# Soliton Solutions, Bäcklund Transformation and Wronskian Solutions for the $(2+1)$-Dimensional Variable-Coefficient Konopelchenko–Dubrovsky Equations in Fluid Mechanics

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This paper is to investigate the $(2+1)$-dimensional variable-coefficient Konopelchenko–Dubrovsky equations, which can be applied to the phenomena in stratified shear flow, internal and shallow-water waves, plasmas, and other fields. The bilinear-form equations are transformed from the original equations, and soliton solutions are derived via symbolic computation. Soliton solutions and collisions are illustrated. The bilinear-form Bäcklund transformation and another soliton solution are obtained. Wronskian solutions are constructed via the Bäcklund transformation and solution.

*Key words:* $(2+1)$-Dimensional Variable-Coefficient Konopelchenko–Dubrovsky Equations; Fluid Mechanics; Soliton Solutions; Bäcklund Transformation; Wronskian Solutions; Symbolic Computation.

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## 1. Introduction

Phenomena in fluid mechanics, physics, chemistry, biology, and other fields can be described by the nonlinear partial differential equations (NLPDEs) [1 – 8]. Investigations on the analytic solutions of the NLPDEs, especially the solitons, have been active in such fields [1 – 8]. Methods applied to constructing the analytic solutions of the NLPDEs have been proposed, such as the Hirota bilinear method [9 – 14], Bäcklund transformation (BT) [9, 15], Wronskian technique [16, 17], Painlevé analysis [18, 19], and Darboux transformation [20 – 26]. The bilinear method can transform some NLPDEs into bilinear equations, e.g., the Korteweg–de Vries (KdV) [10], Gardner [27 – 32], Kadomtsev–Petviashvili (KP) [33 – 36] and modified KP (mKP) [37, 38] equations.The Bäcklund transformation can connect several analytic solutions, and the auto-Bäcklund transformation several analytic solutions for the same equation [9, 15]. The Wronskian technique is used to construct the Wronskian solutions which have the determinant form of the soliton solutions [16, 17].

The variable-coefficient Konopelchenko–Dubrovsky (vcKD) equation [39 – 53],

$$\begin{aligned} &u_t + f_1(t)u_{xxx} + f_2(t)uu_x + f_3(t)u^2u_x \\ &+ f_4(t)\int u_{yy}\,\mathrm{d}x + f_5(t)u_x\int u_y\,\mathrm{d}x = 0, \end{aligned} \tag{1}$$

is the $(2+1)$-dimensional model in fluid mechanics which is the relevant wave model in stratified shear flow, internal and shallow-water waves, plasmas, and other fields [39 – 53]. In (1), $x$ and $y$ are the running coordinates along the propagation directions, $t$ is the time, $u = u(x,y,t)$ is the amplitude of the relevant waves in the stratified shear flow, ocean, and shallow water, the subscripts denote the partial differentiation, and $f_i(t)'s$ $(i = 1,2,3,4,5)$ are the time-dependent coefficients.

In fact, the variable-coefficient NLPDEs are more suitable when the boundaries are considered and more accurate when we consider the description of dynamical and physical phenomena [1 – 8]. Equation (1) covers the Gardner, KP, mKP, and KD equations in ocean dynamics, fluid mechanics, and plasma physics, which are all special cases of (1) with different co-

efficients [10, 27 – 53]. Thereby, (1) with the time-dependent coefficients could be applied to a variety of phenomena in ocean dynamics, fluid mechanics, and plasma physics [39 – 53], with the relevant seen in [54, 55].

In this paper, using the transfomation

$$v(x,y,t) = \int u_y \, \mathrm{d}x$$

and considering the situation $f_5(t) = f_2(t)$, we will rewrite (1) in the following form:

$$\begin{aligned} &u_t + f_1(t)u_{xxx} + f_2(t)[uu_x + u_x v] \\ &+ f_3(t)u^2 u_x + f_4(t)v_y = 0, \\ &u_y = v_x. \end{aligned} \tag{2}$$

With symbolic computation [1 – 5], in Section 2, we will obtain the bilinear-form equations for (2) and give its $N$-soliton solutions. In Section 3, from the bilinear-form equations, we will derive the BT of (2) and obtain new solutions. Furthermore, in Section 4, the Wronskian of $N$-soliton solutions will be constructed from the BT. Finally, Section 5 will be our conclusions.

## 2. Hirota Bilinear Method and Soliton Solutions

The bilinear derivative operator $D_x^m D_t^n$ [9] is defined as

$$\begin{aligned} D_x^m D_t^n f(x,t) \cdot g(x,t) &= \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)^m \\ &\cdot \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\right)^n f(x,t)g(x',t')|_{x'=x,t'=t}. \end{aligned} \tag{3}$$

By truncating the Painlevé expansion at the constant level term, we can obtain the transformations

$$\begin{aligned} u(x,y,t) &= \mathrm{i}\sqrt{\frac{6f_1(t)}{f_3(t)}}\left[\log\left(\frac{g(x,y,t)}{g^*(x,y,t)}\right)\right]_x, \\ v(x,y,t) &= \mathrm{i}\sqrt{\frac{6f_1(t)}{f_3(t)}}\left[\log\left(\frac{g(x,y,t)}{g^*(x,y,t)}\right)\right]_y. \end{aligned} \tag{4}$$

With (3) and (4), (2) becomes the bilinear-form equations

$$\begin{aligned} &\left[D_t + f_1(t)D_x^3 + \sqrt{3f_1(t)f_4(t)}D_xD_y - f_4(t)D_y\right] \\ &\cdot g(x,y,t)\cdot g^*(x,y,t) = 0, \\ &\left[D_y - \sqrt{\frac{3f_1(t)}{f_4(t)}}D_x^2 + D_x\right] g(x,y,t)\cdot g^*(x,y,t) = 0, \end{aligned} \tag{5}$$

where '$*$' is the representation of the complex conjugate.

The single-soliton solution is construct as follows:

$$\begin{aligned} g(x,y,t) = 1 + \exp\Bigg[&p_1 x - p_1 y \\ &- \int [f_1(t)p_1^3 + f_4(t)p_1]\,\mathrm{d}t + \eta_1 + \mathrm{i}\frac{\pi}{2}\Bigg], \end{aligned} \tag{6}$$

where $p_1$ is an arbitrary real parameter.

Now we substitute (6) into (4) which can be written as

$$\begin{aligned} u(x,y,t) &= -2\frac{p_1}{f_2(t)}\sqrt{-3f_1(t)f_4(t)} \\ &\cdot \sec\left[p_1 x - p_1 y - \int [f_1(t)p_1^3 + f_4(t)p_1]\,\mathrm{d}t + \eta_1\right], \end{aligned} \tag{7}$$

$$\begin{aligned} v(x,y,t) &= 2\frac{p_1}{f_2(t)}\sqrt{-3f_1(t)f_4(t)} \\ &\cdot \sec\left[p_1 x - p_1 y - \int [f_1(t)p_1^3 + f_4(t)p_1]\,\mathrm{d}t + \eta_1\right]. \end{aligned} \tag{8}$$

The two-soliton solutions for (2) are described in the following forms:

$$\begin{aligned} g(x,y,t) = 1 &+ \exp\left[\xi_1 + \mathrm{i}\frac{\pi}{2}\right] + \exp\left[\xi_2 + \mathrm{i}\frac{\pi}{2}\right] \\ &- \frac{(p_1 - p_2)^2}{(p_1 + p_2)^2}\exp[\xi_1 + \xi_2 + \mathrm{i}\pi], \end{aligned} \tag{9}$$

$$\xi_1 = p_1 x - p_1 y - \int [f_1(t)p_1^3 + f_4(t)p_1]\,\mathrm{d}t + \eta_1, \tag{10}$$

$$\xi_2 = p_2 x - p_2 y - \int [f_1(t)p_2^3 + f_4(t)p_2]\,\mathrm{d}t + \eta_2. \tag{11}$$

To obtain the expressions of $u(x,y,t)$ and $v(x,y,t)$, we can substitute (9) – (11) into (4).

Hereby, following the preceding formal regularities, we construct the $N$-soliton solutions of (2) which are expressed as

$$g(x,y,t) = \sum_{\mu=0,1} \exp\left[\sum_{j=1}^{N} \mu_j(\xi_j + i\frac{\pi}{2}) + \sum_{j<k}^{(N)} \mu_j\mu_k\delta_{jk}\right], \tag{12}$$

$$\begin{aligned} \xi_j = p_j x - p_j y - \int &\left[f_1(t)p_j^3 + f_4(t)p_j\right]\mathrm{d}t \\ &+ \eta_j \ \ (j = 0,1,2,\ldots,N), \end{aligned} \tag{13}$$

$$\exp(\delta_{jk}) = -\frac{(p_j - p_k)^2}{(p_j + p_k)^2}, \tag{14}$$

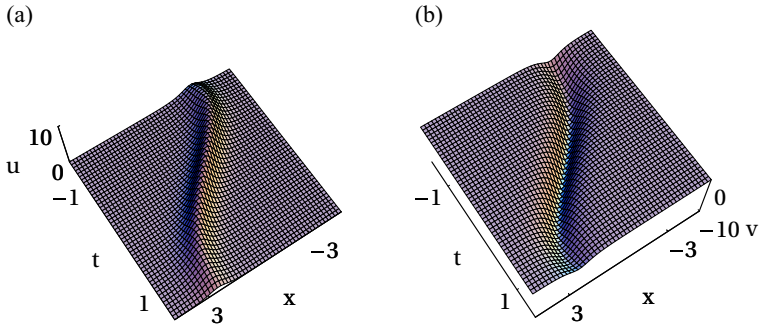


Fig. 1 (colour online). Single-soliton solution at $y = 0$ with parameters $f_1(t) = \cos(t)$, $f_2(t) = 1$, $f_4(t) = -1$, $p_1 = -2$, and $\eta_1 = 0$. (a) $u(x,y,t)$ for (7) and (b) $v(x,y,t)$ for (8).

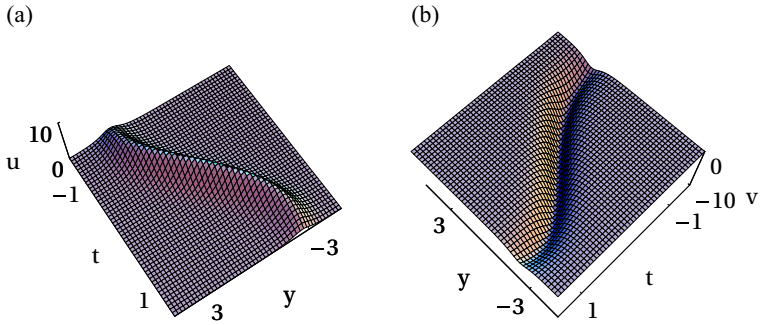


Fig. 2 (colour online). Single-soliton solution at $x = 0$ with parameters $f_1(t) = \cos(t)$, $f_2(t) = 1$, $f_4(t) = -1$, $p_1 = -2$, and $\eta_1 = 0$. (a) $u(x,y,t)$ for (7) and (b) $v(x,y,t)$ for (8).

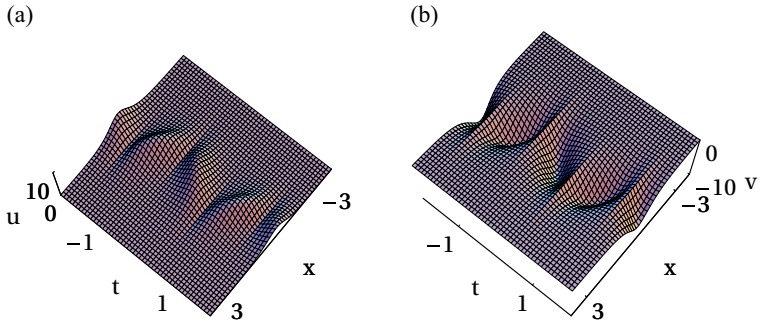


Fig. 3 (colour online). Single-soliton solution at $y = 0$ with parameters $f_1(t) = \cos(3t)$, $f_2(t) = 1$, $f_4(t) = -\cos(3t)$, $p_1 = -1.7$, and $\eta_1 = 0$. (a) $u(x,y,t)$ for (7) and (b) $v(x,y,t)$ for (8).

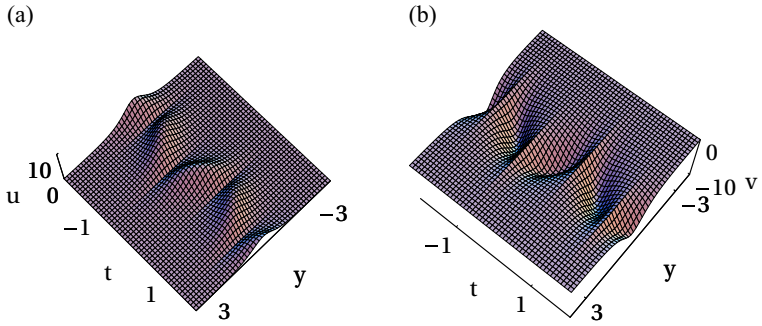


Fig. 4 (colour online). Single-soliton solution at $x = 0$ with parameters $f_1(t) = \cos(3t)$, $f_2(t) = 1$, $f_4(t) = -\cos(3t)$, $p_1 = -1.7$, and $\eta_1 = 0$. (a) $u(x,y,t)$ for (7) and (b) $v(x,y,t)$ for (8).

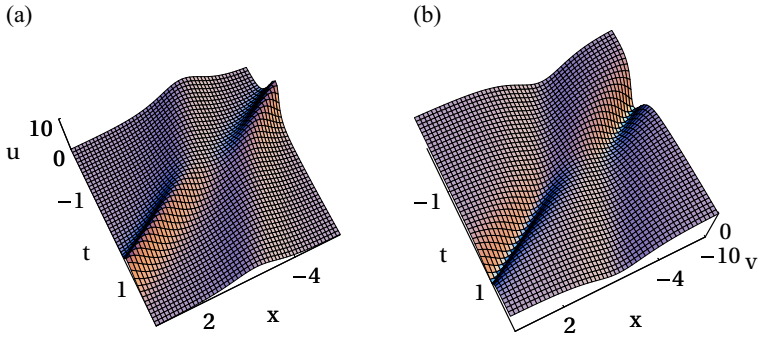


Fig. 5 (colour online). Two-soliton solutions at $y = 0$ for (9) with parameters $f_1(t) = 4$, $f_2(t) = 0.7$, $f_4(t) = -0.7$, $p_1 = -0.7$, $p_2 = -1.2$, $\eta_1 = 0$, and $\eta_2 = 0$. (a) for $u(x,y,t)$ and (b) for $v(x,y,t)$.

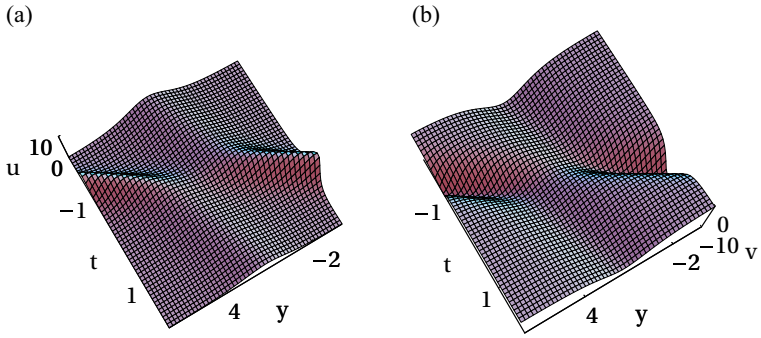


Fig. 6 (colour online). Two-soliton solutions at $x = 0$ for (9) with parameters $f_1(t) = 4$, $f_2(t) = 0.7$, $f_4(t) = -0.8$, $p_1 = -0.7$, $p_2 = -1.2$, $\eta_1 = 0$, and $\eta_2 = 0$. (a) for $u(x,y,t)$ and (b) for $v(x,y,t)$.

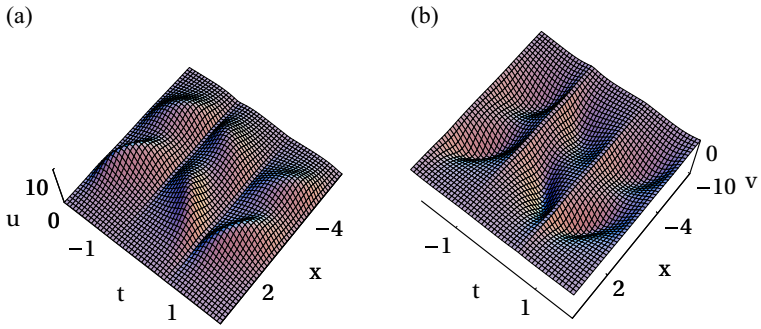


Fig. 7 (colour online). Two-soliton solutions at $y = 0$ for (9) with parameters $f_1(t) = 4\cos(2.5t)$, $f_2(t) = 0.7$, $f_4(t) = -0.5\cos(2.5t)$, $p_1 = -0.7$, $p_2 = -1.2$, $\eta_1 = 0$, and $\eta_2 = 0$. (a) for $u(x,y,t)$ and (b) for $v(x,y,t)$.

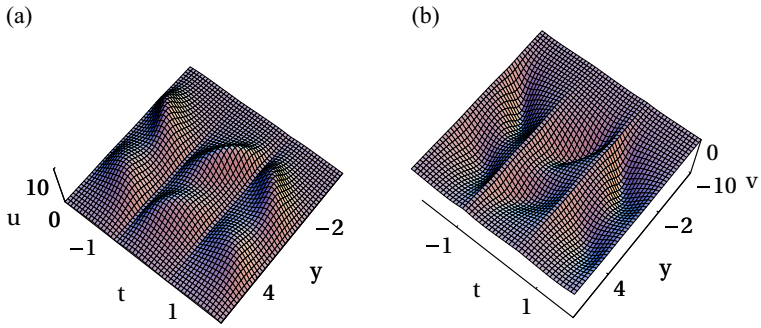


Fig. 8 (colour online). Two-soliton solutions at $x = 0$ for (9) with parameters $f_1(t) = 4\cos(2.5t)$, $f_2(t) = 0.7$, $f_4(t) = -0.5\cos(2.5t)$, $p_1 = -0.7$, $p_2 = -1.2$, $\eta_1 = 0$, and $\eta_2 = 0$. (a) for $u(x,y,t)$ and (b) for $v(x,y,t)$.

where $p_j$ $(j = 0,1,2,\ldots,N)$ are arbitrary real parameters characterizing the $j$th soliton, $\eta_j$ $(j = 0,1,2,\ldots,N)$ are arbitrary real constants, and $\sum_{\mu=0,1}$ contains all the possible combinations for $\mu_1 = 0,1$, $\mu_2 = 0,1,\ldots,\mu_N = 0,1$.

Figures 1–4 present the single-soliton solutions when the appropriate coefficients are selected. In Figures 3 and 4, the periodic solitons appear when we choose the coefficient $f_1(t) = \cos(3t)$ and $f_4(t) = -\cos(3t)$. Figures 5 and 6 illustrate the head-on elastic collisions of $u(x,y,t)$ and $v(x,y,t)$ solitons at $y = 0$ and $x = 0$, respectively. After colliding, the intensity, amplitude, and velocity of the solitons remain the same as before. Figures 7 and 8 illustrate the periodicity of the two-soliton solution with the coefficient $f_1(t) = 4\cos(2.5t)$ and $f_4(t) = -0.5\cos(2.5t)$.

## 3. Bäcklund Transformation

It is known that the BT can connect two solutions of one equation, even two different equations, and we can deduce other solutions from the obtained one [9, 15, 28]. In this section, we will derive the bilinear-form BT by virtue of the bilinear-form (5) with the help of the exchange formula [9] and get two kinds of solutions from the obtained one.

Let us assume the new solutions to be

$$\begin{aligned} u(x,y,t) &= \mathrm{i}\sqrt{\frac{6f_1(t)}{f_3(t)}}\left[\log\left(\frac{f(x,y,t)}{f^*(x,y,t)}\right)\right]_x, \\ v(x,y,t) &= \mathrm{i}\sqrt{\frac{6f_1(t)}{f_3(t)}}\left[\log\left(\frac{f(x,y,t)}{f^*(x,y,t)}\right)\right]_y, \end{aligned} \tag{15}$$

where $f(x,y,t)$ and $f^*(x,y,t)$ are the new solutions of (2).

Now, we consider the equations

$$\begin{aligned} &\Big[\big(D_t + f_1(t)D_x^3 + \sqrt{3f_1(t)f_4(t)}D_xD_y \\ &- f_4(t)D_y\big)f\cdot f^*\Big]gg^* - \Big[\big(D_t + f_1(t)D_x^3 \\ &+ \sqrt{3f_1(t)f_4(t)}D_xD_y - f_4(t)D_y\big)g\cdot g^*\Big]ff^* = 0, \end{aligned} \tag{16}$$

$$\begin{aligned} &\left[\left(D_y - \sqrt{\frac{3f_1(t)}{f_4(t)}}D_x^2 + D_x\right)f\cdot f^*\right]gg^* \\ &- \left[\left(D_y - \sqrt{\frac{3f_1(t)}{f_4(t)}}D_x^2 + D_x\right)g\cdot g^*\right]ff^* = 0. \end{aligned} \tag{17}$$

Then via symbolic computation and the different exchange formula [9], (16) and (17) can be transformed into the following forms:

$$\begin{aligned} &\big[\big(D_t + f_1(t)D_x^3 - f_4(t)D_y\big)f\cdot g\big]f^*g^* \\ &- 3f_1(t)D_x\big(D_xf\cdot g^*\big)\cdot\big(D_xf^*\cdot g\big) \\ &- \big[\big(D_t + f_1(t)D_x^3 - f_4(t)D_y\big)f^*\cdot g^*\big]fg \\ &+ \frac{\sqrt{3f_1(t)f_4(t)}}{2}\big[D_x(D_yf\cdot g^*)\cdot\big(f^*g\big) \\ &- D_x(fg^*)\cdot\big(D_yf^*\cdot g\big) + D_y\big(D_xf\cdot g^*\big) \\ &\cdot(f^*g) - D_y(fg^*)\cdot\big(D_xf^*\cdot g\big)\big] = 0, \end{aligned} \tag{18}$$

$$\begin{aligned} &[(D_y + D_x)f\cdot g]f^*g^* - \big[(D_y + D_x)f^*\cdot g^*\big]fg \\ &- \sqrt{\frac{3f_1(t)}{f_4(t)}}\big[D_x\big(D_xf\cdot g^*\big)\cdot(f^*g) \\ &- D_x(fg^*)\cdot\big(D_xf^*\cdot g\big)\big] = 0. \end{aligned} \tag{19}$$

Further, we decouple (18) and (19) into

$$D_xf(x,y,t)\cdot g^*(x,y,t) = \lambda f^*(x,y,t)g(x,y,t), \tag{20a}$$
$$D_xf^*(x,y,t)\cdot g(x,y,t) = \lambda f(x,y,t)g^*(x,y,t), \tag{20b}$$
$$D_yf(x,y,t)\cdot g^*(x,y,t) = \delta f^*(x,y,t)g(x,y,t), \tag{20c}$$
$$D_yf^*(x,y,t)\cdot g(x,y,t) = \delta f(x,y,t)g^*(x,y,t), \tag{20d}$$
$$\big(D_y + D_x\big)f(x,y,t)\cdot g(x,y,t) = 0, \tag{20e}$$
$$\big(D_y + D_x\big)f^*(x,y,t)\cdot g^*(x,y,t) = 0, \tag{20f}$$
$$\begin{aligned} &\big(D_t + f_1(t)D_x^3 - f_4(t)D_y + 3f_1(t)\lambda^2D_x\big) \\ &\cdot f(x,y,t)\cdot g(x,y,t) = 0, \end{aligned} \tag{20g}$$
$$\begin{aligned} &\big(D_t + f_1(t)D_x^3 - f_4(t)D_y + 3f_1(t)\lambda^2D_x\big) \\ &\cdot f^*(x,y,t)\cdot g^*(x,y,t) = 0, \end{aligned} \tag{20h}$$

where $\lambda$ and $\delta$ are arbitrary constants.

We could also assume the seed solution $g(x,y,t) = 1$. Then (20) can be written as the partial different equations

$$f_x(x,y,t) = \lambda f^*(x,y,t), \tag{21a}$$
$$f_y(x,y,t) = \delta f^*(x,y,t), \tag{21b}$$
$$f_y(x,y,t) + f_x(x,y,t) = 0, \tag{21c}$$
$$\begin{aligned} &f_t(x,y,t) + f_1(t)f_{xxx}(x,y,t) - f_4(t)f_y(x,y,t) \\ &+ 3f_1(t)\lambda^2f_x(x,y,t) = 0, \end{aligned} \tag{21d}$$
$$f_{xx}(x,y,t) = \lambda^2f(x,y,t). \tag{21e}$$

When we solve the linear differential (21), the single-soliton solution of (2) can be constructed as

$$f = a\mathrm{e}^{\lambda x-\lambda y+\int[-4f_1(t)\lambda^3-f_4(t)\lambda]\mathrm{d}t} + bi\mathrm{e}^{-\lambda x+\lambda y-\int[-4f_1(t)\lambda^3-f_4(t)\lambda]\mathrm{d}t}, \tag{22}$$

where $a$ and $b$ are arbitrary real constants.

## 4. Wronskian Solution

Using the BT and the single-soliton solution in Section 3, we can construct the Wronskian solution of (2). Its construction is postulated as follows:

$$\begin{aligned} f &= W(\varphi_1,\varphi_2,\ldots,\varphi_N) \\ &= \begin{vmatrix} \varphi_1 & \varphi_1^{(1)} & \varphi_1^{(2)} & \ldots & \varphi_1^{(N-1)} \\ \varphi_2 & \varphi_2^{(1)} & \varphi_2^{(2)} & \ldots & \varphi_2^{(N-1)} \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ \varphi_N & \varphi_N^{(1)} & \varphi_N^{(2)} & \ldots & \varphi_N^{(N-1)} \end{vmatrix}_{N\times N} \\ &= (\widehat{N-1}), \end{aligned} \tag{23}$$

where $\varphi_j = \varphi_j(x,y,t)$, $\varphi_j^{(i)} = \dfrac{\partial^i\varphi_j(x,y,t)}{\partial x^i}$ $(j = 1,\ldots,N; i = 0,\ldots,N-1)$, and $\varphi_j$ must satisfy the relations

$$\varphi_{j,t} = -4f_1(t)\varphi_{j,xxx} - f_4(t)\varphi_{j,x}, \tag{24a}$$
$$\varphi_{j,y} = -\varphi_{j,x}, \tag{24b}$$
$$\varphi_{j,xx} = \lambda_j^2\varphi_j, \tag{24c}$$
$$\varphi_j^* = \lambda_j\partial^{-1}\varphi_j. \tag{24d}$$

$f$ and $f^*$ can be abbreviated as $f = (\widehat{N-1})$ and $f^* = A(-1,\widehat{N-2})$, where $A = \prod_{j=1}^N \lambda_j$. the derivatives of $f$ and $f^*$ with respect to $x$, $y$ and $t$ can be expressed in the abbreviated denotations

$$f_x = (\widehat{N-2},N), \tag{25}$$
$$f_{xx} = (\widehat{N-3},N-1,N) + (\widehat{N-2},N+1), \tag{26}$$
$$\begin{aligned} f_{xxx} &= (\widehat{N-4},N-2,N-1,N) \\ &\quad + 2(\widehat{N-3},N-1,N+1) + (\widehat{N-2},N+2), \end{aligned} \tag{27}$$
$$f_y = -(\widehat{N-2},N), \tag{28}$$
$$\begin{aligned} f_t &= -4f_1(t)(\widehat{N-4},N-2,N-1,N) \\ &\quad -4f_1(t)(\widehat{N-2},N+2) - f_4(t)(\widehat{N-2},N) \\ &\quad +4f_1(t)(\widehat{N-3},N-1,N+1), \end{aligned} \tag{29}$$
$$f_x^* = A(-1,\widehat{N-3},N-1), \tag{30}$$
$$\begin{aligned} f_{xx}^* &= A[(-1,\widehat{N-4},N-2,N-1) \\ &\quad + (-1,\widehat{N-3},N)], \end{aligned} \tag{31}$$
$$\begin{aligned} f_{xxx}^* &= A[(-1,\widehat{N-5},N-3,N-2,N-1) \\ &\quad + 2(-1,\widehat{N-4},N-2,N) + (-1,\widehat{N-3},N+1)], \end{aligned} \tag{32}$$
$$f_y^* = -A(-1,\widehat{N-3},N-1), \tag{33}$$
$$\begin{aligned} f_t^* &= -4f_1(t)A\Big[(-1,\widehat{N-4},N-2,N) \\ &\quad -(-1,\widehat{N-5},N-3,N-2,N-1) \\ &\quad +(-1,\widehat{N-3},N+1)\Big] \\ &\quad -f_4(t)A(-1,\widehat{N-3},N-1). \end{aligned} \tag{34}$$

Using the relations

$$\left|\widehat{P}\,a\,b\right|\left|\widehat{P}\,c\,d\right| - \left|\widehat{P}\,a\,c\right|\left|\widehat{P}\,b\,d\right| + \left|\widehat{P}\,a\,d\right|\left|\widehat{P}\,b\,c\right| = 0$$

and

$$\sum_{j=1}^N \lambda_j(-1,\widehat{N-2}) = (-1,\widehat{N-3},N-1),$$
$$\sum_{j=1}^N \lambda_j^2(-1,\widehat{N-2}) = (-1,\widehat{N-3},N) - (-1,\widehat{N-4},N-2,N-1),$$
$$\sum_{j=1}^N \lambda_j^2(-1,\widehat{N-3},N-1) = (-1,\widehat{N-3},N+1) - (-1,\widehat{N-5},N-3,N-2,N-1),$$
$$\sum_{j=1}^N \lambda_j(\widehat{N-1}) = (\widehat{N-2},N),$$
$$\sum_{j=1}^N \lambda_j^2(\widehat{N-2},N) = -(\widehat{N-4},N-2,N-1,N) + (\widehat{N-2},N+2),$$
$$\sum_{j=1}^N \lambda_j^2(\widehat{N-1}) = -(\widehat{N-3},N-1,N) + (\widehat{N-2},N+1),$$

and substituting the derivatives of $f$ and $f^*$ into (5), we have

$$
\begin{aligned}
&\left(D_t + f_1(t)D_x^3 + \sqrt{3f_1(t)f_4(t)}D_xD_y - f_4(t)D_y\right) f\cdot f^* \\
&= f_t f^* - f f_t^* - f_4(t)\left(f_y f^* - f f_y^*\right) + f_1(t)\left(f_{xxx}f^* - 3f_{xx}f_x^* + 3f_x f_{xx}^* - f f_{xxx}^*\right) \\
&\quad + \sqrt{3f_1(t)f_4(t)}\left(f_{xy}f^* - f_x f_y^* - f_y f_x^* + f f_{xy}^*\right) \\
&= -6f_1(t)A(-1,\widehat{N-4},N-2,N)(\widehat{N-1}) + 6f_1(t)A(\widehat{N-3},N-1,N+1)(-1,\widehat{N-2}) \\
&\quad + 6f_1(t)A(-1,\widehat{N-3},N+1)(\widehat{N-1}) + 6f_1(t)A(-1,\widehat{N-4},N-2,N-1)(\widehat{N-2},N) \\
&\quad - 6f_1(t)A(\widehat{N-2},N+1)(-1,\widehat{N-3},N-1) - 6f_1(t)A(\widehat{N-4},N-2,N+1,N)(-1,\widehat{N-2}) \\
&\quad - 2A\sqrt{3f_1(t)f_4(t)}\left[(-1,\widehat{N-3},N)(\widehat{N-1}) - (-1,\widehat{N-3},N-1)(\widehat{N-2},N)\right. \\
&\quad \left. + (\widehat{N-3},N-1,N)(-1,\widehat{N-2})\right] \\
&= A(-1)^{N-2}\left\{3f_1(t)\left[\begin{vmatrix} \widehat{N-3} & \mathbf{0} & -1 & N-2 & N-1 & N+1 \\ \mathbf{0} & \widehat{N-3} & -1 & N-2 & N-1 & N+1 \end{vmatrix}\right.\right. \\
&\quad \left. + \begin{vmatrix} \widehat{N-4} & N-2 & \mathbf{0} & 0 & -1 & N-3 & N-1 & N \\ \mathbf{0} & 0 & \widehat{N-4} & N-2 & -1 & N-3 & N-1 & N \end{vmatrix}\right] \\
&\quad \left. + \sqrt{3f_1(t)f_4(t)}\begin{vmatrix} \widehat{N-3} & \mathbf{0} & -1 & N-2 & N-1 & N \\ \mathbf{0} & \widehat{N-3} & -1 & N-2 & N-1 & N \end{vmatrix}\right\} \\
&= 0.
\end{aligned}
$$

In a similar way, the other equation of (5) can be proved:

$$
\begin{aligned}
&\left(D_y - \sqrt{\frac{3f_1(t)}{f_4(t)}}D_x^2 + D_x\right) f(x,y,t)\cdot f^*(x,y,t) \\
&= f_y f^* - f f_y^* - \sqrt{\frac{3f_1(t)}{f_4(t)}}\left(f_{xx}f^* - 2f_x f_x^* + f f_{xx}^*\right) \\
&\quad + f_x f^* - f f_x^* \\
&= -2A\sqrt{\frac{3f_1(t)}{f_4(t)}}\left[(-1,\widehat{N-3},N)(\widehat{N-1})\right. \\
&\quad - (-1,\widehat{N-3},N-1)(\widehat{N-2},N) \\
&\quad \left. + (\widehat{N-3},N-1,N)(-1,\widehat{N-2})\right] \\
&= A(-1)^{N-1}\sqrt{\frac{3f_1(t)}{f_4(t)}} \\
&\quad \cdot \begin{vmatrix} \widehat{N-3} & \mathbf{0} & -1 & N-2 & N-1 & N \\ \mathbf{0} & \widehat{N-3} & -1 & N-2 & N-1 & N \end{vmatrix} \\
&= 0.
\end{aligned}
$$

Therefore, by directly substituting $f = (\widehat{N-1})$ and $f^* = A(-1,\widehat{N-2})$ into the bilinear equations (5), we have proved that the Wronskian solution is the solution of (2). Then the solution of (2) can be expressed as

$$
\begin{aligned}
u(x,y,t) &= \mathrm{i}\sqrt{\frac{6f_1(t)}{f_3(t)}}\left[\log\left(\frac{(\widehat{N-1})}{A(-1,\widehat{N-2})}\right)\right]_x, \\
v(x,y,t) &= \mathrm{i}\sqrt{\frac{6f_1(t)}{f_3(t)}}\left[\log\left(\frac{(\widehat{N-1})}{A(-1,\widehat{N-2})}\right)\right]_y.
\end{aligned}
\tag{35}
$$

## 5. Conclusions

In this paper, we have investigated (2), a variable-coefficient model in fluid mechanics and ocean dynamics. Based on symbolic computation, (2) has been transformed into bilinear forms (5), and $N$-soliton solutions (6)–(14) of (2) have been constructed. Figures 1–8 also illustrate the single soliton, two solitons, and two-soliton collisions including the parallel solitons and the head-on elastic

collisions, after which the intensity, amplitude, and velocity of each soliton remain the same as before, respectively. With the help of exchange formula and symbolic computation, the bilinear-form BT (20) has been derived. From the seed solution, we have obtained solution (22). From BT (20) and soliton solution (22), Wronskian solutions (35) have been constructed and proved to be the solution of (2).

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