# Classical Random Graphs with Unbounded Expected Degrees are Locally Scale-Free

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A common property of many, though not all, massive real-world networks, including the World-Wide Web, the Internet, and social networks, is that the connectivity of the various nodes follows a scale-free distribution, $P(k) \propto k^{-\alpha}$, with typical scaling exponent $2 \le \alpha \le 3$. In this letter, we prove that the Erdős–Rényi random graph with unbounded expected degrees has a scale-free behaviour with scaling exponent $1/2$ in a neighbourhood of expected degree $\langle k \rangle$. This interesting phenomenon shows a discrepancy from the Erdős–Rényi random graph with bounded expected degree, which has a bell shaped connectivity distribution, peaking at $\langle k \rangle$, and decaying exponentially for large $k$.

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The study of complex networks, albeit relatively new, has spanned diverse domains including economy, biology, engineering, physics, and even entire human societies [1 – 3]. An intriguing line of work in the last decade reveals that a common property of a number of massive real-life networks, such as the World-Wide Web [4 – 6], the Internet [7], the call graph [8], and social networks [9, 10], is that the connectivity of the various nodes exhibits a scale-free distribution, $P(k) \propto k^{-\alpha}$. That is, the fraction of nodes of degree $k$ is proportional to $k^{-\alpha}$, where the scaling exponent $\alpha$ is typically in the range $2 \le \alpha \le 3$. On the other hand, the classical random graph model of Erdős and Rényi [11] and Bollobás [12] with bounded expected degree $\langle k \rangle$ has a connectivity distribution which follows a Poisson distribution peaked at $\langle k \rangle$ and decaying exponentially for $k \gg \langle k \rangle$. This property of degrees is shared by some other random graph models, see e.g. [13 – 15].

In addition to the discrepancy in their connectivity distributions, the two canonical models of networks show a quite different performance under error and attack. For example, Albert et al. [16] studied the robustness of networks against two types of attacks: random failure, where nodes are sequentially removed with equal probability, and intentional attack, where hubs (i.e., nodes with large degrees) are preferentially removed. They showed that scale-free networks are highly robust against the random failure in the sense that almost all nodes have to be removed to disintegrate a scale-free network while are fragile to the intentional attack in the sense that the network is destroyed if a small fraction of hubs are removed. In contrast to scale-free networks, the classical random graphs display a low degree of tolerance against both random failure and intentional attack. Further results pertaining to the discrepancy of robustness between these two classes of networks can be found in [17 – 19]. A rigorous mathematical analysis of scale-free graphs, including a coupling with classical random graphs, is contained in [20, 21]. For a brief history and a general overview of scale-free random graphs and related issues, we make a reference to the extensive survey [22].

In this letter, we will show that the gap between scale-free network and classical random graph is not impassable in the sense of connectivity distribution. We prove that (see Theorem 1 below) a Erdős–Rényi random graph with unbounded expected degrees has an approximate scale-free behaviour with scaling exponent $1/2$, i.e., $P(k) \propto k^{-1/2}$, in a neighbourhood of expected degree $\langle k \rangle$. Numerical simulations are performed to validate our theoretical findings. The advances prompt us to wonder how network topology affects a system's survivability.

To begin with, we fix some notations. Let $G(n,\lambda_n/n)$ be a classical random graph containing $n$ nodes and with edge probability $\lambda_n/n$. Throughout the letter, we assume that $\lambda_n\to\infty$ as $n\to\infty$. Therefore, the expected degree of any vertex is $\langle k\rangle\sim\lambda_n$, which is unbounded. For $k=0,1,2,\cdots$, denote by $d_k$ the number of vertices in $G(n,\lambda_n/n)$ of degree $k$. It follows that the connectivity distribution $P(k)=d_k/n$. We will need the following lemma, whose proof is straightforward and hence omitted.

**Lemma 1.** *For any $0<x<y$ and $y>1$,*

$$\left(1+\frac{x}{y}\right)^y<\mathrm{e}^x \text{ and } \left(1-\frac{x}{y}\right)^y<\mathrm{e}^{-x}. \tag{1}$$

The result for connectivity distribution $P(k)$ is established as follows.

**Theorem 1.** *In a Erdős–Rényi random graph $G(n,\lambda_n/n)$ with $\lambda_n\to\infty$ as $n\to\infty$, for any $0<\varepsilon<1$,*

$$P\left(P(k)\le\frac{(1+\varepsilon)}{\sqrt{2\pi}}k^{-1/2}\right)\to 1, \tag{2}$$

*as $n\to\infty$, for all $k$ such that $k>\lambda_n/2$. Furthermore,*

$$P\left(\frac{(1-\varepsilon)}{\sqrt{2\pi}}k^{-1/2}\le P(k)\le\frac{(1+\varepsilon)}{\sqrt{2\pi}}k^{-1/2}\right)\to 1, \tag{3}$$

*as $n\to\infty$, for all $k$ such that $|k-\lambda_n|=o(\sqrt{\lambda_n})$.*

Theorem 1 implies that the tail of the connectivity distribution $P(k)$ (when $k>\langle k\rangle/2$) is upper bounded by a power-law with exponent 1/2, and the bound is tight in a (possibly infinite) neighbourhood of $\langle k\rangle$.

**Proof.** We start with the proof of (2). Given $0<\varepsilon<1$, by Theorem 3 in [20], we have for all $k$

$$P\left((1-\varepsilon)\frac{\lambda_n^k\,\mathrm{e}^{-\lambda_n}}{k!}\le P(k)\le(1+\varepsilon)\frac{\lambda_n^k\,\mathrm{e}^{-\lambda_n}}{k!}\right)\to 1 \tag{4}$$

as $n\to\infty$. In view of the Stirling formula and the fact that $\lambda_n\to\infty$ as $n\to\infty$, we obtain

$$\frac{\lambda_n^k\,\mathrm{e}^{-\lambda_n}}{k!}=(1+o(1))\frac{\lambda_n^k\,\mathrm{e}^{-\lambda_n}\,\mathrm{e}^k}{k^k\sqrt{2\pi k}}. \tag{5}$$

For $\lambda_n/2<k\le\lambda_n$, by using Lemma 1 and (5),

$$\begin{aligned}\frac{\lambda_n^k\,\mathrm{e}^{-\lambda_n}}{k!}&=\frac{(1+o(1))}{\sqrt{2\pi k}}\left(1+\frac{\lambda_n-k}{k}\right)^k\mathrm{e}^{k-\lambda_n}\\&\le\frac{1}{\sqrt{2\pi k}}\,\mathrm{e}^{\lambda_n-k}\,\mathrm{e}^{k-\lambda_n}\\&=\frac{1}{\sqrt{2\pi k}},\end{aligned}\tag{6}$$

for sufficiently large $n$. For $k>\lambda_n$, we similarly have

$$\begin{aligned}\frac{\lambda_n^k\,\mathrm{e}^{-\lambda_n}}{k!}&=\frac{(1+o(1))}{\sqrt{2\pi k}}\left(1-\frac{k-\lambda_n}{k}\right)^k\mathrm{e}^{k-\lambda_n}\\&\le\frac{1}{\sqrt{2\pi k}}\,\mathrm{e}^{-(k-\lambda_n)}\,\mathrm{e}^{k-\lambda_n}\\&=\frac{1}{\sqrt{2\pi k}},\end{aligned}\tag{7}$$

for sufficiently large $n$. Hence, It follows from (4), (6), and (7) that (2) holds.

Next, we turn to the proof of (3). Note that the assumption $|k-\lambda_n|=o(\sqrt{\lambda_n})$ implies that $k>\lambda_n/2$. Therefore, from the above arguments, it suffices to show

$$P\left(\frac{(1-\varepsilon)}{\sqrt{2\pi}}k^{-1/2}\le P(k)\right)\to 1 \tag{8}$$

for all $k$ satisfying $|k-\lambda_n|=o(\sqrt{\lambda_n})$.

If $k\le\lambda_n$, via (5) and Lemma 1, we obtain for any $0<\delta<1$

$$\begin{aligned}\frac{\lambda_n^k\,\mathrm{e}^{-\lambda_n}}{k!}&=\frac{(1+o(1))}{\sqrt{2\pi k}}\left(1-\frac{\lambda_n-k}{\lambda_n}\right)^{-\lambda_n(k/\lambda_n)}\mathrm{e}^{k-\lambda_n}\\&\ge\frac{1}{\sqrt{2\pi k}}\,\mathrm{e}^{(\lambda_n-k)(k/\lambda_n)}\,\mathrm{e}^{k-\lambda_n}\\&\ge(1-\delta)\frac{1}{\sqrt{2\pi k}}\end{aligned}\tag{9}$$

for sufficiently large $n$, where the last inequality holds since

$$(\lambda_n-k)\frac{k}{\lambda_n}+k-\lambda_n\to 0^- \tag{10}$$

as $n\to\infty$. Analogously, for $k>\lambda_n$, we derive that

$$\begin{aligned}\frac{\lambda_n^k\,\mathrm{e}^{-\lambda_n}}{k!}&=\frac{(1+o(1))}{\sqrt{2\pi k}}\left(1+\frac{k-\lambda_n}{\lambda_n}\right)^{-\lambda_n(k/\lambda_n)}\mathrm{e}^{k-\lambda_n}\\&\ge\frac{1}{\sqrt{2\pi k}}\,\mathrm{e}^{-(k-\lambda_n)(k/\lambda_n)}\,\mathrm{e}^{k-\lambda_n}\\&\ge(1-\delta)\frac{1}{\sqrt{2\pi k}}\end{aligned}\tag{11}$$

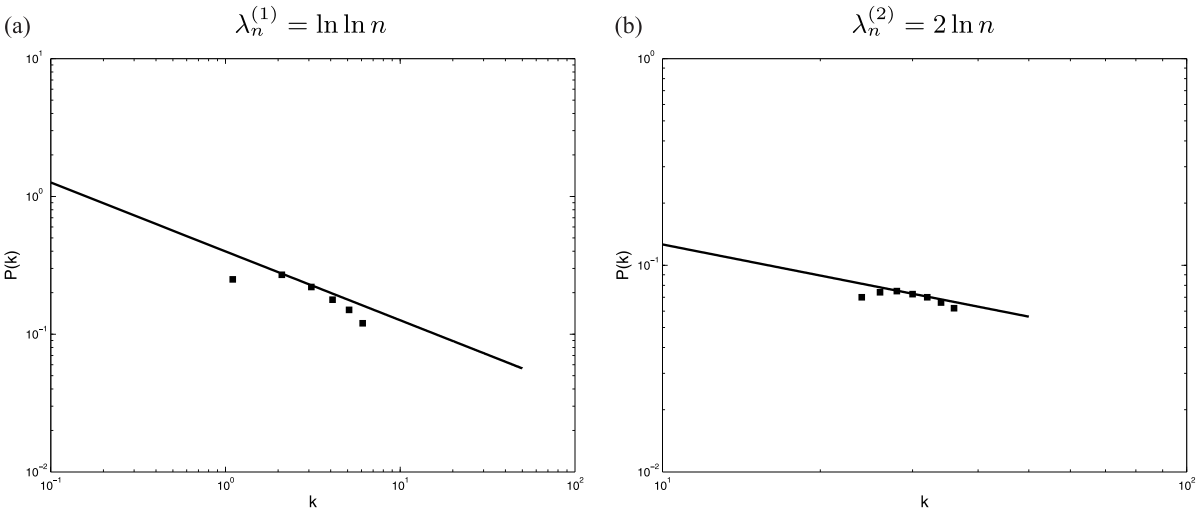


Fig. 1. Connectivity degree $P(k)$ for classical random graph $G(n, \lambda_n^{(i)}/n)$ for $i = 1, 2$ with $n = 10^5$. The solid curves represent the analytical curve $1/\sqrt{2\pi k}$, and the squares are for simulation results.

for sufficiently large $n$. Combining (9) and (11) with (4), we then obtain (8), which finally concludes the proof of Theorem 1. □

To illustrate our result, we perform simulations on a classical random graph $G(n, \lambda_n^{(i)}/n)$ for $i = 1, 2$ with $n = 10^5$ and $\lambda_n^{(1)} = \ln\ln n$ and $\lambda_n^{(2)} = 2\ln n$, respectively. We plot the connectivity distribution $P(k)$ in a neighbourhood of expected degrees $\langle k\rangle \approx 3$ and $\langle k\rangle \approx 28$ in Figure 1a and Figure 1b, respectively. Note that, almost surely, the random graph $G(n, \lambda_n^{(1)}/n)$ is not connected while $G(n, \lambda_n^{(2)}/n)$ is connected [12]. From Figure 1a and Figure 1b, we see that the simulation results agree with their analytical counterparts in the neighbourhood of $\langle k\rangle$ in both cases. In addition, the curve $1/\sqrt{2\pi k}$ turns out to be the upper bound of $P(k)$, as indicated in Theorem 1.

To conclude, we have shown that a classical Erdős–Rényi random graph can also have a scale-free connectivity distribution (although not globally). Specifically, a Erdős–Rényi random graph with unbounded expected degrees has a scale-free connectivity distribution with scaling exponent $1/2$ in an infinite neighbourhood of expected degree. Our results towards local scale-free shed light on the discrepancy in the survivability of scale-free networks and random graphs. The research focus in the present letter is on the connectivity distribution, for future study, it is desirable to investigate the robustness of networks with local scale-free behaviour under random failure or intentional attack. The usual connectivity measure and/or spectral measure [23] may be applied. In addition, the robustness of scale-free networks with an exponent less than 1 is another interesting future work. Are there other network models having local scale-free property? Is it inherent to the topologies? One reviewer mentioned a relevant local scale-free distribution in some subsets of vertex set in networks [24]. Can we quantify the topological characteristics of these types of networks when compared with global scale-free networks?

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