\right)^2 \mathrm{d}x\,\mathrm{d}t. \tag{25}$$

However, it is proven by Liao [18] that the exact square residual error Δ_m defined by (25) needs too much CPU time to calculate even if the order of approximation is not very high, and authors have also seen this during the numerical computations.

Thus to overcome this difficulty, i.e. to decrease the CPU time, we use here the so-called averaged residual error defined by

$$E_m = \frac{1}{5} \sum_{j=1}^{5} \sum_{k=1}^{5} \left(N\left[\sum_{i=0}^{m} u_i\left(\frac{j}{6}, \frac{k}{6}\right)\right]\right)^2. \tag{26}$$

4. Numerical Results and Discussion

In this section, we will discuss the optimal homotopy analysis method with at the most three different convergence control parameters c_0, c_1, c_2 and see how the solution rapidly converges by means of minimizing the so-called averaged residual error E_m defined by (26), corresponding to the nonlinear algebraic equation $E_m' = 0$.

4.1. In the Absence of Absorbent Term (i.e., $a(t) = 0$)

4.1.1. Case I: Optimal c_0 for the Case of $c_1 = c_2 = 0$

In this case, we have only one convergence control parameter c_0. Figures 1–4 are plotted for averaged residual error E_m vs. c_0 for $\alpha = 1$, 0.95, 0.9, and 0.8. Tables 1–4 show the comparison of the averaged residual error for the optimal value of c_0 with the increase in the order of approximation. It is clear from the tables that optimal values of c_0 are -1.20, -1.17, -1.30, -1.30 for $\alpha = 1$, 0.95, 0.9, and 0.8, respectively, for the 9$^{\text{th}}$ order of approximation. According to Tables 1–4, the value of the averaged residual error converges much faster to 0 than the corresponding

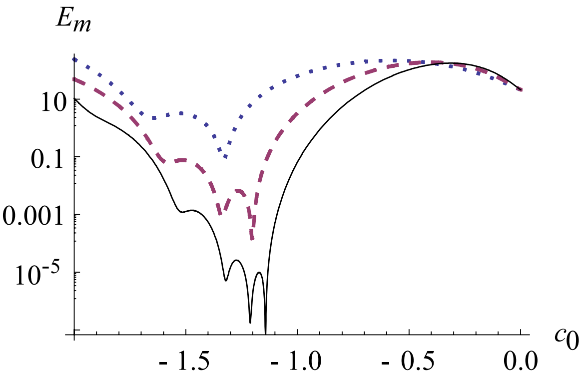

Fig. 1 (colour online). Plot of average residual error E_m versus c_0 for $\alpha = 1$ in the absence of an absorbent term. Dotted line: 5th order approximation; dashed line: 7th order approximation; black line: 9th order approximation.

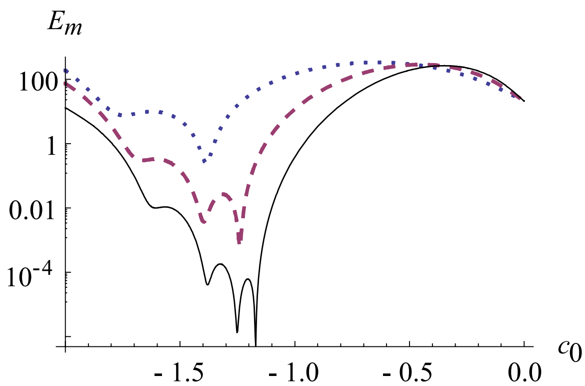

Fig. 2 (colour online). Plot of average residual error E_m versus c_0 for $\alpha = 0.95$ in the absence of an absorbent term. Dotted line: 5th order approximation; dashed line: 7th order approximation; black line: 9th order approximation.

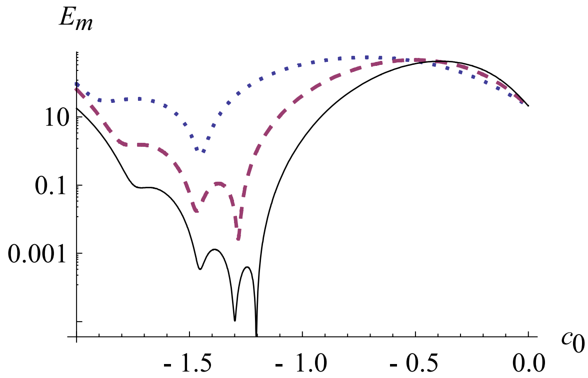

Fig. 3 (colour online). Plot of average residual error E_m versus c_0 for $\alpha = 0.9$ in the absence of an absorbent term. Dotted line: 5th order approximation; dashed line: 7th order approximation; black line: 9th order approximation.

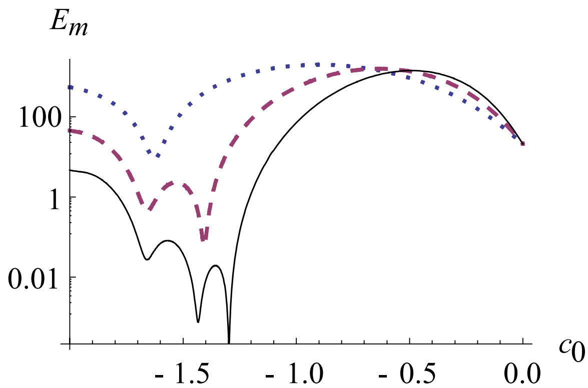

Fig. 4 (colour online). Plot of average residual error E_m versus c_0 for $\alpha = 0.8$ in the absence of an absorbent term. Dotted line: 5th order approximation; dashed line: 7th order approximation; black line: 9th order approximation.

Table 1. Comparison of averaged residual error for different values of c_0 at $\alpha = 1$.

Order of approximation	c_0	E_m	E_m at $c_0 = -1$
3	0.23	12.6526	324.027
5	-1.33	$8.45904 \cdot 10^{-2}$	64.0321
7	-1.20	$1.92603 \cdot 10^{-4}$	3.29301
9	-1.20	$1.89865 \cdot 10^{-6}$	$6.47269 \cdot 10^{-2}$

Table 2. Comparison of averaged residual error for different values of c_0 at $\alpha = 0.95$.

Order of approximation	c_0	E_m	E_m at $c_0 = -1$
3	0.25	11.3819	508.517
5	-1.38	$2.66604 \cdot 10^{-1}$	149.635
7	-1.40	$3.80096 \cdot 10^{-3}$	11.9268
9	-1.17	$6.01029 \cdot 10^{-7}$	$3.82409 \cdot 10^{-1}$

homotopy-series solution given by the usual HAM that is the case $c_0 = -1$, $c_1 = c_2 = 0$. It is also observed from Tables 1–4 that as the value of α decreases the optimal value of c_0 goes away from $c_0 = -1$, the case of usual HAM. Thus, even the one-parameter optimal HAM can give much better approximation.

4.1.2. Case II: Optimal c_0 in Case of $c_1 = c_2 \neq 0$

In this case, we have at most two convergence control parameters, viz. c_0 and c_1. It is seen through Table 5 that the optimal value of c_0 for the case of $c_1 = c_2 = -0.1$ is -1.1. Table 5 also shows that the

Table 3. Comparison of averaged residual error for different values of c_0 at $\alpha = 0.9$.

Order of approximation	c_0	E_m	E_m at $c_0 = -1$
3	0.26	10.1006	794.981
5	-1.45	$7.60891 \cdot 10^{-1}$	348.855
7	-1.28	$4.20015 \cdot 10^{-3}$	42.8465
9	-1.30	$1.05494 \cdot 10^{-5}$	2.21703

Table 4. Comparison of averaged residual error for different values of c_0 at $\alpha = 0.8$.

Order of approximation	c_0	E_m	E_m at $c_0 = -1$
3	0.31	7.59165	1914.86
5	-1.63	9.00506	1886.9
7	-1.41	$6.51259 \cdot 10^{-2}$	544.584
9	-1.30	$3.76640 \cdot 10^{-4}$	71.1987

Table 5. Comparison of averaged residual error E_m at $\alpha = 1$.

Order of approximation	$c_1 = 0$, $c_2 = 0$, $c_0 = -1$	$c_1 = 0$, $c_2 = 0$, $c_0 = -1.2$	$c_1 = -0.2$, $c_2 = -0.2$, $c_0 = -1$	$c_1 = -0.15$, $c_2 = -0.15$, $c_0 = -1.15$	$c_1 = -0.1$, $c_2 = -0.1$, $c_0 = -1.1$
3	324.027	299.699	299.699	248.228	296.478
5	64.0321	7.63927	7.63927	0.10594	6.376
7	3.29301	$1.92603 \cdot 10^{-4}$	$1.92603 \cdot 10^{-4}$	$1.54859 \cdot 10^{-3}$	$4.81571 \cdot 10^{-4}$
9	$6.47269 \cdot 10^{-2}$	$1.89865 \cdot 10^{-6}$	$1.89865 \cdot 10^{-6}$	$5.2041 \cdot 10^{-6}$	$1.99781 \cdot 10^{-7}$

two-parameter optimal HAM gives a slightly better approximation than the one-parameter optimal HAM.

4.1.3. Case III: Optimal $c_1 = c_2$ in Case of $c_0 = -1$

Here, we have only one convergence control parameter c_1. Table 5 shows that the optimal value of c_1 is -0.2. Table 5 demonstrates an interesting thing, that is to say the values of the averaged residual error E_m is same for the two cases $c_1 = c_2 = 0$, $c_0 = -1.2$ and $c_1 = c_2 = -0.2$, $c_0 = -1$ at every order of approximation, which proves that there is flexibility to choose any set of parameters for a better approximation of the solution.

4.2. In the Presence of Absorbent Term (i.e., $a(t) = \frac{-t^\beta}{\Gamma(\beta+1)}$, $\beta = 0.5$)

4.2.1. Case I: Optimal c_0 for the Case of $c_1 = c_2 = 0$

In this case, we have only one convergence control parameter c_0. Figure 5 is plotted for the averaged residual error E_m vs. c_0 for $\alpha = 1$. Table 6 shows the comparison of the results of the averaged residual error for proper choices of c_0 with the increase in the order of approximation. It is clear from the table that the optimal value of c_0 is -1.1 for $\alpha = 1$. According to Table 6, the value of averaged residual error converges much faster to 0 than the corresponding homotopy-series solution given by the usual HAM, that is the case $c_0 = -1$ and $c_1 = c_2 = 0$. It is seen that for this case also the one-parameter optimal HAM can give a much better approximation.

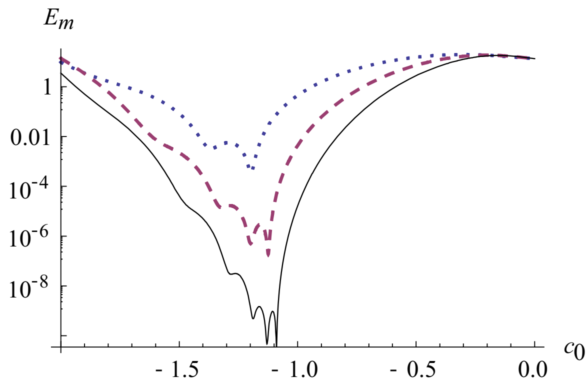

Fig. 5 (colour online). Plot of average residual error E_m versus c_0 for $\alpha = 1$ in the presence of an absorbent term. Dotted line: 5th order approximation; dashed line: 7th order approximation; black line: 9th order approximation.

Table 6. Comparison of averaged residual error E_m at $\alpha = 1$.

Order of approximation	c_0	E_m	E_m at $c_0 = -1$
3	-1.41	$3.64727 \cdot 10^{-1}$	12.5499
5	-1.20	$3.80267 \cdot 10^{-4}$	$4.57172 \cdot 10^{-1}$
7	-1.20	$5.12467 \cdot 10^{-7}$	$4.48427 \cdot 10^{-3}$
9	-1.1	$7.66567 \cdot 10^{-10}$	$1.67189 \cdot 10^{-5}$

4.2.2. Case II: Optimal c_0 in Case of $c_1 = c_2 \neq 0$

In this case, we have at most two convergence control parameters, viz. c_0 and c_1. It is seen through Table 7 that the optimal value of c_0 in the case of $c_1 = c_2 = -0.05$ is -1.08. Table 7 also shows that the two-parameter optimal HAM gives a slightly better approximation than the one-parameter optimal HAM.

Order of approximation	$c_1 = 0$ $c_2 = 0$ $c_0 = -1$	$c_1 = 0$ $c_2 = 0$ $c_0 = -1.1$	$c_1 = -0.1$ $c_2 = -0.1$ $c_0 = -1$	$c_1 = -0.05$ $c_2 = -0.05$ $c_0 = -1$
3	12.5499	8.35172	8.35172	6.97535
5	$4.57172 \cdot 10^{-1}$	$6.05288 \cdot 10^{-2}$	$6.05288 \cdot 10^{-2}$	$1.98207 \cdot 10^{-2}$
7	$4.48427 \cdot 10^{-3}$	$1.80426 \cdot 10^{-5}$	$1.80426 \cdot 10^{-5}$	$9.19528 \cdot 10^{-7}$
9	$1.67189 \cdot 10^{-5}$	$7.66567 \cdot 10^{-10}$	$7.66567 \cdot 10^{-10}$	$1.14947 \cdot 10^{-10}$

Table 7. Comparison of averaged residual error E_m at $\alpha = 1$.

4.2.3. Case III: Optimal $c_1 = c_2$ in Case of $c_0 = -1$

Here, we have only one convergence control parameter c_1. Table 7 shows that the optimal value of c_1 is -0.1. Table 7 clearly demonstrates that the values of the averaged residual error E_m are same for the two cases $c_1 = c_2 = 0$, $c_0 = -1.1$ and $c_1 = c_2 = -0.1$, $c_0 = -1$, which proves the flexibility of choosing any set of parameters for better approximation of the solution.

5. Conclusion

In this article, we have employed the optimal homotopy analysis method to find the solution of the nonlinear diffusion equation with time fractional derivative in the presence/absence of an absorbent term. From the numerical computation given in the tables, it is clear that the optimal homotopy analysis method gives a better approximation than the usual HAM. By minimizing the averaged residual error, we can get the optimal value of the convergence control parameters which gives rise to a rapidly convergent series. This exercise makes the procedure appropriate for solving fractional diffusion equations in different dimensions. From Section 4 it is obvious that in the presence of an absorbent term the residual error more rapidly tends to zero than in the absence of an absorbent term, which physically supports the diffusion process. Thus we may conclude that the study of finding the 'best' deformation function among all the existing ones for getting faster convergent series solution has been very useful.

Applying the method successfully in solving the nonlinear diffusion equation with fractional time derivative, we may also conclude that the present method is very effective and efficient for solving this class of nonlinear fractional PDEs.

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