Exact Chirped Soliton Solutions for the One-Dimensional Gross–Pitaevskii Equation with Time-Dependent Parameters

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The Gross–Pitaevskii equation (GPE) describing the dynamics of a Bose–Einstein condensate at absolute zero temperature, is a generalized form of the nonlinear Schrödinger equation. In this work, the exact bright one-soliton solution of the one-dimensional GPE with time-dependent parameters is directly obtained by using the well-known Hirota method under the same conditions as in S. Rajendran et al., Physica D **239**, 366 (2010). In addition, the two-soliton solution is also constructed effectively.

Key words: Hirota Method; Gross–Pitaevskii Equation; Chirped Soliton Solution.
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1. Introduction

The nonlinear Schrödinger equation (NLSE) has been triggered immense interest in the modelling of many physical phenomena [1] such as propagation of laser beams in nonlinear media [2, 3], plasma dynamics [4], mean field dynamics of Bose–Einstein condensates [5 – 16], condensed matter [17], etc. The Gross–Pitaevskii equation (GPE), describing the dynamics of a Bose–Einstein condensate (BEC) at absolute zero temperature [18], is a generalized form of the NLSE [19 – 21]. On the other hand, the NLSE is a completely integrable soliton system [19] whereas the GPE is not integrable in general, but can admit exact solutions only in very special cases [22, 23].The GPE can be reduced to the effective one-dimensionl (1D) GPE by assuming the kinetic energy of the longitudinal excitations and the two-body interaction energy of the atoms.

The 1D GPE is given as [24]

$$\mathrm{i}\Phi_t = -\frac{1}{2}\Phi_{xx} - \tilde{R}(t)|\Phi|^2\Phi + \frac{\Omega^2(t)}{2}x^2\Phi + \mathrm{i}\frac{\gamma(t)}{2}\Phi, \tag{1}$$

which arises as a model for the dynamics of Bose–Einstein condensates in the mean field approximation, where the nonlinear coefficient $\tilde{R}(t)$ is physically controlled by acting on the so-called Feschbach resonances, $\Omega(t)$ stands for the strength of the quadratic potential as a function of time, and $\gamma(t)$ is the gain/loss term which is phenomenologically incorporated to account for the interaction of the atomic or the thermal cloud. Equation (1) can be transformed into the standard NLSE by means of the similar transformation [24]. Under the conditions $\Omega(t) = \gamma(t) = \beta = \text{const}$, the N-solitary solution of (1) has been obtained by Darboux transformation [25]. Modulation instability and solitons on a continuous-wave background in inhomogeneous optical fiber media have been discussed in [15]. Additionally, when $\Omega^2(t) = 2\mathrm{sech}^2(t) - 1$, $\tilde{R}(t) = \mathrm{sech}(t)$, $\gamma(t) = \tanh(t)$ or $\Omega^2(t) = 2\mathrm{sech}^2(t+t_0)$, $\tilde{R}(t) = \mathrm{sech}(t)$, $\gamma(t) = \tanh(t)$, the N-solitary solution of (1) has been also obtained by Darboux transformation [26].

It is well known that the Hirota bilinear method is a established method for obtaining multi-soliton expressions in nonlinear evolution equations [27, 28]. But the bilinear form guarantees only the existence of two-soliton solutions. The purpose of this work is to develop the usage of this method to the GPE with the aim of producing one- and new two-soliton solutions, see (22) and (36) below. The new two-soliton solutions are obtained by a novel factorization procedure, which will reduce the algebraic manipulations involved in the intermediate calculations considerably.

The structure of the paper can be explained as follows. In terms of the reasonable assumption, the GPE can be decoupled into two equations, and then the one-soliton solution of (1) is presented in Section 2. A comparison of our results with those from [24] is also made. The new two-soliton solution is analytically constructed in Section 3. Conclusions are drawn in Section 4.

2. Chirped One-Soliton Solution

First, the main idea of the Hirota bilinear method is introduced briefly. Applying the transformation

$$\Phi(x,t) = \exp\left[\int \frac{\gamma(t)}{2}\,\mathrm{d}t\right]\phi(x,t), \tag{2}$$

(1) can be rewritten in terms of the new variable $\phi(x,t)$ as

$$\mathrm{i}\phi_t + \frac{1}{2}\phi_{xx} - \frac{1}{2}x^2\Omega^2(t)\phi + 1R(t)|\phi|^2\phi = 0, \tag{3}$$

where $R(t)$ is a time-dependent parameter and can be expressed as

$$R(t) = \exp\left[\int \gamma(t)\,\mathrm{d}t\right]\tilde{R}(t). \tag{4}$$

Now, we consider the following transformation:

$$\phi(x,t) = \frac{G(x,t)}{F(x,t)}, \tag{5}$$

where $G(x,t)$ is a complex function, and $F(x,t)$ is a real function. With this transformation, (3) is decoupled as bilinear form

$$\left(\mathrm{i}D_t + \frac{1}{2}D_x^2 - \frac{x^2}{2}\Omega^2\right)G\cdot F = 0, \tag{6}$$

$$\frac{1}{2}D_x^2 F\cdot F = RG\overline{G}, \tag{7}$$

where $\overline{G}$ is the conjugate function of G, and the D-operator is defined by

$$D_x^m D_t^n F(x,t)\cdot G(x,t) = \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)^m \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\right)^n \cdot \left[F(x,t)G(x',t')\right]\big|_{x'=x,t'=t}$$

in which the spatial and time-dependent terms $x^2\Omega^2(t)$ and $R(t)$ will give a difficulty for getting solutions as shown below.

In order to obtain the bright chirped one-soliton solution of (1), we proceed in the standard assumption

$$F(x,t) = 1 + \varepsilon^2 F_1(x,t),\quad G(x,t) = \varepsilon G_1(x,t), \tag{8}$$

where ε is an arbitrary parameter which will be absorbed in expressing the soliton solution in the following sections. Substituting (8) into (6)–(7), then collecting the coefficients with same power in ε yields

$$\varepsilon: \mathrm{i}G_{1t} + \frac{1}{2}G_{1xx} - \frac{1}{2}x^2\Omega^2(t)G_1 = 0, \tag{9}$$

$$\varepsilon^2: F_{1xx} - RG_1\overline{G_1} = 0, \tag{10}$$

$$\varepsilon^3: \left(\mathrm{i}D_t + \frac{1}{2}D_x^2 - \frac{1}{2}x^2\Omega^2(t)\right)G_1\cdot F_1 = 0, \tag{11}$$

$$\varepsilon^4: D_x^2 F_1\cdot F_1 = 0. \tag{12}$$

Firstly, we can assume $G_1(x,t)$ has the form

$$G_1 = \mathrm{e}^{\eta},\quad \eta = \mathrm{i}p(t)x^2 + q(t)x + \omega(t), \tag{13}$$

where $p(t), q(t), \omega(t)$ are the time-dependent functions to be determined, and $p(t)$ is a chirp function. Substituting (13) into (9), a system of algebraic or first-order ordinary differential equations for the parameter functions is achieved:

$$\begin{aligned} &2p_t + 4p^2 + \Omega^2 = 0,\quad q_t + 2pq = 0,\\ &\omega_t = \frac{\mathrm{i}}{2}q^2 - p. \end{aligned} \tag{14}$$

From (10), we get $F_1(x,t)$ in the following form:

$$F_1(x,t) = c(t)\mathrm{e}^{\eta+\overline{\eta}}, \tag{15}$$

where $c(t)$ is a time-dependent function. Inserting this form into (10)–(13), the relations between these parameters are given by

$$R = c(q+\overline{q})^2,\quad c_t - 2pc = 0. \tag{16}$$

Solving above equations shows that

$$q = \frac{q_0 R}{R_0},\quad c = \frac{c_0 R_0}{R}, \tag{17}$$

$$\omega = \omega_0 + \int_0^t \left(\frac{\mathrm{i}}{2}q^2 - p\right)\mathrm{d}z, \tag{18}$$

$$R_0 = c_0(q_0 + \overline{q_0})^2, \tag{19}$$

$$p(t) = -\frac{R_t}{2R}. \tag{20}$$

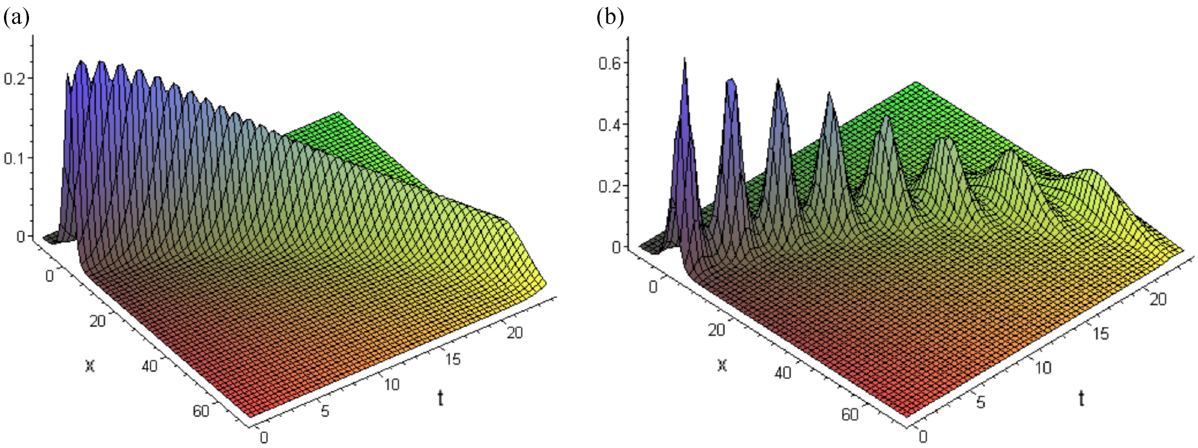

Fig. 1 (colour online). Temporal evolution of one-soliton solution (22) for variable coefficient 1D GPE, (a) with the parameters $\gamma(t)=0$, $R_0=1$, $q_0=0.5+\mathrm{i}$, $\omega_0=0$, $c_0=1$, $R(t)=\mathrm{sech}(0.1t)$. (b) under conditions identical to those of (a) except that $\gamma(t)=2\cos(2t)$.

Here the subscript 0 denotes the value of the given function at $z=0$. Equation (20) is thought to be a condition constraint of the initial value for the parameter functions. $p(t)$ is a normalized chirp function; it is related to the nonlinear coefficient $R(t)$ and the gain/loss coefficient $\gamma(t)$. It is noted that $R(t)$ and $\Omega(t)$ satisfy the following Riccati equation:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{R_t}{R}\right)-\left(\frac{R_t}{R}\right)^2-\Omega^2(t)=0. \tag{21}$$

It is not possible to find explicit solutions of this Riccati equation, however, for a given form of $R(t)$, the corresponding form of the trap frequency $\Omega(t)$ can be calculated as $\Omega(t)=\sqrt{\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{R_t}{R}\right)-\left(\frac{R_t}{R}\right)^2}$. From (17) to (20), the other functions can be figured out when the function $R(t)$ is fixed. Mathematically, the choices of $R(t)$ must allow for the term $\int_0^t q^2\,\mathrm{d}t$ in (18). After taking $\varepsilon=1$, the exact chirped one-soliton solution of (1) can be derived as

$$\Phi(x,t)=\frac{\mathrm{e}^{\int\frac{\gamma(t)}{2}\,\mathrm{d}t}\,\mathrm{e}^{\eta}}{1+c(t)\,\mathrm{e}^{\eta+\overline{\eta}}}, \tag{22}$$

where $\eta=\mathrm{i}p(t)x^2+q(t)x+\omega(t)$, the parameters $p(t)$, $q(t)$, $\omega(t)$, and $c(t)$ are determined by (17) to (21). In (22), after taking

$$q_0=\sqrt{2}a+\mathrm{i}(c+2br_0),\quad R_0=\frac{1}{r_0},\qquad w_0=-\frac{1}{2}\ln\left(\frac{R_0}{8a^2}\right), \tag{23}$$

the one soliton solution is reduced to (23) [24]. Here the soliton evolutions under the condition (23) are omitted deliberately. So here we only want to show the different phenomenon when $R(t)=\mathrm{sech}(0.1t)$, $\gamma(t)=0$ and $\gamma(t)=2\cos(2t)$. Figure 1a and 1b show that the soliton amplitude and width are affected by the external potential and the nonlinear term. Figure 1a is depicted with vanishing gain/loss term. Figure 1b illustrates that the soliton will breathe more and more lightly with the propagation distance. From the physical point of view, the collapse and revival of the atomic condensate through periodic exchange of atoms with the background is due to the gain/loss term $\gamma(t)=2\cos(2t)$. It is noted here that Strecker et al. have experimentally observed the formation of the bright soliton of a lithium atom in a quasi-1D optical trap by magnetically turning the interactions in a stable BEC from repulsive to attractive [29].

3. New Chirped Two-Soliton Solution

In this section, we show how to obtain analytically a chirped two-soliton solution of the GPE with time-dependent parameters. Similar to Section 2, to obtain two-soliton, we now assume that

$$\begin{aligned}F(x,t)&=1+\varepsilon^2F_1(x,t)+\varepsilon^4F_2(x,t),\\ G(x,t)&=\varepsilon\,G_1(x,t)+\varepsilon^3G_2(x,t).\end{aligned} \tag{24}$$

Inserting (24) into (6) and (7), requiring the coefficients of power term of ε be equal to zero, a series of equations are obtained

$$\varepsilon : \mathrm{i}G_{1t} + \frac{1}{2}G_{1xx} - \frac{1}{2}x^2\Omega^2(t)G_1 = 0, \tag{25}$$

$$\varepsilon^2 : F_{1xx} - RG_1\overline{G_1} = 0, \tag{26}$$

$$\begin{aligned}\varepsilon^3 : &\mathrm{i}G_{2t} + \frac{1}{2}G_{2xx} - \frac{1}{2}x^2\Omega^2(t)G_2 \\ &+ (\mathrm{i}D_t + \frac{1}{2}D_x^2 - \frac{1}{2}x^2\Omega^2(t))G_1 \cdot F_1 = 0,\end{aligned} \tag{27}$$

$$\varepsilon^4 : \frac{1}{2}D_x^2 F_1 \cdot F_1 + F_{2xx} - RG_2\overline{G_1} - RG_1\overline{G_2} = 0, \tag{28}$$

$$\begin{aligned}\varepsilon^5 : &(\mathrm{i}D_t + \frac{1}{2}D_x^2 - \frac{1}{2}x^2\Omega^2(t)) \\ &\cdot (G_1 \cdot F_2 + G_2 \cdot F_1) = 0,\end{aligned} \tag{29}$$

$$\varepsilon^6 : D_x^2 F_1 \cdot F_2 - RG_2\overline{G_2} = 0, \tag{30}$$

$$\varepsilon^7 : (\mathrm{i}D_t + \frac{1}{2}D_x^2 - \frac{1}{2}x^2\Omega^2(t))G_2 \cdot F_2 = 0, \tag{31}$$

$$\varepsilon^8 : D_x^2 F_2 \cdot F_2 = 0. \tag{32}$$

We set

$$G_1(x,t) = \mathrm{e}^{\eta_1} + \mathrm{e}^{\eta_2}$$

together with

$$\eta_1 = \mathrm{i}p(t)x^2 + q_1(t)x + \omega_1(t),$$
$$\eta_2 = \mathrm{i}p(t)x^2 + q_2(t)x + \omega_2(t).$$

By employing the same procedure as in Section 2, we can choose F_1, G_2, and F_2 according to (25)–(28) as

$$\begin{aligned}F_1(x,t) = &c_1(t)\,\mathrm{e}^{\eta_1+\overline{\eta_1}} + c_2(t)\,\mathrm{e}^{\eta_2+\overline{\eta_1}} \\ &+ c_3(t)\,\mathrm{e}^{\eta_1+\overline{\eta_2}} + c_4(t)\,\mathrm{e}^{\eta_2+\overline{\eta_2}},\end{aligned} \tag{33}$$

$$G_2(x,t) = b_1(t)\,\mathrm{e}^{\eta_1+\overline{\eta_1}+\eta_2} + b_2(t)\,\mathrm{e}^{\eta_2+\overline{\eta_2}+\eta_1}, \tag{34}$$

$$F_2(x,t) = c_5(t)\,\mathrm{e}^{\eta_1+\overline{\eta_1}+\eta_2+\overline{\eta_2}}. \tag{35}$$

In this way, a system of algebraic or fist-order ordinary differential equations for the parameter functions is obtained. Solving the obtained system, after a long and tedious calculation and taking $\varepsilon = 1$, the new chirped two-soliton solution of (1) with time-dependent parameters reads as

$$\Phi(x,t) = \frac{\mathrm{e}^{\int \frac{\gamma(t)}{2}dt}\left(\mathrm{e}^{\eta_1} + \mathrm{e}^{\eta_2} + b_1\,\mathrm{e}^{\eta_1+\overline{\eta_1}+\eta_2} + b_2\,\mathrm{e}^{\eta_2+\overline{\eta_2}+\eta_1}\right)}{1 + c_1\,\mathrm{e}^{\eta_1+\overline{\eta_1}} + c_2\,\mathrm{e}^{\eta_2+\overline{\eta_1}} + c_3\,\mathrm{e}^{\eta_1+\overline{\eta_2}} + c_4\,\mathrm{e}^{\eta_2+\overline{\eta_2}} + c_5\,\mathrm{e}^{\eta_1+\overline{\eta_1}+\eta_2+\overline{\eta_2}}}, \tag{36}$$

where

$$\eta_1 = \mathrm{i}px^2 + q_1x + \omega_1\,,\quad \eta_2 = \mathrm{i}px^2 + q_2x + \omega_2\,,$$
$$c_1 = \frac{c_{10}R_0}{R}\,,\quad c_2 = \frac{c_{20}R_0}{R}\,,\quad c_3 = \frac{c_{30}R_0}{R}\,,$$
$$c_4 = \frac{c_{40}R_0}{R}\,,$$
$$q_1 = \frac{q_{10}R}{R_0}\,,\quad q_2 = \frac{q_{20}R}{R_0}\,,\quad p = -\frac{R_t}{2R}\,,$$
$$\omega_1 = \omega_{10} + \int_0^t \left(\frac{1}{2}q_1^2 - p\right)\mathrm{d}z\,,$$
$$\omega_2 = \omega_{20} + \int_0^t \left(\frac{1}{2}q_2^2 - p\right)\mathrm{d}z\,,$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{R_t}{R}\right) - \left(\frac{R_t}{R}\right)^2 - \Omega^2(t) = 0\,,$$
$$b_1 = \frac{c_{20}(q_{10}-q_{20})^2R_0}{(q_{10}+\overline{q_{10}})^2R}\,,$$
$$b_2 = \frac{(q_{10}-q_{20})^2R_0c_{30}c_{40}}{c_{10}(q_{10}+\overline{q_{10}})^2R}\,,$$
$$c_5 = \frac{c_{30}c_{40}R_0^2(\overline{q_{10}}-\overline{q_{20}})^2(q_{10}-q_{20})^2}{(q_{10}+\overline{q_{10}})^2(q_{20}+\overline{q_{10}})^2R^2}\,.$$

Here the subscript 0 denotes the value of the given function at $z = 0$. The choices for the parameters c_{10}, c_{20}, c_{30}, c_{40}, q_{10}, and q_{20} should satisfy

$$\begin{aligned}&c_{10}(q_{10}+\overline{q_{10}})^2 = c_{20}(q_{20}+\overline{q_{10}})^2 = c_{30}(q_{10}+\overline{q_{20}})^2 \\ &= c_{40}(q_{20}+\overline{q_{20}})^2 = R_0,\ \ c_{20} = c_{30}\,.\end{aligned} \tag{37}$$

An interaction between two solitons is interesting for the 1D GPE with time-dependent parameters. Figure 2a and 2b show that the interaction between the two solitons is affected by the periodic nonlinear gain or loss. Similarly to Figure 1, Figure 2a is depicted with vanishing gain/loss term. Figure 2b illustrates that the new chirped two-soliton will breathe more and more lightly with the propagation distance. From the physical point of view, the collapse and revival of the atomic condensate through periodic exchange of atoms with the background is due to the gain/loss term $\gamma(t) = 2\cos(2t)$. The two-soliton does not collide or attract

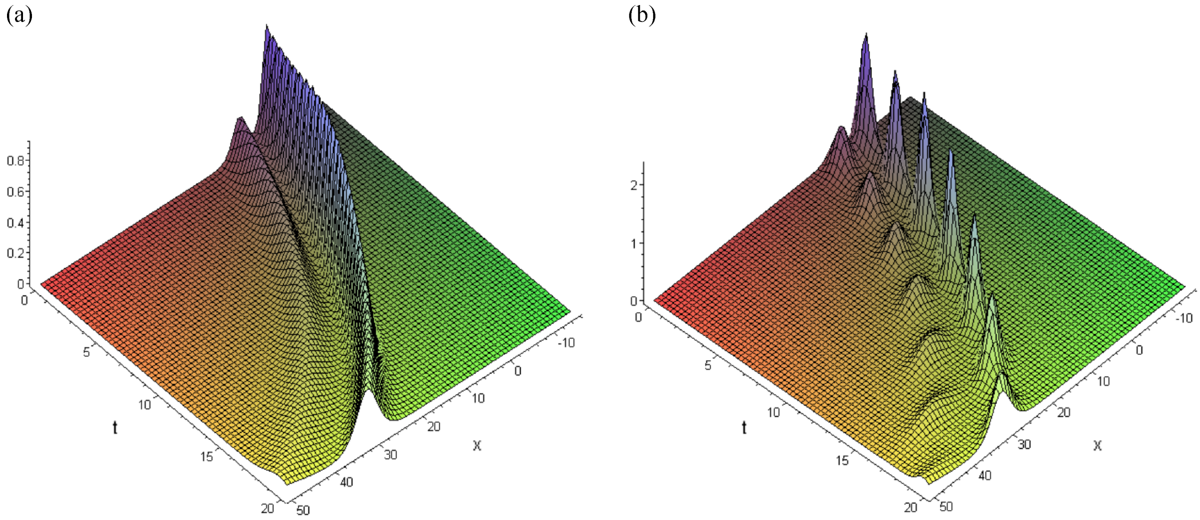

Fig. 2 (colour online). Temporal evolution of two-soliton solution (36) for 1D GPE, (a) under conditions $\gamma = 0$, $R_0 = 1$, $q_{10} = -0.5 + \mathrm{i}$, $q_{20} = 1 + \mathrm{i}$, $\omega_{10} = 0$, $\omega_{20} = 0$, $R(t) = \mathrm{sech}(0.1t)$, (b) with the same parameters as in (a) except that $\gamma(t) = 2\cos(2t)$.

each other and propagate parallel as the time evolves. Furthermore, from the above phenomenon, it is worthwhile to note that the gain/loss term does not affect the width and motion of the soliton but changes its peak height.

4. Conclusions

In this work, the soliton solutions with chirp of the GPE with time-dependent parameters are investigated. A developed Hirota method is applied carefully to the GPE. In terms of this technique, we decoupled the GPE into two equations. Additionally, with a reasonable assumption, the exact chirped one- and new two-soliton solutions are constructed effectively. The finding of a new mathematical algorithm to discover soliton solutions in nonlinear dispersive systems with parameter variations is helpful on future research. On the other hand, the results are useful not only in Bose–Einstein condensates, but also in nonlinear optical systems.

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