Electromagnetic Waves in Variable Media

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Two methods are explained to exactly solve Maxwell's equations where permittivity, permeability, and conductivity may vary in space. In the constitutive relations, retardation is regarded. If the material properties depend but on one coordinate, general solutions are derived. If the properties depend on two coordinates, geometrically restricted solutions are obtained. Applications to graded reflectors, especially to dielectric mirrors, to filters, polarizers, and to waveguides, plain and cylindrical, are indicated. New foundations for the design of optical instruments, which are centered around an axis, and for the design of invisibility cloaks, plain and spherical, are proposed. The variability of material properties makes possible effects which cannot happen in constant media, e.g. stopping the flux of electromagnetic energy without loss. As a consequence, spherical devices can be constructed which bind electromagnetic waves.

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1. Exact Solutions of Maxwell's Equations

This work originated from a scrutiny for the foundations of quantum mechanics. Do the equations, which Schrödinger considered as the description of the propagation of light, rest upon Maxwell's equations? The answer is no. The study produced analytic methods for the exact solution of Maxwell's equations even if permittivity, permeability, and conductivity vary in space. These methods should be useful to everyone concerned with electromagnetic fields. Thus readers interested in basic physics might read Section 1.1, whereas technicians may begin with Section 1.2.

1.1. The Foundations of Quantum Mechanics

Let us trace the way Schrödinger walked to find the Schrödinger equation. He thought that the propagation of light, if it is construed as propagation of particles, is best described by the eikonal equation

$$(\nabla s(\mathbf{r}))^2 = n^2(\mathbf{r})\,. \tag{1}$$

Surfaces of equal eikonal $s(\mathbf{r})$ are perpendicular to the light rays everywhere in the space described by the vector of location $\mathbf{r}$. $n(\mathbf{r})$ is the index of refraction. Schrödinger compared this with the Helmholtz equation

$$\nabla^2\psi(\mathbf{r}) + \frac{n^2(\mathbf{r})}{c^2}\omega^2\psi(\mathbf{r}) = 0 \tag{2}$$

which he considered as the best description of waves. ω denotes the frequency of that wave and c is the velocity of light. The meaning of ψ is not known. Next Schrödinger remembered that there is an eikonal equation for massive particles, too, the Hamilton–Jacobi equation

$$(\nabla S(\mathbf{r}))^2 = 2m(E - V(\mathbf{r}))\,. \tag{3}$$

The mechanical eikonal $S(\mathbf{r})$ has a similar meaning as in ray optics, but its dimension is different. Hence Schrödinger deduced from a comparison of (1) and (3) a mechanical index of refraction

$$n(\mathbf{r}) = \sqrt{\frac{c^2}{\hbar^2\omega}2m(E - V(\mathbf{r}))}\,. \tag{4}$$

The factor in front of $2m(E - V(\mathbf{r}))$ is an adjustable constant to get dimensions right. That it is related to Planck's constant $\hbar$, Schrödinger realized when he solved the first problems. However, when the guess (4) is used in the Helmholtz equation (2), an equation for the wavy propagation of massive particles is established, the Schrödinger equation

$$\nabla^2 \Psi(\mathbf{r}) + \frac{2m}{\hbar^2}(E - V(\mathbf{r}))\Psi(\mathbf{r}) = 0\,. \tag{5}$$

Here also the meaning of Ψ is not clear.

The problem with this type of approach is the Helmholtz equation (2). The wavy propagation of light is reigned by Maxwell's equations. In the analytic solution of these equations, the Helmholtz equation occurs as a mathematical auxiliary [1], but this is only true when material properties as permittivity ε and permeability μ are constant. What is the replacement of the Helmholtz equation if these properties and thus the index of refraction

$$n(\mathbf{r}) = \sqrt{\frac{\varepsilon_0 \mu_0}{\varepsilon(\mathbf{r})\mu(\mathbf{r})}} \tag{6}$$

vary in space? This question will be answered in Sections 3 and 4. The modifications will turn out so severe that the Helmholtz equation (2) and all the more the eikonal equation (1) can be considered only in rare cases as approximations. It is not even possible to formulate the true equations using the index of refraction only. Permittivity and permeability enter individually.

The true companion of Maxwell's equations is Dirac's equation. Maxwell's is for vectors, Dirac's for spinors. Yet both systems carry similar information, namely equations for divergences and curls related to time derivatives. Dirac's equation is a linear system of partial differential equations with variable coefficients, the electrodynamic potentials. In Maxwell's equations, variable coefficients appear when permittivity, permeability, and conductivity depend on location. Dirac's equation can be solved analytically if the coefficients vary just one-dimensionally or if they vary central-symmetrically. The analog for Maxwell's equations, and more, is the main result of this article, see Sections 3.1 and 3.2.

1.2. Two Steps Towards Reality

The impact of this article might be even larger on practical problems. In modern times, people fabricate graded materials or so-called metamaterials within which permittivity and permeability vary in space almost arbitrarily. Therefore analytic solutions that predict effects of such variations will be useful.

Yet usefulness for practicians coerces the consideration of dissipation and dispersion. Most materials have finite conductivity. Ohmic currents must be included in the theory. Moreover inertia and friction within the materials modify permittivity ε, permeability μ, and conductivity σ. The simple constants must be upgraded, in a minimum approach to reality, to response functions which vary in space and describe retardation:

$$\mathbf{D}(\mathbf{r},t) = \int_0^t \varepsilon(\mathbf{r},t-\tau)\mathbf{E}(\mathbf{r},\tau)\,\mathrm{d}\tau\,, \tag{7}$$

$$\mathbf{B}(\mathbf{r},t) = \int_0^t \mu(\mathbf{r},t-\tau)\mathbf{H}(\mathbf{r},\tau)\,\mathrm{d}\tau\,, \tag{8}$$

$$\mathbf{j}(\mathbf{r},t) = \int_0^t \sigma(\mathbf{r},t-\tau)\mathbf{E}(\mathbf{r},\tau)\,\mathrm{d}\tau\,. \tag{9}$$

$\mathbf{D}(\mathbf{r},t)$, $\mathbf{E}(\mathbf{r},t)$, $\mathbf{B}(\mathbf{r},t)$, $\mathbf{H}(\mathbf{r},t)$, and $\mathbf{j}(\mathbf{r},t)$ denote dielectric displacement, electric force field, magnetic force field, magnetic field strength, and electric current density, respectively. They all are vector fields depending on space $\mathbf{r}$ and time t.

With the constitutive relations (7) – (9) the evolution of the electrodynamic field is completely conceived by Maxwell's equations

$$\nabla \times \mathbf{E}(\mathbf{r},t) = -\partial_t \mathbf{B}(\mathbf{r},t)\,, \tag{10}$$

$$\nabla \mathbf{B}(\mathbf{r},t) = 0\,, \tag{11}$$

$$\nabla \times \mathbf{H}(\mathbf{r},t) = \partial_t \mathbf{D}(\mathbf{r},t) + \mathbf{j}(\mathbf{r},t)\,, \tag{12}$$

$$\nabla \mathbf{D}(\mathbf{r},t) = \rho(\mathbf{r},t)\,, \tag{13}$$

written in an unfamiliar sequence for reasons that will become clear in Section 2.

So these are the two steps to reality: First, solutions of Maxwell's equations shall be found with material properties that vary in space. Second, retardation shall be taken into account.

Yet generality will be restricted in two ways: First, only the homogeneous problem will be tackled. For example, externally driven currents will be omitted. The reason to keep nevertheless the charge density $\rho(\mathbf{r},t)$ and the current density $\mathbf{j}(\mathbf{r},t)$ in Maxwell's equations is to admit Ohmic currents. The exclusion of nonhomogeneities is not a serious limitation as there are standard procedures to construct the solutions of nonhomogeneous equations from the solutions of the homogeneous system.

By contrast, the second lack cannot be cured and can be justified only by the desire to produce exact solutions of Maxwell's equations: All material properties will be restricted to depend on one or two spatial coordinates only. To be specific, introduce coordinates ξ,η,ζ to describe the vector of position $\mathbf{r}$. They may be the Cartesian coordinates x,y,z, but generally these greek letters are meant to describe curvilinear yet orthogonal coordinates, for example spherical or cylindrical ones. The normalized basis vectors shall be denoted as $\mathbf{e}_\xi$, $\mathbf{e}_\eta$, $\mathbf{e}_\zeta$ and the line element $\mathrm{d}s$ be given as

$$(\mathrm{d}s)^2 = g_{\xi\xi}(\mathrm{d}\xi)^2 + g_{\eta\eta}(\mathrm{d}\eta)^2 + g_{\zeta\zeta}(\mathrm{d}\zeta)^2 \quad (14)$$

with elements $g_{\xi\xi}$, $g_{\eta\eta}$, $g_{\zeta\zeta}$ of the metric tensor [2]. The response functions are supposed to depend on ζ only,

$$\varepsilon(\zeta,t),\ \ \mu(\zeta,t),\ \ \sigma(\zeta,t), \quad (15)$$

see Section 3, or only on η and ζ,

$$\varepsilon(\eta,\zeta,t),\ \ \mu(\eta,\zeta,t),\ \ \sigma(\eta,\zeta,t), \quad (16)$$

see Section 4. In the second case (16), which appears to be more general, we will have to impose restrictions upon the solutions of Maxwell's equations. Nevertheless the realm of exactly solvable problems will be extended immensely.

This is the plan of this article: In Section 2 Maxwell's and the constitutive equations will be rewritten to facilitate a simple description of retardation. In Sections 3 and 4 the main results will be produced and proven, namely two theorems of representation. They reduce the eight mingled Maxwellian equations to two uncoupled partial differential equations each for one unknown only. Applications of these theorems are sketched in the Sections 3.1, 3.1.1, 3.1.2, 3.1.3, 3.2, 3.2.1, 4.1, and 4.2. Finally in Section 5 attempts are made to do justice to precursors of the ideas and the results presented here.

2. Reshaping Maxwell's Equations

The first two equations (10) and (11) are ideally simple. The aim is to rewrite the last two equations (12) and (13) until they take the same shape as the first two. The clue is the *complete displacement*

$$\mathbf{C}(\mathbf{r},t) = \mathbf{D}(\mathbf{r},t) + \int_0^t \mathbf{j}(\mathbf{r},\tau)\,\mathrm{d}\tau\,. \quad (17)$$

This takes (12) to

$$\nabla\times\mathbf{H}(\mathbf{r},t) = \partial_t\mathbf{C}(\mathbf{r},t) \quad (18)$$

having up to a sign the same structure as (10). Similarly (13) appears as

$$\nabla\mathbf{C}(\mathbf{r},t) = 0 \quad (19)$$

having the exactly same structure as (11). Because of the continuitity equation

$$\rho(\mathbf{r},t) = -\nabla\int_0^t \mathbf{j}(\mathbf{r},\tau)\,\mathrm{d}\tau + \rho(\mathbf{r},0) \quad (20)$$

(19) is true if there is no initial bunching of charges $\rho(\mathbf{r},0)=0$. The effects of an initial bunching of charges can be covered by a scalar potential in a manner explained in [1, Sec. 2]. It is not extraordinary enough to be treated here.

The reshaped Maxwell equations are now (10) and (11) and (18) and (19). To close the system, we have to combine from (7) and (9) the constitutive equation for the complete displacement. It is

$$\mathbf{C}(\mathbf{r},t) = \int_0^t \varepsilon(\mathbf{r},t-\tau)\mathbf{E}(\mathbf{r},\tau)\,\mathrm{d}\tau \quad (21)$$

$$\varepsilon(\mathbf{r},t) = \varepsilon(\mathbf{r},t) + \int_0^t \sigma(\mathbf{r},\tau)\,\mathrm{d}\tau \quad (22)$$

with the *complete permittivity* $\varepsilon(\mathbf{r},t)$.

The reshaped Maxwell equations (10) and (11) and (18) and (19) together with the constitutive equations (8) and (21) form a closed system, but it is a system of integro-differential equations. Performing Laplace transforms [3]

$$f_\omega(\mathbf{r}) = \int_0^\infty f(\mathbf{r},t)\,\mathrm{e}^{\mathrm{i}\omega t}\,\mathrm{d}t \quad (23)$$

where $f(\mathbf{r},t)$ may denote any component of the vector fields or any response function, one gets rid of the integrals. The convolution theorem converts the integrals in the constitutive relations (8) and (21) to products:

$$\mathbf{B}_\omega(\mathbf{r}) = \mu_\omega(\mathbf{r})\,\mathbf{H}_\omega(\mathbf{r})\,, \quad (24)$$

$$\mathbf{C}_\omega(\mathbf{r}) = \varepsilon_\omega(\mathbf{r})\,\mathbf{E}_\omega(\mathbf{r})\,. \quad (25)$$

The Laplace transform of the complete permittivity follows from (22):

$$\varepsilon_\omega(\mathbf{r}) = \varepsilon_\omega(\mathbf{r}) + \mathrm{i}\sigma_\omega(\mathbf{r})/\omega\,. \quad (26)$$

It is just a linear combination of the Laplace transforms of the ordinary permittivity and the conductivity.

Quite a few microscopic models feature the dependence of permeability and complete permittivity on ω. The deviations from the vacuum values are attributed to atoms, especially to the electrons surrounding them [4, Chaps. 3.5, 4.8]. In microwave and optics applications, the atoms are much smaller than the wavelengths of the electromagnetic radiation. Thus these theories are local. One can use the results of these models in (24) – (26) to establish permittivities and permeabilities which depend on ω and on $\mathbf{r}$.

By the Laplace transform one also gets rid of the derivatives with respect to time t:

$$\int_0^\infty \partial_t f(\mathbf{r},t)\,\mathrm{e}^{\mathrm{i}\omega t}\,\mathrm{d}t = -\mathrm{i}\omega f_\omega(\mathbf{r}) - f(\mathbf{r},0)\,. \tag{27}$$

The second term on the right-hand side is a valuable peculiarity of the Laplace transform as it facilitates straightforward solutions of initial-value problems.

The Maxwell equations (10) and (11) and (18) and (19) are transformed to

$$\nabla\times\mathbf{E}_\omega(\mathbf{r}) = \mathrm{i}\omega\,\mathbf{B}_\omega(\mathbf{r})\,, \tag{28}$$

$$\nabla\mathbf{B}_\omega(\mathbf{r}) = 0\,, \tag{29}$$

$$\nabla\times\mathbf{H}_\omega(\mathbf{r}) = -\mathrm{i}\omega\,\mathbf{C}_\omega(\mathbf{r})\,, \tag{30}$$

$$\nabla\mathbf{C}_\omega(\mathbf{r}) = 0\,. \tag{31}$$

Here two other nonhomogeneities were omitted, viz. $\mathbf{B}(\mathbf{r},0)$ and $-\mathbf{C}(\mathbf{r},0)$ on the right-hand sides of (28) and (30) which arise from the Laplace transforms of $-\partial_t\mathbf{B}(\mathbf{r},t)$ and $\partial_t\mathbf{C}(\mathbf{r},t)$, respectively, according to (27).

Most people would consider the reshaped Maxwell equations (28) – (31) as obtained from the orginal Maxwell equations just by separation of $\exp(-\mathrm{i}\omega t)$. Yet this point of view hides the origin of the permittivity $\varepsilon_\omega(\mathbf{r})$ and permeability $\mu_\omega(\mathbf{r})$ depending on *frequency* and it aggravates the solution of initial-value problems, i.e. it impedes a rational theory of pulses. To do this, one has, first, to solve the reshaped Maxwell equations (28) – (31) with the reshaped constitutive relations (24) and (25), second, to introduce the initial-values as nonhomogeneities in (28) and (30) and to solve the nonhomogeneous system and, third, to calculate the pulses from the inverse Laplace transform

$$f(\mathbf{r},t) = \frac{1}{2\pi}\int_{-\infty+\mathrm{i}r}^{\infty+\mathrm{i}r} f_\omega(\mathbf{r})\,\mathrm{e}^{-\mathrm{i}\omega t}\,\mathrm{d}\omega\,. \tag{32}$$

r denotes a real number big enough such that all locations of singularities of $f_\omega(\mathbf{r})$ in the complex plane of ω have smaller real parts.

The customary variable of Laplace transforms is $p = \mathrm{i}\omega$. The author introduced ω instead in order to pacify conservative readers. It they want to believe that the equations (28) – (31) are just the ordinary Maxwell equations with $\exp(-\mathrm{i}\omega t)$ separated off, they can do so. The calculations to be presented right now, however, do not depend on this point of view. So let us abbreviate:

$$\begin{aligned}&\mathbf{C} = \mathbf{C}_\omega(\mathbf{r}),\ \mathbf{E} = \mathbf{E}_\omega(\mathbf{r}),\ \varepsilon = \varepsilon_\omega(\mathbf{r}),\\ &\mathbf{B} = \mathbf{B}_\omega(\mathbf{r}),\ \mathbf{H} = \mathbf{H}_\omega(\mathbf{r}),\ \mu = \mu_\omega(\mathbf{r}).\end{aligned} \tag{33}$$

The emphasis will be to obtain analytic solutions of Maxwell's equations, i.e. permittivity and permeability will be parametrized, and the dependences on these parameters will appear in the solutions explicitly. Generally the parameters will be functions of ω. Hence the dependence on ω can be taken into account by straightforward algebraic insertion after an analytic solution is found.

3. Triple Curl Again

The aim is to reduce all Maxwell equations (28) – (31) to one partial differential equation for one scalar auxiliary, the representative $b = b_\omega(\mathbf{r})$. The approach is the same as in [1, Sec. 2] looking for certain equations with triple curl.

The ansatz

$$\mathbf{B} = -\nabla\times(\nabla\times\mathbf{v}b)\,, \tag{34}$$

$$\mathbf{E} = -\mathrm{i}\omega\nabla\times\mathbf{v}b \tag{35}$$

solves two Maxwell equations immediately, viz. (28) and (29). The vector field $\mathbf{v}$, the *carrier*, shall be chosen such that the remaining two equations (30) and (31), too, can be solved. Such a choice will be possible if the response functions depend only on one spatial variable, say ζ, as declared in (15), and thus

$$\varepsilon = \varepsilon_\omega(\zeta),\ \ \mu = \mu_\omega(\zeta). \tag{36}$$

Inserting the constitutive relations (24) and (25) into the ansatz (34) and (35) gives

$$\mathbf{H} = -\frac{1}{\mu}\nabla\times(\nabla\times\mathbf{v}b)\,, \tag{37}$$

$$\mathbf{C} = -\mathrm{i}\omega\nabla\times\mathbf{v}\varepsilon b\,. \tag{38}$$

In the last equation the permittivity ε was drawn under the curl though it is not constant. This can be justified if the carrier $\mathbf{v}$ is chosen to point into the direction of the basis vector $\mathbf{e}_\zeta$.

$$\mathbf{v} = |\mathbf{v}|\mathbf{e}_\zeta\,. \tag{39}$$

Then, because of (36), the carrier points into the same direction as the gradient of ε

$$\nabla\varepsilon = \frac{\mathbf{e}_\zeta}{\sqrt{g_{\zeta\zeta}}}\frac{\partial\varepsilon}{\partial\zeta} = \frac{\mathbf{v}}{|\mathbf{v}|\sqrt{g_{\zeta\zeta}}}\frac{\mathrm{d}\varepsilon}{\mathrm{d}\zeta}\,. \tag{40}$$

Consequently in the identity

$$\varepsilon\nabla\times\mathbf{v}b = \nabla\times\mathbf{v}\varepsilon b + \mathbf{v}b\times\nabla\varepsilon \tag{41}$$

the last term is zero and thus (38) proven.

Because of (38), Maxwell's equation (31) is automatically fulfilled, too. So we just have to care for (30). Insertion of (37) and (38) produces an equation of triple curl

$$\nabla\times\left(-\frac{1}{\mu}\nabla\times(\nabla\times\mathbf{v}b) + \mathbf{v}\varepsilon\omega^2 b\right) = 0\,. \tag{42}$$

The second term behind the leading curl is proportional to the carrier. All that remains to be done in order to obtain the desired scalar equation is to show that the first term is a gradient plus a term which also aligns with the carrier. First, we replace the double curl with the Laplacian ∇^2:

$$-\frac{1}{\mu}\nabla\times(\nabla\times\mathbf{v}b) = \frac{1}{\mu}(\nabla^2\mathbf{v}b - \nabla(\nabla\mathbf{v}b))\,. \tag{43}$$

Second, we commute the carrier $\mathbf{v}$ with the Laplacian and require that the commutation does not produce more than a gradient. This can be done only if the carrier varies at most linearly

$$\mathbf{v} = \mathbf{v}_0 + v_1\mathbf{r} \;\Leftrightarrow\; \nabla^2\mathbf{v}b = \mathbf{v}\nabla^2 b + 2v_1\nabla b\,, \tag{44}$$

$\mathbf{v}_0$ denoting a constant vector and v_1 a constant number. The equivalence is valid only if the dependence of b on the coordinates is not restricted. For details of the proof see [5] or [6]. If μ were constant, we had completed the task. Then the second terms on the right-hand sides of (43) and (44) were gradients which the leading curl in (42) would discard. When μ varies, we must effect a third transformation

$$\begin{aligned}\frac{1}{\mu}(-\nabla(\nabla\mathbf{v}b) + 2v_1\nabla b) &= \left(\nabla\frac{1}{\mu}\right)(\nabla\mathbf{v}b - 2v_1 b)\\ &-\nabla\frac{1}{\mu}(\nabla\mathbf{v}b - 2v_1 b)\,.\end{aligned} \tag{45}$$

Here, at last, the second term on the right-hand side is a gradient, while the first aligns with the carrier because of (36). Using again (39), we find

$$\nabla\frac{1}{\mu} = \frac{\mathbf{v}}{|\mathbf{v}|\sqrt{g_{\zeta\zeta}}}\frac{\mathrm{d}}{\mathrm{d}\zeta}\frac{1}{\mu} \tag{46}$$

similar to (40). Collecting (43), (44), and (45), we derive from (42) the scalar equation

$$\begin{aligned}&\nabla^2 b + \frac{\mu}{|\mathbf{v}|\sqrt{g_{\zeta\zeta}}}\left(\frac{\mathrm{d}}{\mathrm{d}\zeta}\frac{1}{\mu}\right)(\nabla\mathbf{v}b - 2v_1 b)\\ &+\varepsilon\mu\omega^2 b = 0\,.\end{aligned} \tag{47}$$

To find formulae for the other polarization, we impose the ansatz

$$\mathbf{C} = \nabla\times(\nabla\times\mathbf{v}a)\,, \tag{48}$$

$$\mathbf{H} = -\mathrm{i}\omega\nabla\times\mathbf{v}a \tag{49}$$

with the representative $a = a_\omega(\mathbf{r})$, which automatically satisfies all Maxwell equations except (28). The demand to have also this one solved produces a triple-curl equation similar to (42). a replaces b, ε and μ are interchanged. Performing the same transformations as before, we arrive at a scalar equation similar to (47).

Thus we finished the proof of the

Three-Dimensional Representation Theorem. *Solutions of Maxwell's equations (28)–(31) are provided by the representations*

$$\mathbf{E} = \frac{1}{\varepsilon}\nabla\times(\nabla\times\mathbf{v}a) - \mathrm{i}\omega\nabla\times\mathbf{v}b\,, \tag{50}$$

$$\mathbf{H} = -\mathrm{i}\omega\nabla\times\mathbf{v}a - \frac{1}{\mu}\nabla\times(\nabla\times\mathbf{v}b) \tag{51}$$

if the representatives a and b obey the scalar partial differential equations

$$\nabla^2 a - \frac{1}{\mathbf{v}^2}(\mathbf{v}\nabla\log\varepsilon)(\nabla\mathbf{v}a - 2v_1 a) + \varepsilon\mu\omega^2 a = 0\,, \tag{52}$$

$$\nabla^2 b - \frac{1}{\mathbf{v}^2}(\mathbf{v}\nabla\log\mu)(\nabla\mathbf{v}b - 2v_1 b) + \varepsilon\mu\omega^2 b = 0\,, \tag{53}$$

and the carrier is chosen such that

$$\mathbf{v} = \mathbf{v}_0 + v_1\mathbf{r} \text{ and } \nabla\varepsilon \propto \mathbf{v} \text{ and } \nabla\mu \propto \mathbf{v}\,, \tag{54}$$

$\mathbf{v}_0$ being a constant vector and v_1 a constant number.

In most optical instruments, different media meet. Often a graded medium is surrounded by air. Usually material properties jump discontinously where the media touch. Therefore boundary conditions for the representatives are necessary. The conditions are especially simple when permittivity and permeability take constant values on the boundary. In this case the normal vector $\mathbf{n}$ of the boundary aligns with the carrier $\mathbf{v}$.

Corollary on Boundary-Value Conditions. *Let S denote the surface where different media meet, $\mathbf{n}$ the normal on this surface, and $\partial/\partial n$ the differentiation along this normal. If $\mathbf{v} \propto \mathbf{n}$, the representatives a and b must satisfy*

$$a|_{S-} = a|_{S+}, \quad \frac{1}{\varepsilon}\frac{\partial|\mathbf{v}|a}{\partial n}\bigg|_{S-} = \frac{1}{\varepsilon}\frac{\partial|\mathbf{v}|a}{\partial n}\bigg|_{S+}, \tag{55}$$

$$b|_{S-} = b|_{S+}, \quad \frac{1}{\mu}\frac{\partial|\mathbf{v}|b}{\partial n}\bigg|_{S-} = \frac{1}{\mu}\frac{\partial|\mathbf{v}|b}{\partial n}\bigg|_{S+}. \tag{56}$$

The symbols $S-$ and $S+$ indicate that the values of the functions and their derivatives are to be calculated via an approach on the one side of S, say, the low side $S-$, or on the other side, say, the high side $S+$.

Proof. It follows from the structure of the Maxwell equations (28) and (30) that the tangential components of the magnetic field strength $\mathbf{H}$ and the electric force field $\mathbf{E}$ don't jump on transition through S. Evaluating these facts in the representation formulae (50) and (51) produces the proof of the corollary. Details of the calculational procedure are similar as in [1, Sec. 3]. □

In optics it is difficult to observe the electromagnetic field directly. Instead one measures the flux of energy which can be calculated as the Pointing vector $\mathbf{S}(\mathbf{r},t) = \mathbf{E}(\mathbf{r},t) \times \mathbf{H}(\mathbf{r},t)$. This is a general formula for physical fields $\mathbf{E}(\mathbf{r},t)$ and $\mathbf{H}(\mathbf{r},t)$. Before we can use it for the mathematical fields handled here, we must calculate the dependence on time from (32) and extract the real parts Re. However, if the dependence on time can be described by the factor $\exp(-\mathrm{i}\,\omega t)$ with real frequency ω, we may apply the

Corollary on the Energy Flux. *The time-averaged Pointing vector $\bar{\mathbf{S}}$ can be calculated from the representatives a and b according to*

$$\begin{aligned}
\bar{\mathbf{S}} &= \frac{1}{2}\mathrm{Re}(\mathbf{E} \times \mathbf{H}^*) \qquad (57)\\
&= \mathrm{Re}\left(\frac{\mathrm{i}\,\omega}{2\varepsilon}(\nabla \times (\nabla \times \mathbf{v}a)) \times (\nabla \times \mathbf{v}a^*)\right)\\
&\quad + \mathrm{Re}\left(\frac{\mathrm{i}\,\omega}{2\mu}(\nabla \times \mathbf{v}b) \times (\nabla \times (\nabla \times \mathbf{v}b^*))\right)\\
&\quad + \mathrm{Re}\left(\frac{\omega^2}{2}(\nabla \times \mathbf{v}b) \times (\nabla \times \mathbf{v}a^*)\right)\\
&\quad - \mathrm{Re}\left(\frac{1}{2\varepsilon\mu}(\nabla\times(\nabla\times\mathbf{v}a))\times(\nabla\times(\nabla\times\mathbf{v}b^*))\right),
\end{aligned}$$

the asterisk * denoting complex conjugation.

The proof follows immediately from the representation formulae (50) and (51).

When there is only one polarization, i.e. either $a = 0$ or $b = 0$, the mixed terms in the third and the forth lines of (57) do not apply. In the first and second lines, reader's attention shouldn't miss the inconspicuous imaginary units i and the factors $1/\varepsilon$ as well as $1/\mu$. The former are indispensable for a weird stop of energy flux in dielectric materials, whereas the latter may alter the type of the flux considerably when they are non-constant.

Due to the condition (54), two limiting cases stand out, namely when the material properties vary one-dimensionally, see Section 3.1, or centralsymmetrically, see Section 3.2.

3.1. One-Dimensional Variations of Material Properties

z be the name of the coordinate along which permittivity, permeability, and conductivity are allowed to vary. It is the same z which is costumary in the cartesian system and all cylindrical coordinate systems. In the equations of Sections 1 and 3, we got to set $\zeta = z$. The unit vector along z is chosen as carrier, i.e. $\mathbf{v} = \mathbf{e}_z$. According to condition (54), we have $\mathbf{v}_0 = \mathbf{e}_z$ and $v_1 = 0$. The differential equations (52) and (53) become

$$\nabla^2 a - \frac{\mathrm{d}\log\varepsilon}{\mathrm{d}z}\frac{\partial a}{\partial z} + \varepsilon\mu\omega^2 a = 0, \tag{58}$$

$$\nabla^2 b - \frac{\mathrm{d}\log\mu}{\mathrm{d}z}\frac{\partial b}{\partial z} + \varepsilon\mu\omega^2 b = 0. \tag{59}$$

These differential equations are valid in all coordinate systems which incorporate a cartesian direction.

The most elementary example is the cartesian system x, y, z. (58) appears as

$$\begin{aligned}
&\frac{\partial^2 a}{\partial x^2} + \frac{\partial^2 a}{\partial y^2} + \frac{\partial^2 a}{\partial z^2} - \frac{\mathrm{d}\log\varepsilon}{\mathrm{d}z}\frac{\partial a}{\partial z}\\
&+ \varepsilon\mu\omega^2 a = 0,
\end{aligned} \tag{60}$$

which is separated by the ansatz

$$a = XYZ_a \text{ with } X = X(x), \quad Y = Y(y),\ Z_a = Z_{a\omega}(z) \tag{61}$$

to yield three ordinary differential equations

$$\frac{\mathrm{d}^2X}{\mathrm{d}x^2} + k_x^2 X = 0\,, \tag{62}$$

$$\frac{\mathrm{d}^2Y}{\mathrm{d}y^2} + k_y^2 Y = 0\,, \tag{63}$$

$$\frac{\mathrm{d}^2Z_a}{\mathrm{d}z^2} - \frac{\mathrm{d}\log\varepsilon}{\mathrm{d}z}\frac{\mathrm{d}Z_a}{\mathrm{d}z} + (\varepsilon\mu\,\omega^2 - k^2)Z_a = 0 \tag{64}$$

with separation constants k_x and k_y meaning physically wave numbers and with $k^2 = k_x^2 + k_y^2$. The solutions of the first two equations (62) and (63) are, of course, exponentials or sines and cosines

$$a = \mathrm{e}^{\mathrm{i}(k_x x + k_y y)} Z_a\,, \tag{65}$$

but the solutions of (64) can be weird and convey physical information never considered before.

According to (59), the representative b is subject to another partial differential equation

$$\frac{\partial^2 b}{\partial x^2} + \frac{\partial^2 b}{\partial y^2} + \frac{\partial^2 b}{\partial z^2} - \frac{\mathrm{d}\log\mu}{\mathrm{d}z}\frac{\partial b}{\partial z} + \varepsilon\mu\omega^2 b = 0\,. \tag{66}$$

When this is separated using the ansatz $b = XYZ_b$ similar to (61), the ordinary differential equations for X and Y are the same as (62) and (63), respectively, but the differential equation for Z_b,

$$\frac{\mathrm{d}^2Z_b}{\mathrm{d}z^2} - \frac{\mathrm{d}\log\mu}{\mathrm{d}z}\frac{\mathrm{d}Z_b}{\mathrm{d}z} + (\varepsilon\mu\,\omega^2 - k^2)Z_b = 0 \tag{67}$$

is different from (64). This reflects a responsiveness of graded materials to polarization. Examples will be discussed in the next sections.

It should never be forgotten that the partial differential equations (58) and (59) hold in any cylindrical coordinate system. For a less trivial example let us select elliptic-cylinder coordinates ξ, η, ζ [2, Sec. 1]

$$x = c\cosh\xi\cos\eta\,, \tag{68}$$

$$y = c\sinh\xi\sin\eta\,, \tag{69}$$

$$z = \zeta\,, \tag{70}$$

c being a positive constant. The ordinary differential equations obtained by separation of

$$a \text{ or } b = \Xi\,\mathrm{H}\,Z_{a \text{ or } b} \text{ with } \Xi = \Xi(\xi), \quad \mathrm{H} = \mathrm{H}(\eta),\ Z_{a \text{ or } b} = Z_{a\omega \text{ or } b\omega}(z) \tag{71}$$

are

$$\frac{\mathrm{d}^2\Xi}{\mathrm{d}\xi^2} - (q - k^2c^2\cosh^2\xi)\Xi = 0\,, \tag{72}$$

$$\frac{\mathrm{d}^2\mathrm{H}}{\mathrm{d}\eta^2} + (q - k^2c^2\cos^2\eta)\mathrm{H} = 0\,. \tag{73}$$

The separation constants are here q and k^2. The solutions of these two equations are Mathieu functions [7]. The third equation was not written because it is identical with (64) or (67). The elliptic-cylinder coordinates are especially interesting as they allow to exactly predict the diffraction by a strip or a slit, not just by an edge, see [1, Secs. 10,11] and the references therein. Moreover in the theory presented here, the strip need not to be homogeneous. So we can devise novel ways to bunching and debunching of electromagnetic waves.

3.1.1. Graded Reflectors, Transmitters, and Polarizers

As a first example of application, consider a medium homogenous and isotropic for $z < 0$:

$$\varepsilon = \varepsilon_-\,,\ \mu = \mu_- \tag{74}$$

with constant permittivity ε_- and permeability μ_-. The solutions of the differential equations (60) and (66) are almost trivial:

$$a \text{ or } b = \mathrm{e}^{\mathrm{i}(k_x x + k_y y)} Z_{a \text{ or } b} \tag{75}$$

with $Z_{a \text{ or } b}$ depending only on z,

$$Z_{a \text{ or } b} = \mathrm{e}^{\mathrm{i}\sqrt{\varepsilon_-\mu_-\omega^2 - k^2}\,z} + R_{a \text{ or } b}\,\mathrm{e}^{-\mathrm{i}\sqrt{\varepsilon_-\mu_-\omega^2 - k^2}\,z}, \tag{76}$$

and the abbreviation $k^2 = k_x^2 + k_y^2$. These solutions describe waves of different polarization intruding from negative infinity and being partially reflected at $z = 0$. R_a and R_b are complex constants to fix the strengths and phases of the reflected waves. These constants have to be determined from the solution of the boundary-value problem; see below.

For $z > 0$, the permittivity is supposed to vary

$$\varepsilon = \frac{\varepsilon_\infty}{1 - (1 - \varepsilon_\infty/\varepsilon_+)\exp(-z/z_+)}\,,\quad \mu = \mu_+\,, \tag{77}$$

whereas the permeability is supposed to stay constant μ_+. The permittivity (77) takes the value ε_+ at $z = 0$

and increases or decreases with constant positive decay length z_+ while it approaches ε_∞ for $z \to \infty$. Again we have solutions of the differential equations (60) and (66) as in (75), but the functions $Z_{a\text{ or }b}$ are different. For $z > 0$ the ordinary differential equations (64) and (67) both can be solved in terms of the hypergeometric function

$$Z_{a\text{ or }b} = T_{a\text{ or }b}\, \mathrm{e}^{\mathrm{i}\sqrt{\varepsilon_\infty \mu_+ \omega^2 - k^2}\, z} \cdot F(\alpha_{a\text{ or }b}, \beta_{a\text{ or }b}, \gamma_{a\text{ or }b}, (1-\varepsilon_\infty/\varepsilon_+)\exp(-z/z_+))\,. \tag{78}$$

T_a and T_b are complex constants to seize the strengths and phases of the transmitted wave to be determined from the boundary conditions, see below. For Z_a, we have to use the parameters

$$\begin{aligned}
\alpha_a &= 1/2\left(1+\sqrt{1+4k^2z_+^2}\right) - \mathrm{i}\sqrt{\varepsilon_\infty\mu_+\omega^2-k^2}\, z_+\,,\\
\beta_a &= 1/2\left(1-\sqrt{1+4k^2z_+^2}\right) - \mathrm{i}\sqrt{\varepsilon_\infty\mu_+\omega^2-k^2}\, z_+\,,\\
\gamma_a &= 1-2\mathrm{i}\sqrt{\varepsilon_\infty\mu_+\omega^2-k^2}\, z_+\,,
\end{aligned} \tag{79}$$

but for Z_b

$$\begin{aligned}
\alpha_b &= +kz_+ - \mathrm{i}\sqrt{\varepsilon_\infty\mu_+\omega^2-k^2}\, z_+\,,\\
\beta_b &= -kz_+ - \mathrm{i}\sqrt{\varepsilon_\infty\mu_+\omega^2-k^2}\, z_+\,,\\
\gamma_b &= 1-2\mathrm{i}\sqrt{\varepsilon_\infty\mu_+\omega^2-k^2}\, z_+\,.
\end{aligned} \tag{80}$$

The very fact that the parameters in (79) and (80) differ shows that graded materials act discriminatorily towards waves with different polarization.

There are second solutions of the second-degree equations (64) and (67), but we don't need them here. Namely

$$F(\alpha,\beta,\gamma,(1-\varepsilon_\infty/\varepsilon_+)\exp(-z/z_+)) \to 1 \quad \text{for } z \to \infty \tag{81}$$

is a property of the hypergeometric function for all values of α, β, γ. Therefore (78) is sufficient to describe a wave running to positive infinity.

The four coefficients R_a, T_a and R_b, T_b are determined by the conditions at the boundary S. The surface S is given here as $z = 0$. The normal $\mathbf{n}$ coincides with the vector $\mathbf{e}_z$. Specializing (55) and (56) yields

$$\begin{aligned}
&Z_a|_{z=-0} = Z_a|_{z=+0}\,,\\
&\frac{1}{\varepsilon}\frac{\mathrm{d}Z_a}{\mathrm{d}z}\bigg|_{z=-0} = \frac{1}{\varepsilon}\frac{\mathrm{d}Z_a}{\mathrm{d}z}\bigg|_{z=+0}\,,
\end{aligned} \tag{82}$$

$$\begin{aligned}
&Z_b|_{z=-0} = Z_b|_{z=+0}\,,\\
&\frac{1}{\mu}\frac{\mathrm{d}Z_b}{\mathrm{d}z}\bigg|_{z=-0} = \frac{1}{\mu}\frac{\mathrm{d}Z_b}{\mathrm{d}z}\bigg|_{z=+0}\,.
\end{aligned} \tag{83}$$

The boundary conditions (82) and (83) yield both two linear equations for R_a, T_a and R_b, T_b, respectively, and are solvable elementarily. Next one differentiates the electromagnetic field from the representatives (75) according to (50) and (51) and calculates from the electromagnetic field the Pointing vector to obtain quantities directly comparable to experimental results.

These formulae for the reflection and transmission coefficients are generalizations of Fresnel's equations. The classical case is recovered when surface effects are wiped out, i.e. $\varepsilon_+ = \varepsilon_\infty$ in (77). Absorption is taken into account if one admits complex ε_-, ε_+, and ε_∞; see (26) for the physical meaning of the complex values. The properties of Zenneck waves alias surface plasmon polaritons [8] can be derived from the poles of R_a, T_a or R_b, T_b because these waves are eigenmodes. Equivalently one may omit the leading term on the right-hand side of (76) and solve the eigenvalue problem posed by (82) or (83) directly. The solutions wherein the imaginary parts of $\sqrt{\varepsilon_-\mu_-\omega^2-k^2}$ and $\sqrt{\varepsilon_\infty\mu_+\omega^2-k^2}$ both acquire positive values describe waves clinging to the boundary $z = 0$.

Several modifications are at hand. The waves may run in the opposite directions. In this case the second solutions of (64) and (67) are needed to describe reflected waves. They too are expressible in terms of the hypergeometric function. We may also study the transition of waves from one graded medium to other graded materia. In this case the purely exponential waves (76) must be replaced with expressions containing hypergeometric functions.

In the definition of gradation (77), it was assumed that only the permittivity varies. This is resonable for many materials, but for independent variations both of permittivity and permeability it is advisable to use other parametrizations. Power laws, for example,

$$\varepsilon = \varepsilon_\alpha z^\alpha\,,\quad \mu = \mu_\beta z^\beta \quad \text{with } \alpha+\beta = -2,-1,0,2 \tag{84}$$

lead to solvable differential equations (64) and (67) - solvable by functions not more complicated than the confluent hypergeometric function.

Another useful variability is

$$\varepsilon = \varepsilon_0 \exp\frac{z}{z_\varepsilon}\,,\quad \mu = \mu_0 \exp\frac{z}{z_\mu} \tag{85}$$

with positive constants ε_0, μ_0 and positive or negative constant decay lengths z_ε and z_μ. Apt solutions of (64) and (67) are Hankel functions $H_\nu^{(1)}$ and $H_\nu^{(2)}$ of weird index and weird argument:

$$Z_a = \exp\left(\frac{z}{2z_\varepsilon}\right) \cdot H_\nu^{(1,2)}\left(\frac{2z_\varepsilon z_\mu \sqrt{\varepsilon_0\mu_0}\,\omega}{z_\varepsilon + z_\mu} \exp\frac{(z_\varepsilon + z_\mu)z}{2z_\varepsilon z_\mu}\right) \quad \text{with } \nu = \frac{z_\mu\sqrt{1+4k^2 z_\varepsilon^2}}{z_\varepsilon + z_\mu}, \tag{86}$$

$$Z_b = \exp\left(\frac{z}{2z_\mu}\right) \cdot H_\nu^{(1,2)}\left(\frac{2z_\varepsilon z_\mu \sqrt{\varepsilon_0\mu_0}\,\omega}{z_\varepsilon + z_\mu} \exp\frac{(z_\varepsilon + z_\mu)z}{2z_\varepsilon z_\mu}\right) \quad \text{with } \nu = \frac{z_\varepsilon\sqrt{1+4k^2 z_\mu^2}}{z_\varepsilon + z_\mu}. \tag{87}$$

Hankel functions, also denoted as Bessel functions of third kind [9, Chap. III], are more appropriate for the description of running waves than ordinary Bessel and Neumann functions.

The parametrizations (84) or (85) should be applied to slabs as there is no medium with infinitely large or infinitely small material properties. Instead of one block of boundary-value conditions as in (82) and (83) there should be two blocks, one for a boundary at, say, $z = z_1$ and the other block for a surface at $z = z_2$. The somewhat larger linear systems do not essentially aggravate the solution of the entire problem.

3.1.2. Stopping the Energy Flux

A peculiar special case of (85) is

$$\varepsilon = \varepsilon_0 \exp\frac{z}{z_\varepsilon}, \quad \mu = \mu_0 \exp\frac{-z}{z_\varepsilon}. \tag{88}$$

The index of refraction (6) is just 1. Believers in the Helmholtz equation (2) should expect just ordinary propagation of waves, but no spectacular effect.

The true differential equations (64) and (67) have constant coefficients:

$$\frac{d^2 Z_a}{dz^2} - \frac{1}{z_\varepsilon}\frac{dZ_a}{dz} + (\varepsilon_0\mu_0\omega^2 - k^2)Z_a = 0, \tag{89}$$

$$\frac{d^2 Z_b}{dz^2} + \frac{1}{z_\varepsilon}\frac{dZ_b}{dz} + (\varepsilon_0\mu_0\omega^2 - k^2)Z_b = 0. \tag{90}$$

Hence their solutions are readily found:

$$Z_a = \exp\left(\frac{z}{2z_\varepsilon}\right) \cdot \exp\left(\pm\sqrt{\frac{1}{4z_\varepsilon^2} - (\varepsilon_0\mu_0\omega^2 - k^2)}\,z\right), \tag{91}$$

$$Z_b = \exp\left(\frac{-z}{2z_\varepsilon}\right) \cdot \exp\left(\pm\sqrt{\frac{1}{4z_\varepsilon^2} - (\varepsilon_0\mu_0\omega^2 - k^2)}\,z\right). \tag{92}$$

The exponential functions directly behind the equal signs impress much, but for the energy flux they don't matter at all. They only exist to compensate the factors $1/\varepsilon$ and $1/\mu$ in (57). The astonishing item is the square root in the second exponentials. Normally, i.e. for $z_\varepsilon \to \infty$ as it holds for any almost homogeneous medium, the value of the root is imaginary and the exponential function carrying it is complex. It is just the description of a plane wave propagating we are used to. Yet if

$$|z_\varepsilon| < \frac{1}{2\sqrt{\varepsilon_0\mu_0\omega^2 - k^2}}, \tag{93}$$

the functions (91) and (92) become real and the energy flux against the gradation is stopped. The condition means that the slope constant must be smaller than the wavelength the wave would have in a medium without gradation divided by 4π. Though this is a short length, it can be constructed in modern labs.

Proof. Evaluation of (57) using (91) and (65) yields

$$\mathbf{S} \propto \mathbf{e}_x k_x + \mathbf{e}_y k_y + \mathbf{e}_z \begin{cases} \sqrt{\varepsilon_0\mu_0\omega^2 - k^2 - 1/(2z_\varepsilon)^2} & \text{if } \varepsilon_0\mu_0\omega^2 > k^2 + 1/(2z_\varepsilon)^2 \\ 0 & \text{otherwise.} \end{cases} \tag{94}$$

The result is the same when the electromagnetic wave is represented by b with Z_b from (92).

The most surprising feature of the effect is its independence of the sign of the slope constant z_ε. It doesn't matter if permittivity increases or decreases. Important is only a sufficiently steep change.

The finding (94) differs fundamentally from the behaviour of electromagnetic waves in conducting materials. There the energy intrudes and is dissipated.

The finding is also fundamentally different from the behaviour in homogeneous dielectric materials. Even when there is a discontinuity, part of the wave is maybe reflected, but the remainder goes on to transport energy. The waves found here differ as well from the evanescent waves which make possible dielectric waveguides. In these waveguides, k must be greater than a positive cut-off wavenumber, whereas the stopping described here works also for $k = 0$.

The effect is moreover not singular. The case (88) need not be fulfilled exactly. One can derive this from the asymptotic expansions of the Hankel functions in (86) and (87). When both index and argument get great, the Hankel functions become exponential functions with real argument if the index is greater than the argument, but they become exponential functions with imaginary argument in the opposite case [9, Sec. 3.14.2], similar to the elementary functions in (91) and (92). Generally, however, there is some dependence of the stopping on polarization.

Time-dependent analysis reveals that the energy density oscillates. During one half of the period $2\pi/\omega$ it is pushed into the graded medium, during the other half it is withdrawn. The depth of the penetration is approximately described by the second exponential functions in (91) and (92) taken with negative signs before the roots. Therefore, in a slab of finite thickness, the stopping is not perfect. Waves impinging on the one boundary of the graded medium decrease in the medium, but the mechanism just described may excite waves, though weak ones, on the other boundary. Exact amplitudes and phases follow from the boundary conditions (82) and (83).

3.1.3. Dielectric Mirrors

Having read the previous section one may argue that monotonous exponential growth cannot be maintained on long distances. However, one can realize similar stopping with zigzagging material properties. This section is devoted to periodic variations of the permittivity. Let

$$\varepsilon = \varepsilon_0 - \varepsilon_1 \cos k_0 z\,, \quad \mu = \mu_0 \tag{95}$$

with positive constants ε_0, ε_1, μ_0, and k_0. When we compare the resulting equation (67)

$$\frac{\mathrm{d}^2 Z_b}{\mathrm{d}z^2} + (\varepsilon_0\mu_0\omega^2 - k^2 - \varepsilon_1\mu_0\omega^2\cos k_0 z) Z_b = 0 \tag{96}$$

to the equation of the Mathieu functions $\mathrm{me}_\nu(w;q)$

$$\frac{\mathrm{d}^2 \mathrm{me}_{\pm\nu}(w;q)}{\mathrm{d}w^2} + (\lambda - 2q\cos 2w)\mathrm{me}_{\pm\nu}(w;q) = 0 \tag{97}$$

with constant λ and q as defined by Meixner and Schäfke [7, p. 105], see [10, p. 404] for a slightly different definition, we find $Z_b = \mathrm{me}_{\pm\nu}(w;q)$ with

$$\begin{aligned} &w = k_0 z/2, \;\; \lambda = 4(\varepsilon_0\mu_0\omega^2 - k^2)/k_0^2, \\ &q = 2\varepsilon_1\mu_0\omega^2/k_0^2\,. \end{aligned} \tag{98}$$

The Mathieu functions possess properties transcending the flexibility of the hypergeometric function and all its descendants. It follows from Floquet's theorem [10, p. 412] that one can compute the Mathieu functions from a Fourier series times an exponential factor

$$\mathrm{me}_\nu(w;q) = \mathrm{e}^{\mathrm{i}\nu w} \sum_{n=-\infty}^{n=+\infty} c_{2n}^{\nu}(q)\,\mathrm{e}^{\mathrm{i}2nw}\,. \tag{99}$$

with coefficients $c_{2n}^{\nu}(q)$ for which Hill's theory [10, p. 413] provides handy expressions.

The characteristic exponent ν is a surprising function of λ and q. Imagine $q > 0$ fixed while λ varies. Then ν assumes, for certain bands of λ, only real values. This is what one should expect for physical reasons: A periodic perturbation in the differential equation causes periodic or rather quasi-periodic perturbations in the solutions. However, in the complements of these bands, ν acquires complex values. The phenomenon is known as parametric amplification, but it is often forgotten that only one solution describes amplification, whereas the other describes attenuation. This type of damping, which comes about without friction or spatial dissipation, is the effect we will consider here.

Floquet's theorem and Hill's theory were reinvented and generalized in solid-state physics where the theory is known as Bloch's theorem [11, 12]. In solid-state physics, λ plays the role of the energy in Schrödinger's equation and the complementary bands are denoted as *forbidden*. More or less the same mathematical content reappeared more recently in the theory of photonic crystals [13].

The lowest forbidden band of Mathieu's equation (97) is characterized [7, p. 120], see also [7, Fig. 6], by

$$|\lambda - 1| < |q| + O(q^2)\,. \tag{100}$$

Maximum attenuation, described by the imaginary part Im of the characteristic exponent ν, takes place approximately in its median [7, p. 165]

$$\operatorname{Im}\nu = |q|/2 + O(q^2) \quad \text{where } \lambda = 1 + O(q^2)\,. \tag{101}$$

To estimate the attenuation of the Pointing vector (57), we must take the absolute square of the leading factor on the right-hand side of (99). The attenuation of energy flux is thus

$$\bar{\mathbf{S}} \propto \exp(-|q|w + O(q^2))\,. \tag{102}$$

Applying this to the problem at hand (98), we see the first forbidden band approximately defined by

$$|4(\varepsilon_0\mu_0\omega^2 - k^2) - k_0^2| < 2\varepsilon_1\mu_0\omega^2\,. \tag{103}$$

The median of the forbidden band is where the left-hand side is zero. One may write this as

$$\frac{1}{k_0} \approx \frac{1}{2\sqrt{\varepsilon_0\mu_0\omega^2 - k^2}} \tag{104}$$

meaning that the wave length of the intruding light must be twice the wave length of the dielectric zigzagging. In other words: We must have two bilayers of different media for every spatial period of light. The reader is encouraged to compare (104) with (93).

When (104) is used in the formula for q in (98) and if it is assumed, just for simplicity, that $k = 0$, i.e. the light impinges vertically, we obtain as a crude estimate

$$q \approx \frac{\varepsilon_1}{2\varepsilon_0} \tag{105}$$

and thus for the attenuation (102)

$$\bar{\mathbf{S}} \propto \exp\left(-\frac{\varepsilon_1}{\varepsilon_0}\frac{k_0 z}{4}\right). \tag{106}$$

A variation of permittivity $\varepsilon_1 = 0.13\varepsilon_0$ is realistic viz. for cryolite on zinc sulfide. Then, according to (106), it takes approximately 11 bilayers to attain attenuation by a factor of 10, it takes less than 23 bilayers to attain attenuation by a factor of 100 and so forth. An electromagnetic wave that impinges on a periodic structure cannot penetrate. It is reflected.

The dielectric mirror just described is selective. Light that has not the suitable wave length (104) passes. Yet selectiveness isn't overly sharp. According to (100), the width of the band is $2|q|$. If q is estimated according to (105), we find that the relative width of the forbidden band is $\varepsilon_1/\varepsilon_0$, i.e. 13% in the present example. This is enough to cover a considerable part of the spectrum visible to human eyes, but not enough to produce a mirror for all colors. Moreover the dielectric mirror alters its properties with the angle of incidence. The angle is contained in the transverse wave number k and enters the theory via the parameter λ in (98).

It remains to check the dielectric mirror for its dependence on polarization. To this end the solution of the differential equation (64) has to be compared with the solution of (67) which we just discussed.

The term with the first derivatives in (64) can be eliminated introducing the auxiliary function $\tilde{Z}_a$, yet at the cost of more complications in the factor of $\tilde{Z}_a$:

$$\tilde{Z}_a = Z_a/\sqrt{\varepsilon}\,, \tag{107}$$
$$\frac{\mathrm{d}^2\tilde{Z}_a}{\mathrm{d}z^2} + \left(\varepsilon\mu\omega^2 - k^2 + \frac{1}{2\varepsilon}\frac{\mathrm{d}^2\varepsilon}{\mathrm{d}z^2} - \frac{3}{4\varepsilon^2}\left(\frac{\mathrm{d}\varepsilon}{\mathrm{d}z}\right)^2\right)\tilde{Z}_a = 0.$$

Generally this is not exactly a Mathieu equation, but it is, because of its periodic coefficient, of Hill's type. It can be solved in the same way as Mathieu's and exhibits the same features, namely allowed and forbidded bands. Nevertheless for small oscillations of the permittivity $\varepsilon_1 \ll \varepsilon_0$, (107) can be approximated by the Mathieu equation

$$\begin{aligned}\frac{\mathrm{d}^2\tilde{Z}_a}{\mathrm{d}z^2} + \Bigg(&\varepsilon_0\mu_0\omega^2 - k^2 \\ &- \left(\varepsilon_1\mu_0\omega^2 - \frac{\varepsilon_1}{\varepsilon_0}\frac{k_0^2}{2}\right)\cos k_0 z\Bigg)\tilde{Z}_a = 0\,,\end{aligned} \tag{108}$$

i.e. $\tilde{Z}_a$ is represented by a Mathieu function, too, where w and λ is same as in (98), but

$$q = \frac{2\varepsilon_1\mu_0\omega^2}{k_0^2} - \frac{\varepsilon_1}{\varepsilon_0}\,. \tag{109}$$

Repeating the same deliberations as those following (98), we find that the median of the forbidden band is at the same position (103) and $q \approx -\varepsilon_1/(2\varepsilon_0)$. The parameter q has now, apart from its sign, approximately the same magnitude as in (105), but the sign of q can be compensated by an unimportant phase shift of the argument in Mathieu's function. Therefore both the width of the forbidden band and the attenuation are approximately the same as for Z_b. Thus, surprisingly enough, the dielectric mirror depends but weakly on polarization.

3.2. *Central-Symmetric Variations of Material Properties*

For this case we specialize the condition (54) by $\mathbf{v}_0 = \mathbf{0}$ and $v_1 = 1$. The carrier is just the vector of position $\mathbf{v} = \mathbf{r}$ and hence $\mathbf{v}^2 = r^2$. In all equations of Section 2, we got to replace ζ with the r customary in the spherical coordinate system r, θ, φ. The differential equations (52) and (53) appear as

$$\begin{aligned}&\frac{1}{r}\frac{\partial^2 ra}{\partial r^2} + \frac{1}{r^2}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial a}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 a}{\partial\varphi^2}\right)\\ &-\frac{\mathrm{d}\log\varepsilon}{\mathrm{d}r}\frac{1}{r}\frac{\partial ra}{\partial r} + \varepsilon\mu\omega^2 a = 0,\end{aligned} \tag{110}$$

$$\begin{aligned}&\frac{1}{r}\frac{\partial^2 rb}{\partial r^2} + \frac{1}{r^2}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial b}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 b}{\partial\varphi^2}\right)\\ &-\frac{\mathrm{d}\log\mu}{\mathrm{d}r}\frac{1}{r}\frac{\partial rb}{\partial r} + \varepsilon\mu\omega^2 b = 0.\end{aligned} \tag{111}$$

Both equations can be separated by similar ansatzes

$$\begin{aligned}&a = \frac{1}{r}R_a Y_{lm} \quad\text{and}\quad b = \frac{1}{r}R_b Y_{lm} \quad\text{with}\\ &R_a = R_{a\omega}(r),\ R_b = R_{b\omega}(r),\ Y_{lm} = Y_{lm}(\theta,\varphi).\end{aligned} \tag{112}$$

The equation for the angular factor Y_{lm} is the same for both representatives a and b:

$$\begin{aligned}&\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y_{lm}}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 Y_{lm}}{\partial\varphi^2}\\ &+ l(l+1)Y_{lm} = 0.\end{aligned} \tag{113}$$

It is the differential equation of the familiar spherical harmonics. They must be unique. So l and m must be integers, in fact $l = 1, 2, 3, \ldots$, and $|m| \le l$. $l = 0$ is excluded because Y_{00} is a constant, and a representative not depending at all on the angles is annihilated by the curls in (50) or (51).

The differential equations for the radial parts are extraordinary:

$$\begin{aligned}&\frac{\mathrm{d}^2 R_a}{\mathrm{d}r^2} - \frac{\mathrm{d}\log\varepsilon}{\mathrm{d}r}\frac{\mathrm{d}R_a}{\mathrm{d}r}\\ &+\left(\varepsilon\mu\omega^2 - \frac{l(l+1)}{r^2}\right)R_a = 0,\end{aligned} \tag{114}$$

$$\begin{aligned}&\frac{\mathrm{d}^2 R_b}{\mathrm{d}r^2} - \frac{\mathrm{d}\log\mu}{\mathrm{d}r}\frac{\mathrm{d}R_b}{\mathrm{d}r}\\ &+\left(\varepsilon\mu\omega^2 - \frac{l(l+1)}{r^2}\right)R_b = 0.\end{aligned} \tag{115}$$

Although they look like their one-dimensional analogs in (64) and (67), the last terms on the left-hand sides are different. Nevertheless also the present differential equations can be solved for all power laws conforming to

$$\begin{aligned}&\varepsilon = \varepsilon_\alpha r^\alpha,\quad \mu = \mu_\beta r^\beta\\ &\text{with } \alpha+\beta = -2, -1, 0, 2,\end{aligned} \tag{116}$$

i.e. solved in terms of functions not more complicated than the confluent hypergeometric function.

3.2.1. *Bound Electromagnetic Waves*

In Schrödinger's quantum mechanics, electrons can be bound in a spherical well. Some eigenvalues of Schrödinger's solutions are discrete. A similar construction for photons isn't known. For instance

$$\begin{aligned}\varepsilon &= \begin{cases}\varepsilon_0 & \text{if } 0 \le r < r_1,\\ \varepsilon_1 & \text{if } r_1 \le r < \infty,\end{cases}\\ \mu &= \begin{cases}\mu_0 & \text{if } 0 \le r < r_1,\\ \mu_1 & \text{if } r_1 \le r < \infty,\end{cases}\end{aligned} \tag{117}$$

with positive constants ε_0, μ_0 inside the spherical core $r < r_1$ and other positive constants ε_1, μ_1 outside admits as solution of (114) only

$$\frac{1}{r}R_a = \begin{cases}A\, j_l(\sqrt{\varepsilon_0\mu_0}\,\omega r) & \text{if } 0 \le r < r_1,\\ B\,h_l^{(1)}(\sqrt{\varepsilon_1\mu_1}\,\omega r) + C\,h_l^{(2)}(\sqrt{\varepsilon_1\mu_1}\,\omega r) & \text{if } r_1 \le r < \infty.\end{cases} \tag{118}$$

with spherical Bessel functions j_l, spherical Hankel functions $h_l^{(1)}, h_l^{(1)}$, $l = 1, 2, 3\ldots$ [14, Sec. 10], and constants A, B, and C. All functions describe running waves. One might think of materials where the real permittivity or the permeability are negative. Such materials exist, but those negative values take place only in narrow bands of ω and come always with considerable conductivity. So there is not the least chance to establish bound electromagnetic waves and discrete values of ω with spatially constant material properties.

With graded materials, however, we can construct a home of bound waves. Consider instead of (117)

$$\begin{aligned}\varepsilon &= \begin{cases}\varepsilon_0 & \text{if } 0 \le r < r_1,\\ \varepsilon_1 r_1/r & \text{if } r_1 \le r < \infty,\end{cases}\\ \mu &= \begin{cases}\mu_0 & \text{if } 0 \le r < r_1,\\ \mu_1 r_1/r & \text{if } r_1 \le r < \infty.\end{cases}\end{aligned} \tag{119}$$

The respective solution of (114) is

$$R_a = \begin{cases} A\, r\, j_l(\sqrt{\varepsilon_0\mu_0}\,\omega r) & \text{if } 0 \le r < r_1, \\ B\, r^{-\sqrt{l(l+1)-\varepsilon_1\mu_1\omega^2 r_1^2}} & \text{if } r_1 \le r < \infty. \end{cases} \tag{120}$$

The solution on the flank of the wall $r_1 \le r$ can be a wave, though a weird one

$$\begin{aligned} & r^{\pm\sqrt{l(l+1)-\varepsilon_1\mu_1\omega^2 r_1^2}} \\ & = \exp\left(\pm \mathrm{i}\sqrt{\varepsilon_1\mu_1\omega^2 r_1^2 - l(l+1)}\log r\right) \end{aligned} \tag{121}$$

if ω is sufficiently high. Yet, if

$$\Omega = \sqrt{\varepsilon_0\mu_0}\,\omega r_1 < \sqrt{\frac{\varepsilon_0\mu_0}{\varepsilon_1\mu_1} l(l+1)} \tag{122}$$

with Ω as nondimensional substitute of ω, the flank function in (120) just decreases without any variation of phase such that the same considerations apply as in Section 3.1.2: The transfer of energy through the flanks is stopped. One can check this explicitly evaluating the Pointing vector (57) with (120) and (112).

To find the eigenvalues of Ω and thus of ω, we must satisfy the boundary conditions (55). The surface S is now the sphere $r = r_1$ and the normal vector is $\mathbf{n} = \mathbf{r}/r$. Hence

$$\begin{aligned} & R_a\,|_{r=r_1-0} = R_a\,|_{r=r_1+0}\,, \\ & \frac{1}{\varepsilon}\frac{\mathrm{d}R_a}{\mathrm{d}r}\bigg|_{r=r_1-0} = \frac{1}{\varepsilon}\frac{\mathrm{d}R_a}{\mathrm{d}r}\bigg|_{r=r_1+0}. \end{aligned} \tag{123}$$

This produces a homogenous linear system for A and B. It has a non-trivial solution if its determinant is zero:

$$\sqrt{\frac{\varepsilon_0\mu_1}{\varepsilon_1\mu_0}}\sqrt{\frac{\varepsilon_0\mu_0}{\varepsilon_1\mu_1}l(l+1) - \Omega^2} = -\frac{(\Omega\, j_l(\Omega))'}{j_l(\Omega)}, \tag{124}$$

the prime indicating differentiation with respect to the argument Ω. The function on the left-hand side is a parabola open to the left. We need its positive branch at positive values of Ω. The parabola disappears for Ω greater than the cut-off given on the right-hand side of (122). The function on the ride-hand side of (124) takes the value $-(l+1)$ at $\Omega = 0$. It increases with Ω and crosses the Ω-axis at the zero of $(\Omega\, j_l(\Omega))'$. The function continues to increase until it approaches its pole at the zero of $j_l(\Omega)$. To secure the existence of a solution of (124), it would be sufficient to demand that the cut-off in (122) be greater than the zero of $(\Omega\, j_l(\Omega))'$. But these zeros are not tabulated. So let us be generous and demand that the cut-off be greater than the first zero of $j_l(\Omega)$. This yields a condition

$$\sqrt{\frac{\varepsilon_0\mu_0}{\varepsilon_1\mu_1}} > \frac{\Omega_l}{\sqrt{l(l+1)}} \quad \text{where } j_l(\Omega_l) = 0 \tag{125}$$

which warrants the existence of at least one positive solution of (124). The expression on the right-hand side tends to 1 as l tends to infinity. Therefore the restriction on the ratio of the indices of refraction is unimportant at high multipolarities. Yet even for $l = 1$ the condition (125) can be fulfilled. The first zero of the first spherical Bessel function is $\Omega_1 \approx 4.5$ [14, Sec. 10] such that ratios of the indices must be greater than 3.2. In modern times where indices of refraction can be made as big as 38.6 [15], this is a moderate requirement.

Waves of the other polarization can found replacing the representative a with b and interchanging ε and μ. The characteristic equation of this case differs from (124) just by a different leading factor. Therefore it depends on the polarization whether an electromagnetic wave can be bound, but it does not depend much.

One might compare the construction explained here with a hydrogen atom. Rather it is similar to a nucleon bound in a collective nuclear potential as the spectrum of eigenvalues is finite. The essential difference, however, is that every electron always carries the same charge which necessitates a normalization of its wave. Here, by contrast, the energy of the bound electromagnetic wave is arbitrary. The only necessity to confine the energy is a possible breakdown of material properties (7)–(9). It is therefore blameworthy to speak about ‘photonics’, an ‘atom for photons’, and so forth. Nevertheless, if quantum electrodynamics were true, the energy stored in the construction just described should be discrete.

As in Section 3.1.2, people might argue that the system just constructed is not realistic. Fortunately there is no singularity at the origin at $r = 0$, but it is certainly questionable to require permittivity and permeability approaching zero as in (119). The solution of this problem, however, is known. One must replace the monotonous decrease with zigzagging as explained in Section 3.1.3. This will work. For the differential equations (64) and (114) are the same for $r \to \infty$.

Effects of finite conductivity can be studied if one assumes complex values of ε_0 and ε_1 and solves (124) for complex Ω. The inverse of the imaginary part of

Ω would correspond to the finite lifetime of the wave stored in the sphere.

4. An Alternative Theorem of Representation

In the study of graded fibers, one cannot use the theorem of representation provided in Section 3. It is possible to analyze electromagnetic fields in cylindrical bodies, but the material properties must not vary except in the direction of the axis of the cylinder. For graded waveguides, one needs permittivity and permeability varying with the distance from the axis, i.e. $\varepsilon = \varepsilon_\omega(\rho)$, $\mu = \mu_\omega(\rho)$ in circular cylindrical coordinates ρ, φ, z. Yet there is no carrier according to (54) that would be proportional to $\nabla\varepsilon$ and $\nabla\mu$. Fortunately we can rely on the

Two-Dimensional Representation Theorem.
In a system of orthogonal coordinates ξ, η, ζ where the elements of the metric tensor

$$g_{\xi\xi} = g_{\xi\xi}(\eta,\zeta),\ \ g_{\eta\eta} = g_{\eta\eta}(\eta,\zeta),\\ g_{\zeta\zeta} = g_{\zeta\zeta}(\eta,\zeta), \tag{126}$$

cf. the line element (14), do not depend on the distinguished coordinate ξ and where permittivity and permeability

$$\varepsilon = \varepsilon_\omega(\eta,\zeta),\ \ \mu = \mu_\omega(\eta,\zeta) \tag{127}$$

do not depend on ξ, the fields

$$\mathbf{E} = \frac{1}{\varepsilon}\nabla\times\mathbf{e}_\xi\frac{a}{\sqrt{g_{\xi\xi}}} - \mathrm{i}\omega\mathbf{e}_\xi\frac{b}{\sqrt{g_{\xi\xi}}}, \tag{128}$$

$$\mathbf{H} = -\mathrm{i}\omega\mathbf{e}_\xi\frac{a}{\sqrt{g_{\xi\xi}}} - \frac{1}{\mu}\nabla\times\mathbf{e}_\xi\frac{b}{\sqrt{g_{\xi\xi}}} \tag{129}$$

solve Maxwell's equations (28)–(31) including the constitutive relations (24) and (25) if the representatives a and b

$$a = a_\omega(\eta,\zeta),\ \ b = b_\omega(\eta,\zeta) \tag{130}$$

do not depend on ξ and obey the differential equations

$$\sqrt{\frac{g_{\xi\xi}}{g_{\eta\eta}g_{\zeta\zeta}}}\left[\left(\frac{\partial}{\partial\eta}\sqrt{\frac{g_{\zeta\zeta}}{g_{\eta\eta}g_{\xi\xi}}}\frac{\partial a}{\partial\eta}\right)\right.\\ \left.+\left(\frac{\partial}{\partial\zeta}\sqrt{\frac{g_{\eta\eta}}{g_{\zeta\zeta}g_{\xi\xi}}}\frac{\partial a}{\partial\zeta}\right)\right] - \frac{1}{g_{\eta\eta}}\frac{\partial\log\varepsilon}{\partial\eta}\frac{\partial a}{\partial\eta}\\ -\frac{1}{g_{\zeta\zeta}}\frac{\partial\log\varepsilon}{\partial\zeta}\frac{\partial a}{\partial\zeta} + \varepsilon\mu\,\omega^2 a = 0, \tag{131}$$

$$\sqrt{\frac{g_{\xi\xi}}{g_{\eta\eta}g_{\zeta\zeta}}}\left[\left(\frac{\partial}{\partial\eta}\sqrt{\frac{g_{\zeta\zeta}}{g_{\eta\eta}g_{\xi\xi}}}\frac{\partial b}{\partial\eta}\right)\right.\\ \left.+\left(\frac{\partial}{\partial\zeta}\sqrt{\frac{g_{\eta\eta}}{g_{\zeta\zeta}g_{\xi\xi}}}\frac{\partial b}{\partial\zeta}\right)\right] - \frac{1}{g_{\eta\eta}}\frac{\partial\log\mu}{\partial\eta}\frac{\partial b}{\partial\eta}\\ -\frac{1}{g_{\zeta\zeta}}\frac{\partial\log\mu}{\partial\zeta}\frac{\partial b}{\partial\zeta} + \varepsilon\mu\,\omega^2 b = 0. \tag{132}$$

For the proof, let us start with the representative a only. The ansatz

$$\mathbf{C} = \nabla\times\mathbf{e}_\xi\frac{a}{\sqrt{g_{\xi\xi}}}, \tag{133}$$

$$\mathbf{H} = -\mathrm{i}\omega\mathbf{e}_\xi\frac{a}{\sqrt{g_{\xi\xi}}} \tag{134}$$

is the special case of (128) and (129) with $b = 0$. It solves at once the Maxwell equations (30) and (31). Yet under the geometrical restrictions (126), (127), and (130), it also solves Maxwell's equation (29), namely

$$\nabla\mathbf{B} = \frac{-\mathrm{i}\omega}{\sqrt{g_{\xi\xi}g_{\eta\eta}g_{\zeta\zeta}}}\frac{\partial\sqrt{g_{\eta\eta}g_{\zeta\zeta}}\,\mu\,a/\sqrt{g_{\xi\xi}}}{\partial\xi} = 0, \tag{135}$$

because nothing behind the differentiation depends on ξ.

Thus the only Maxwell equation that still expects solution is (28). Using the ansatz (133) and (134), it is transformed to

$$\varepsilon\nabla\times\left(\frac{1}{\varepsilon}\nabla\times\mathbf{e}_\xi\frac{a}{\sqrt{g_{\xi\xi}}}\right) = \varepsilon\mu\omega^2\mathbf{e}_\xi\frac{a}{\sqrt{g_{\xi\xi}}} \tag{136}$$

which is equivalent, as we will see soon, to the differential equation (131). It is apparent that the right-hand side of (136) is proportional to $\mathbf{e}_\xi/\sqrt{g_{\xi\xi}}$. We will discover that the same is true for the left-hand side. To this end, we sever the differentiation of ε in (136):

$$\varepsilon\nabla\times\left(\frac{1}{\varepsilon}\nabla\times\mathbf{e}_\xi\frac{a}{\sqrt{g_{\xi\xi}}}\right) = -(\nabla\log\varepsilon)\\ \times\left(\nabla\times\mathbf{e}_\xi\frac{a}{\sqrt{g_{\xi\xi}}}\right) + \nabla\times\left(\nabla\times\mathbf{e}_\xi\frac{a}{\sqrt{g_{\xi\xi}}}\right). \tag{137}$$

The nabla operator applied to $\log\varepsilon$ yields

$$-\nabla\log\varepsilon = -\frac{\mathbf{e}_\eta}{\sqrt{g_{\eta\eta}}}\frac{\partial\log\varepsilon}{\partial\eta} - \frac{\mathbf{e}_\zeta}{\sqrt{g_{\zeta\zeta}}}\frac{\partial\log\varepsilon}{\partial\zeta}. \tag{138}$$

The simple curl in (137) is

$$\nabla\times\mathbf{e}_\xi\frac{a}{\sqrt{g_{\xi\xi}}} = \frac{\mathbf{e}_\eta}{\sqrt{g_{\zeta\zeta}g_{\xi\xi}}}\frac{\partial a}{\partial\zeta} - \frac{\mathbf{e}_\zeta}{\sqrt{g_{\eta\eta}g_{\xi\xi}}}\frac{\partial a}{\partial\eta}. \tag{139}$$

Evaluating the cross product of (138) and (139) as required in (137) produces the contributions to (131) with the logarithms times $\mathbf{e}_\xi/\sqrt{g_{\xi\xi}}$. The double curl on the right-hand side of (137) gives the first contribution to (131) times $\mathbf{e}_\xi/\sqrt{g_{\xi\xi}}$. So it is shown that solution of (131) completes the solution of Maxwell's equations.

The truth of (132) can be proven when we start from the ansatz (128) and (129) with the representative b only, putting $a=0$. All Maxwell equations turn out to be automatically solved except (30). This one is evaluated as described in (136)–(139) where a is interchanged with b and ε with μ.

Finally we remember the linearity of Maxwell's equations. The full proof of the alternative theorem of representation is just the superposition of the two proofs produced in the last paragraphs. □

The consistent setup of boundary-value problems is described in the

Corollary on Boundary-Value Conditions. *Let S denote the line where different media meet,* $\mathbf{n}$ *the normal on this line with* $\mathbf{n}\mathbf{e}_\xi = 0$*, and $\partial/\partial n$ the differentiation along this normal. The representatives a and b must satisfy*

$$a|_{S-} = a|_{S+}\,, \quad \left.\frac{1}{\varepsilon}\frac{\partial a}{\partial n}\right|_{S-} = \left.\frac{1}{\varepsilon}\frac{\partial a}{\partial n}\right|_{S+}\,, \tag{140}$$

$$b|_{S-} = b|_{S+}\,, \quad \left.\frac{1}{\mu}\frac{\partial b}{\partial n}\right|_{S-} = \left.\frac{1}{\mu}\frac{\partial b}{\partial n}\right|_{S+}\,. \tag{141}$$

The symbols $S-$ and $S+$ indicate that the values of the functions and their derivatives are to be calculated via an approach on the one side of S, say, the low side $S-$, or on the other side, say, the high side $S+$.

The proof is nearly the same as the proof of the corollary on boundary-value conditions in Section 3. We must use now the representation formulae (128) and (129). The difference is: Here ε and μ needn't be constant on the boundary.

The reader is kindly asked not to misunderstand the denotation 'two-dimensional'. A propagating electromagnetic field always spans the three-dimensional space. 'Two-dimensional' means just that all components of the field depend only on two coordinates.

Yet quite a few problems can be declared to be two-dimensional by a judicious choice of coordinates. In all these cases it is advantageous to apply the two-dimensional representation theorem. Namely the calculation of the electromagnetic field using (128) and (129) takes less work than using (50) and (51) as two curls less need to be computed.

When permittivity and permeability are constant in space, the two-dimensional theorem of representation doesn't offer anything which is not included in the three-dimensional theorem given in Section 3. With constant material properties, the theorem already presented in [1, Sec. 10] grants the best systematic approach.

4.1. Examples in a Plane

The most straightforward applications of the foregoing theorem take place in cartesian coordinates x, y, z. None of the components of the metric tensor depends on any coordinate:

$$g_{xx} = 1\,,\ \ g_{yy} = 1\,,\ \ g_{zz} = 1\,. \tag{142}$$

As distinguished coordinate ξ, we may select either x or y or z. Let us identify $\xi = z, \eta = x, \zeta = y$. The partial differential equations (131) and (132) appear as

$$\begin{aligned} &\frac{\partial^2 a}{\partial x^2} + \frac{\partial^2 a}{\partial y^2} - \frac{\partial \log\varepsilon}{\partial x}\frac{\partial a}{\partial x} \\ &- \frac{\partial \log\varepsilon}{\partial y}\frac{\partial a}{\partial y} + \varepsilon\mu\,\omega^2 a = 0\,, \end{aligned} \tag{143}$$

$$\begin{aligned} &\frac{\partial^2 b}{\partial x^2} + \frac{\partial^2 b}{\partial y^2} - \frac{\partial \log\mu}{\partial x}\frac{\partial b}{\partial x} \\ &- \frac{\partial \log\mu}{\partial y}\frac{\partial b}{\partial y} + \varepsilon\mu\,\omega^2 b = 0\,. \end{aligned} \tag{144}$$

The electromagnetic field is represented according to (128) and (129) through

$$\mathbf{E} = \frac{1}{\varepsilon}\nabla\times\mathbf{e}_z a - \mathrm{i}\omega\mathbf{e}_z b\,, \tag{145}$$

$$\mathbf{H} = -\mathrm{i}\omega\mathbf{e}_z a - \frac{1}{\mu}\nabla\times\mathbf{e}_z b\,, \tag{146}$$

showing that the electromagnetic wave extends in three dimensions while the representatives $a = a_\omega(x,y)$ and $b = b_\omega(x,y)$ depend on two coordinates x and y. Moreover it should be noticed that both permittivity and permeability may depend on both coordinates x and y. Therefore for the so-called two-dimensional problem, the equations (143) and (144) constitute the most general reduction of the coupled Maxwellian system to two uncoupled equations.

Even when the partial differential equations (143) and (144) are not separable, they vastly simplify the solution of Maxwell's equations as all methods which people learn in the ordinary courses of quantum mechanics can be applied directly, for example, Born's approximation and the JWKB approximation, denoted also as semi-classical approximation. In the latter case, however, the classical eikonal equation (1) will turn out to be only of restricted usefulness. Also numerical methods will profit from the reduction.

When ε and μ depend only on one coordinate, x or y, the partial differential equations (143) and (144) can be separated and produce ordinary differential equations similar to (64) and (67). Also when ε and μ are products of functions which depend on one coordinate only, separation is possible, but only under certain circumstances. We will see an example below.

In cartesian coordinates, we can choose x, y or z as the distinguished coordinate ξ, but a changed choice does not alter the geometrical situation. In circular cylindrical coordinates ρ, φ, z the elements of the metric tensor depend neither on z nor on φ:

$$g_{\rho\rho} = 1, \; g_{\varphi\varphi} = \rho^2, \; g_{zz} = 1. \tag{147}$$

Hence we may select either φ or z as distinguished coordinate, but now the choice varies the geometrical situation.

Begin with $\xi = z$ as distinguished coordinate. The partial differential equation (131) appears as

$$\begin{aligned} &\frac{1}{\rho}\left(\frac{\partial}{\partial\rho}\rho\frac{\partial a}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 a}{\partial\varphi^2} - \frac{\partial\log\varepsilon}{\partial\rho}\frac{\partial a}{\partial\rho} \\ &- \frac{1}{\rho^2}\frac{\partial\log\varepsilon}{\partial\varphi}\frac{\partial a}{\partial\varphi} + \varepsilon\mu\,\omega^2 a = 0, \end{aligned} \tag{148}$$

(132) for b is identical up to an interchange of ε with μ and is therefore not written.

The partial differential equation (148) is the equation of plane scattering. It can be separated if ε and μ are functions of ρ only. The ansatz

$$a = \mathrm{P}_a \Phi \;\text{ with } \mathrm{P}_a = \mathrm{P}_{a\omega}(\rho)\,, \Phi = \Phi(\varphi) \tag{149}$$

generates the ordinary differential equations

$$\frac{\mathrm{d}^2\Phi}{\mathrm{d}\varphi^2} + m^2\Phi = 0, \tag{150}$$

$$\begin{aligned} &\frac{\mathrm{d}^2\mathrm{P}_a}{\mathrm{d}\rho^2} + \left(\frac{1}{\rho} - \frac{\mathrm{d}\log\varepsilon}{\mathrm{d}\rho}\right)\frac{\mathrm{dP}_a}{\mathrm{d}\rho} \\ &+ \left(\varepsilon\mu\,\omega^2 - \frac{m^2}{\rho^2}\right)\mathrm{P}_a = 0. \end{aligned} \tag{151}$$

The first equation has the familiar solutions $\Phi = \exp(\mathrm{i}m\varphi)$. The separation constant m^2 must be the square of an integer $m = 0, \pm1, \pm2, \pm3, \ldots$. Otherwise a and hence the electromagnetic field would not be unique. The second equation can be solved using functions not more complicated than the confluent hypergeometric function if

$$\begin{aligned} &\varepsilon = \varepsilon_\alpha \rho^\alpha\,, \;\; \mu = \mu_\beta \rho^\beta \\ &\text{with} \;\;\; \alpha + \beta = -2, -1, 0, 2 \end{aligned} \tag{152}$$

and with constant α, β, ε_α, and μ_β. This together with the boundary conditions (140) gives ample freedom to model graded centers of scattering around $\rho = 0$.

Interestingly the partial differential equation (148) can also be separated if permittivity and permeability depend on both variables, e.g.

$$\begin{aligned} &\varepsilon = \rho^\alpha \varepsilon_\varphi, \; \mu = \rho^\beta \mu_\varphi \text{ with } \alpha + \beta = -2 \\ &\text{and } \varepsilon_\varphi = \varepsilon_\varphi(\varphi), \; \mu_\varphi = \mu_\varphi(\varphi). \end{aligned} \tag{153}$$

The functions ε_φ and μ_φ depend only on the angle φ and must be periodic for the uniqueness, else they are arbitrary. The ansatz

$$\begin{aligned} &a = \mathrm{P}_a \Phi_a \;\text{ with } \mathrm{P}_a = \mathrm{P}_a(\rho), \\ &\Phi_a = \Phi_{a\omega}(\varphi) \end{aligned} \tag{154}$$

produces

$$\begin{aligned} &\frac{\mathrm{d}^2\Phi_a}{\mathrm{d}\varphi^2} - \frac{\mathrm{d}\log\varepsilon_\varphi}{\mathrm{d}\varphi}\frac{\mathrm{d}\Phi_a}{\mathrm{d}\varphi} \\ &+ (q + \varepsilon_\varphi\mu_\varphi\omega^2)\Phi_a = 0, \end{aligned} \tag{155}$$

$$\frac{\mathrm{d}^2\mathrm{P}_a}{\mathrm{d}\rho^2} + \frac{1-\alpha}{\rho}\frac{\mathrm{dP}_a}{\mathrm{d}\rho} - \frac{q}{\rho^2}\mathrm{P}_a = 0 \tag{156}$$

with the separation constant q. The latter equation is now of Eulerian type simply solved by

$$\mathrm{P}_a = \rho^{\alpha/2 \pm \sqrt{q + \alpha^2/4}}\,, \tag{157}$$

whereas the former equation is of Hill's type. When the average values of ε_φ and μ_φ are denoted by $\bar{\varepsilon}_\varphi$ and $\bar{\mu}_\varphi$, respectively, and when ε_φ and μ_φ oscillate around their average values but weakly, then (155) is solved by periodic Mathieu functions and the separation constant can be estimated as

$$q \approx m^2 - \bar{\varepsilon}_\varphi\bar{\mu}_\varphi\omega^2\,, \tag{158}$$

$m = 0, \pm 1, \pm 2, \pm 3, \ldots$. Because of (157), this determines whether we see a wave or monotonous variation along ρ.

There is no sizeable difficulty to solve (156) even if the oscillations of ε_φ and μ_φ are large. Use, for example, Hill's theory.

Presently the devices to simulate invisibility cloaks are mostly plain. Therefore this is the section with the best formulas to design them. It takes the solution of a scattering problem in a loop of minimization.

One starts with a guess of permittivity $\varepsilon = \varepsilon(\rho)$ and permeability $\mu = \mu(\rho)$ demanding that these functions converge quickly to constants ε_∞ and μ_∞ as ρ tends to infinity. Using these functions, one solves the differential equation (148), for example by a partial-wave expansion based on (150) and (151), fulfilling the boundary condition

$$a \sim \exp(\mathrm{i}k\rho\cos\varphi) + A(\varphi)\exp(\mathrm{i}k\rho)/\sqrt{\rho} \quad \text{for } \rho \to \infty \tag{159}$$

with $k = \sqrt{\varepsilon_\infty \mu_\infty}\,\omega$. The dependence of the scattering amplitude $A(\varphi)$ on the angle φ is obtained as a part of the solution. Invisibility would appear when there is an object in a circle, say $\rho < \rho_0$, while the scattering amplitude is zero, at least in some angular range, say $\varphi_1 < \varphi < \varphi_2$. ρ_0, φ_1, and φ_2 are given constants. Perfect invisibility cannot be attained. So $\varepsilon(\rho)$ and $\mu(\rho)$ must be varied until one finds

$$\begin{aligned} &|a| \ \text{ for } \rho < \rho_0 \ \text{ and} \\ &|A(\varphi)| \ \text{ for } \varphi_1 < \varphi < \varphi_2 \end{aligned} \tag{160}$$

small enough.

For a first guess of $\varepsilon(\rho)$ and $\mu(\rho)$, one may fall back to another approach to invisibility. People apply non-bijective coordinate transforms to Maxwell's equations. The transforms are such that they approach identity far away from the origin whereas they dig up the space around the center. In this way the plane-wave solution of Maxwell's equations is transformed to a wave that stays plane in infinity while it wafts around the center without touching it [16, 17]. The idea is much the same as in Joukowsky's classical theory of airfoils [18]. Yet every non-bijective transform implies singularities. Moreover Maxwell's equations cannot be reduced to the Laplace equation. The main difference to Joukowsky's theory is thus that the coordinate transform converts permittivity and permeability to tensors and that these tensors exhibit proportional dependences on the coordinates as there are two material properties but only one coordinate transform. To verify this theory, called transformation optics, experimentalists must try and construct metamaterials with, first, suitable singularities, second, proportional spatial variations and, third, tensorial characteristics of permittivity and permeability. Naturally they cannot, but they may find materials wherein singularities are approximated by smooth functions and tensors are approximated by effective scalars. This is where the present theory starts. One may insert theoretically suggested, yet realistically modified values of the material properties and find the belonging exact solutions of Maxwell's equations.

Visibility depends on polarization. Thus one must solve the scattering problem for the representative b, too, and minimize its size around the origin and its scattering amplitude as indicated in (160) for a.

The best formulas to design central-symmetrical cloaks can be found in Section 3.2. The scattering problem for a finite center in three-dimensional space is very well studied [19]. Instead of the axial distance ρ, one uses the radial distance r from that center. The asymptotic behaviour is slightly different from that described in (159); $\sqrt{\rho}$ must be replaced with r and the scattering amplitude depends on the angle θ rather than on φ.

4.2. Examples around an Axis

When we stay with circular cylindrical coordinates, but select $\xi = \varphi$ as distinguished coordinate, we obtain from (131) and (132)

$$\rho\left(\frac{\partial}{\partial\rho}\frac{1}{\rho}\frac{\partial a}{\partial\rho}\right) + \frac{\partial^2 a}{\partial z^2} - \frac{\partial \log\varepsilon}{\partial\rho}\frac{\partial a}{\partial\rho} - \frac{\partial\log\varepsilon}{\partial z}\frac{\partial a}{\partial z} + \varepsilon\mu\,\omega^2 a = 0\,, \tag{161}$$

$$\rho\left(\frac{\partial}{\partial\rho}\frac{1}{\rho}\frac{\partial b}{\partial\rho}\right) + \frac{\partial^2 b}{\partial z^2} - \frac{\partial \log\mu}{\partial\rho}\frac{\partial b}{\partial\rho} - \frac{\partial\log\mu}{\partial z}\frac{\partial b}{\partial z} + \varepsilon\mu\,\omega^2 b = 0\,. \tag{162}$$

The reader might notice that the second-order operator in these equations cannot be understood as a part of the Laplace operator. Also one needs to get used to the equations of representation

$$\mathbf{E} = \frac{1}{\varepsilon}\left(-\frac{\mathbf{e}_\rho}{\rho}\frac{\partial a}{\partial z} + \frac{\mathbf{e}_z}{\rho}\frac{\partial a}{\partial\rho}\right) - \mathrm{i}\omega\frac{\mathbf{e}_\varphi}{\rho}b\,, \tag{163}$$

$$\mathbf{H} = -\mathrm{i}\omega\frac{\mathbf{e}_\varphi}{\rho}a + \frac{1}{\mu}\left(\frac{\mathbf{e}_\rho}{\rho}\frac{\partial b}{\partial z} - \frac{\mathbf{e}_z}{\rho}\frac{\partial b}{\partial \rho}\right) \qquad (164)$$

which follow, despite of their weird appearance, directly from (128) and (129). For $b = 0$ the magnetic field forms rings around the axis $z = 0$, for $a = 0$ the electric field builds clings to circular lines.

Most optical instruments are centered around an axis. The equations (161) – (164) provide a better foundation to design them than anything known up to now. Ray-tracing methods, for example, are based on the eikonal equation (1). They cannot produce correct results in media where material properties vary continuously but steeply. The equations (161) – (164), by contrast, yield exact solutions of Maxwell's equations. The second advantage is one has to solve partial differential equations only for one unknown. This is much simpler than solving the multiply coupled Maxwell equations. One may, for example, insert a singularity a or $b \sim \log((\rho-\rho_\mathrm{o})^2 + (z-z_\mathrm{o})^2)$ at a point of an object ρ_o, z_o and compute where other (quasi-)singularities arise. The inserted singularity would represent the object, the (quasi-)singularities would indicate the images. One may simulate lenses by regions in the ρ-z-plane where permittivity ε and permeability μ are increased. At the same time, one may simulate metallic stops of finite thickness by regions with complex ε, cf. (26), and study the interaction between lenses and stops. Many properties of imaging can thus be predicted, however, with an important exception: astigmatism can not be observed because the dependence on the azimuth φ is missing.

The first example is the graded mono-mode fiber. There we have permittivity $\varepsilon = \varepsilon_\omega(\rho)$ and permeability $\mu = \mu_\omega(\rho)$ as functions of the axial distance ρ only. Both partial differential equations (161) and (162) can be separated. Let us select the first for example. The ansatz with leading ρ,

$$a = \rho\,\mathrm{P}_a Z \;\text{ with } \mathrm{P}_a = \mathrm{P}_{a\omega}(\rho),\; Z = Z(z)\,, \qquad (165)$$

is advantageous because it permits simple boundary conditions for $\rho \to 0$ and $\rho \to \infty$, namely $\mathrm{P}_a = 0$ in both cases. Why? We just have to consider the electromagnetic field in (163) and (164) and to demand that all its components should stay finite on the axis and decrease towards infinity. The ansatz (165) generates the ordinary differential equations

$$\frac{\mathrm{d}^2 Z}{\mathrm{d}z^2} + k^2 Z = 0\,, \qquad (166)$$

$$\begin{aligned}&\frac{\mathrm{d}^2\mathrm{P}_a}{\mathrm{d}\rho^2} + \left(\frac{1}{\rho} - \frac{\mathrm{d}\log\varepsilon}{\mathrm{d}\rho}\right)\frac{\mathrm{d}\mathrm{P}_a}{\mathrm{d}\rho}\\ &+\left(\varepsilon\mu\,\omega^2 - k^2 - \frac{1}{\rho^2} - \frac{1}{\rho}\frac{\mathrm{d}\log\varepsilon}{\mathrm{d}\rho}\right)\mathrm{P}_a = 0\,.\end{aligned} \qquad (167)$$

The relation between the real wavenumber k and the frequency ω, the so-called dispersion relation, is the desired item. It is found from the solution of (167) satisfying the boundary conditions for $\rho \to 0$ and for $\rho \to \infty$ as explained in the previous paragraph. One can solve the ordinary differential equation (167) for permittivities and permeabilities obeying power laws as in (152) or one can solve it numerically. A boundary-value problem with one ordinary differential equation is by orders of magnitude simpler than the same problem with the Maxwell equations and it is much simpler to attain high accuracy.

The considerations produced here for circular cylindrical coordinates can be transferred to all coordinate system which embody an axis of rotation, e.g. spherical coordinates, prolate and oblate spheroidal coordinates, parabolic coordinates, and all rotational systems [2, Secs. I, IV]. Especially interesting appear at first glance the oblate spheriodal coordinates because they allow an easy study what an electromagnetic wave does in a bottleneck, and the toroidal coordinates because they allow the study of electromagnetic waves in a tokamak. Perhaps fusion research, too, might profit from the methods developed here.

5. Retro and Prospects

We have now two systematic approaches to exact solutions of Maxwell's equations when material properties vary in space. What was known before this article was written?

The interest in electromagnetically variable media increased dramatically with the advent of dielectric waveguides, i.e. in the seventies of the previous century. Attempts were made to improve the fibers using dielectrics with a graded index of refraction (GRIN), see [20] for references. In 1975, Kogelnik noted down a Helmholtz equation with a variable coefficient as in (2) which, according to his belief, would found a theory of electromagnetic waves in GRIN media [21, Sec. 2.4]. He remarked the similarity of his equation with Schrödinger's and rewrote some solutions found in textbooks on quantum mechanics for his purpose.

In their classic monography, Born and Wolf [22, Sec. 1.6] reproduce some deliberations probably first thought by Abelés [23] in 1950. Maxwell's equations are written in a cartesian coordinate system and permittivity as well as permeability are allowed to depend on one coordinate. Maxwell's system, which usually couples all components of the electromagnetic field, is shuffled until there is one equation for only one component. This equation has the same shape as (64) although its physical content is different. Unfortunately this useful equation comes with a second, more complicated equation which has to be solved at the same time if the electromagnetic field is to be calculated. In summary, the theory put forward by Born and Wolf is practically useless.

So Born and Wolf haste to a theory which construes the continuum as a sequence of small steps. The graded medium is replaced with a pile of thin layers. The reflection and the transmission in a single layer are calculated from Snell's law and Fresnel's formulae. The results are entered in simple matrices such that the reflections and the transmissions in the pile can be computed as matrix multiplication. This is the transfer-matrix method. It is, with many technical improvements, most popular with practicians. Software packages that help to construct the matrices and to execute their multiplication can be found and downloaded in the internet. Have a look, for example, at 'Freesnell' or 'RP Coating'!

In 2010 Turakulov presented a preprint [24] and in 2011 an article [25] wherein vector potentials were used. Ordinary differential equations similar to the correct equations (64) and (67) were found although in a different mathematical context and a useless scalar potential was introduced [24] which obscures the calculation of the electromagnetic field. Turakulov also communicated that the equations (64) and (67) can actually be solved for the example (85). Unfortunately the solutions presented by him were not correct.

The awkwardness of the approaches mentioned so far is caused by the arbitrary eliminations. One of the 12 components of the electromagnetic field and one of 8 Maxwellian equations is selected whereupon lots of unsystematic attempts are made to eliminate the other 11 components from the arbitrarily selected equation. This is sometimes feasible in the cartesian coordinate system. It is cumbersome in cylindrical coordinates and becomes a nightmare in spherical coordinates.

Most physical laws are formulated as partial differential equations between vector fields and scalars. Thus we have several independent variables, denoted by physicists as *coordinates*, and several dependent ones, denoted by physicists as *fields*. Given are partial differential equations where all these variables are mingled. Wanted are ordinary differential equations each with one dependent and one independent variable only. To reach the wanted end, we must perform two separations: a separation of the dependent variables and a separation of the independent ones. These two kinds of separation must be kept separate. The fields must be uncoupled without reference to special coordinates. This is the idea not comprehended in previous work.

What one must do is this: In a problem with vector fields one must decompose them into their longitudinal and their transverse parts. The former can be represented by a scalar potential, the latter by two vector potentials, the one being a simple vector potential, the other a vector potential's vector potential. Both vector potentials must consist of a predetermined vector field times an amplitude which describes the dynamics. Otherwise there is no chance to arrive at a differential equation for one scalar quantity only.

This is the recipe that works in all vector-field theories, e.g. in the theory of sound, hydrodynamics, elasticity, and electrodynamics [26]. It was applied in [1] for a general method solving Maxwell's equations when the material properties are constant. Finally it was applied here, namely in (50) and (51) and in (128) and (129).

The solutions of the equations (60) and (66) in Sections 3.1 appear all to be new.

At any surface the value of permittivity must deviate from that in the bulk. The most natural way to describe this is (77). The exact solution yields coefficients of reflection and transmission which differ from Fresnel's just by factors in terms of the hypergeometric function. Therefore the formulae given in Section 3.1.1 constitute a more realistic foundation for the comparison of standard optical measurements with theories that try to relate the measured values to microscopic properties of matter.

The stopping exemplified in Section 3.1.2 must not be confused with tunnelling. Tunnelling takes a barrier to penetrate. Stopping happens at barriers and holes.

Seemingly nobody has noticed that the theory of dielectric mirrors is based on the Mathieu equation and its generalizations as described in Section 3.1.3. Calculations up to now were done using the transfer-matrix method described above; consider the paper by Fink et al. [27] as an anchor of references.

Appearently there were not the least precursors for the solutions of the spherical problem in Sections 3.2 ff. nor were there harbingers of the two-dimensional theorem of representation in Section 4, especially of the solution of the cylindrical problem in Section 4.2.

The broad scope of this article is at the same time its weakness. The applications were here only indicated, but must be elaborated in detail to be useful for the design of optical instruments. For example, in the theory of dielectric mirrors, the permittivity must be formulated as a Fourier series

$$\varepsilon = \sum_{n=-N}^{N} \varepsilon_n \, \mathrm{e}^{\mathrm{i}nk_0 z} \tag{168}$$

with positive wavenumber k_0 and coefficients ε_n, $n = 0, \pm 1, \pm 2, \ldots \pm N$, which model the real medium. Solving the respective equations (60) and (66) costs some work, but it can be done as precisely as wanted. The reward are exact solutions of Maxwell's equations. These solutions should be better than those obtained using the transfer-matrix method because gradual transitions between the layers in the mirror can be taken into account.

Another area worthy of study is the dependence of all material properties on the frequency ω. What kind of dependence should one select? The obvious choice are the functions of ω known in the bulk, possibly corrected for variations of mass density. For example in (77) one may utilize Drude's model or an oscillator model with damping for the permittivity ε_∞ in the bulk. Moreover one might select the same model for the permittivity ε_+ at the surface as a first approximation. In the next step of research one should look for trade-offs between surface effects as described in Section 3.1.1 and microscopic effects. It may happen that a dependence on ω, which presently is attributed to some fancy microscopic mechanism, is just due to a gradual surface. In the long run one must face the need of a microscopic theory which takes the difference between bulk and surface into account.

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