Travelling-Wave Solution of Volterra Lattice by the Optimal Homotopy Analysis Method

Qi Wang

Department of Applied Mathematics, Shanghai University of Finance and Economics, Shanghai 200433, PR China

Reprint requests to Q. W.; E-mail: wangqee@gmail.com

Z. Naturforsch. **67a,** 15 – 20 (2012) / DOI: 10.5560/ZNA.2011-0062
Received August 17, 2011

The travelling-wave solution of the Volterra lattice has been constructed by the optimal homotopy analysis method. The optimal method used here contains three auxiliary convergence-control parameters to adjust and control the convergence region of the solution. By minimizing the averaged residual error, the optimal convergence-control parameters can be obtained, which give much better approximations than those given by the usual homotopy analysis method. The obtained results show that the optimal homotopy analysis method is also very efficient for differential-difference equations.

Key words: Volterra Lattice; Optimal Homotopy Analysis Method; Travelling-Wave Solution.
PACS numbers: 02.30.Xx; 02.30.Mv; 02.60.Lj

1. Introduction

For better understanding the meaning of nonlinear partial differential equations (PDEs), it is crucial to search for their exact analytic solutions. The exact solutions, if available, of those nonlinear PDEs can facilitate the verification of numerical solvers and aid in the stability analysis of solutions. Many powerful methods have been generalized to construct solutions of PDEs such as the inverse scattering method [1, 2], the Bäcklund transformation [3, 4], the Darboux transformation [5], the Lie group method [6], the Hirota method [7], etc. Among them, the homotopy analysis method (HAM) which is based on the idea of homotopy in topology, is a general analytic method for nonlinear problems [8]. Unlike the traditional methods (for example, perturbation techniques and so on), HAM contains many auxiliary parameters which provide us with a simple way to adjust and control the convergence region and rate of convergence of the series solution, and it has been successfully employed to solve explicit analytic solutions for many types of nonlinear problems [9 – 15].

However, unlike the widely applications in continuous cases of HAM, there is very few work on constructing solutions for differential-difference equations (DDEs) [16, 17] which are semi-discretized with some (or all) of their spacial variables while time is usually kept continuous. They play an important role in numerical simulations of nonlinear PDEs, queuing problems, and discretization in solid state and quantum physics [18 – 22]. Moreover as illustrated in [23], the usual HAM has only one convergence-control parameter c_0 but unfortunately the curves for the convergence-control parameter (i.e. c_0-curves) can not tell us which value of $c_0 \in \mathbb{R}$ gives the fastest convergent series. Recently, to overcome this shortcoming, Liao [23] proposed an optimal HAM with more than one convergence-control parameter. In this optimal method, Liao also introduced the so called averaged residual error to get the optimal convergence-control parameters efficiently, which compared with the exact square residual error can greatly decrease the computing time (CPU time) and also give a good enough approximation. In general, the optimal HAM can greatly modify the convergence of homotopy series solution for PDEs [23, 24].

The aim of this paper is to extend the optimal HAM to consider the travelling-wave solutions of the Volterra lattice. The method proposed here contains three convergence-control parameters to accelerate the convergence of homotopy series solution. The optimal convergence-control parameters can be determined by minimizing the averaged residual error. The results ob-

tained in this paper show that the optimal HAM is also very efficient for DDEs. The solutions obtained by the optimal HAM give much better approximations and convergence much faster than those obtained by the usual HAM.

2. Optimal HAM for Volterra Lattice

The Volterra lattice system [25]

$$\begin{aligned}\frac{\partial a_n}{\partial t} &= a_n(b_n - b_{n-1}),\\ \frac{\partial b_n}{\partial t} &= b_n(a_{n+1} - a_n),\end{aligned} \tag{1}$$

in which $a_n(t)$ and $b_n(t)$ are functions of continuous variable t and discrete variable $n \in \mathbb{Z}^+$. The Volterra type equations are discretizations of the Korteweg–de Vries (KdV) and modified KdV equations [26, 27]. The exact solutions of (1) have been obtained by the tanh method [28]. Set

$$a_n = u_{2n-1}, \;\; b_n = u_{2n}, \tag{2}$$

the Volterra lattice system (1) become the more convenient form of a single polynomial equation

$$\frac{\partial u_n}{\partial t} = u_n(u_{n+1} - u_{n-1}). \tag{3}$$

To find the travelling-wave solutions of (3), it is convenient to introduce a new dependent variable $w_n(\xi_n)$ defined by

$$u_n(t) = a\, w_n(\xi_n), \tag{4}$$

where $\xi_n = kt + qn$, a is the amplitude, k is the wave speed, and q is an arbitrary non-zero constant. Substitution of u_n given by (4) into (3) gives

$$kw_n' = aw_n(w_{n+1} - w_{n-1}), \tag{5}$$

where the prime denotes the differentiation with respect to ξ_n. Assume that the dimensionless wave solution $w_n(\xi_n)$ arrives its maximum at the origin. Obviously, $w_n(\xi_n)$ and its derivatives tend to zero when $\xi_n \to \infty$. Thus, the boundary conditions of the travelling-wave solutions are

$$w_n(0) = 1, \;\; w_n(\infty) = 0, \;\; w_n'(\infty) = 0. \tag{6}$$

According to (5) and the boundary conditions (6), the travelling-wave solution can be expressed by

$$w_n(\xi_n) = \sum_{m=1}^{+\infty} d_m \, \mathrm{e}^{-m\xi_n}, \tag{7}$$

where d_m $(m = 1, 2, \ldots)$ are coefficients to be determined. Moreover, according to the rule of solution expression denoted by (7) and the boundary conditions (6), it is natural to choose $w^*(\xi_n) = \mathrm{e}^{-\xi_n}$ as the initial approximation of $w_n(\xi_n)$.

Let $p \in [0,1]$ denote the embedding parameter, $c_0 \neq 0$ denote an auxiliary parameter, called the convergence-control parameter, and $\phi_n(\xi_n; p)$ denote a kind of continuous mapping of $w_n(\xi_n)$, respectively. We can construct such a generalized homotopy

$$\begin{aligned}&(1 - C(p))\mathcal{L}[\phi_n(\xi_n; p) - w^*(\xi_n)]\\ &\quad = c_0 B(p)\mathcal{N}[\phi_n(\xi_n; p)],\end{aligned} \tag{8}$$

where

$$\mathcal{L}[\phi_n(\xi_n; p)] = \left(2\frac{\partial^2}{\partial \xi_n^2} + \frac{\partial}{\partial \xi_n}\right)\phi_n(\xi_n; p) \tag{9}$$

is an auxiliary linear operator, with the property

$$\mathcal{L}[C_1 \, \mathrm{e}^{-\xi_n/2} + C_2] = 0, \tag{10}$$

where C_1 and C_2 are constants. From (5), we define the nonlinear operator

$$\begin{aligned}\mathcal{N}[\phi_n(\xi_n; p)] &= k\frac{\partial \phi_n(\xi_n; p)}{\partial \xi_n}\\ &- a\phi_n(\xi_n; p)(\phi_{n+1}(\xi_{n+1}; p) - \phi_{n-1}(\xi_{n-1}; p)).\end{aligned} \tag{11}$$

In (11), $B(p)$ and $C(p)$ are the so-called deformation functions satisfying

$$B(0) = C(0) = 0, \;\; B(1) = C(1) = 1, \tag{12}$$

whose Taylor series

$$B(p) = \sum_{m=1}^{+\infty} \nu_m p^m, \;\; C(p) = \sum_{m=1}^{+\infty} \sigma_m p^m \tag{13}$$

exist and are convergent for $|p| \leq 1$.

Then when $p = 0$, according to the definition of $\mathcal{L}$ and $w^*(\xi_n)$, it is obvious that $\phi_n(\xi_n; 0) = w^*(\xi_n)$. When $p = 1$, according to the definition (11), (8) is equivalent to the original (3), provided $\phi_n(\xi_n; 1) = w_n(\xi_n)$. Thus, as p increases from 0 to 1, the solution $\phi_n(\xi_n; p)$

varies (or deforms) continuously from the initial guess $w^*(\xi_n)$ to the solution $w_n(\xi_n)$ of (3).

According to [23], there are an infinite number of deformation functions satisfying the properties (12) and (13). And in theory, the more convergence-control parameters are used, the better approximation one should obtain by this generalized HAM. But for the sake of computation efficiency, we just use the following one-parameter deformation functions:

$$\begin{aligned} B(c_1;p) &= \sum_{m=1}^{+\infty} \nu_m(c_1)p^m, \\ C(c_2;p) &= \sum_{m=1}^{+\infty} \sigma_m(c_2)p^m, \end{aligned} \tag{14}$$

where $|c_1| < 1$ and $|c_2| < 1$ are constants, which are convergence-control parameters too, and

$$\nu_1(c_1) = 1 - c_1, \quad \nu_m(c_1) = (1-c_1)c_1^{m-1}, \quad m > 1, \tag{15}$$

$$\sigma_1(c_2) = 1 - c_2, \quad \sigma_m(c_2) = (1-c_2)c_2^{m-1}, \quad m > 1. \tag{16}$$

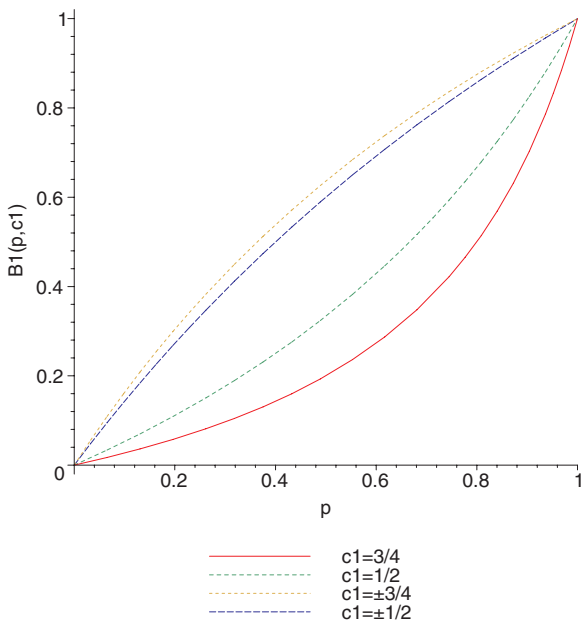

Fig. 1 (colour online). Deformation function $B_1(p;c_1)$ defined by (17) and (18). Solid line: $c_1 = 3/4$; dotted line: $c_1 = 1/2$; long-dashed line: $c_1 = -3/4$; space-dashed line: $c_1 = -1/2$.

The different values of c_1 give different paths of $B(c_1;p)$, as shown in Figure 1. Note that $B(c_1;p)$ and $C(c_2;p)$ contain the convergence-control parameters c_1 and c_2, respectively. So, we have at most three unknown convergence-control parameters c_0, c_1, and c_2, which can be used to ensure the convergence of solutions series, as shown later.

Then the so-called zeroth-order deformation equation becomes

$$\begin{aligned} &(1 - C(c_2;p))\mathcal{L}[\phi_n(\xi_n;p) - w^*(\xi_n)] \\ &= c_0 B(c_1;p)\mathcal{N}[\phi_n(\xi_n;p)], \end{aligned} \tag{17}$$

and according to (6), it should subject to following boundary conditions:

$$\begin{aligned} &\phi_n(0;p) = 1, \quad \phi_n(\infty;p) = 0, \\ &\frac{\partial \phi_n(\xi_n;p)}{\partial \xi_n}\Big|_{\xi_n=\infty} = 0. \end{aligned} \tag{18}$$

Obviously, $\phi_n(\xi_n;p)$ is determined by the auxiliary linear operator $\mathcal{L}$, the initial guess $w^*(\xi_n)$ and the convergence-control parameters c_0, c_1, and c_2. Note that we have great freedom to choose all of them. Assuming that all of them are so properly chosen that the Taylor series

$$\phi_n(\xi_n;p) = w^*(\xi_n) + \sum_{m=1}^{+\infty} w_{m,n}(\xi_n)p^m, \tag{19}$$

exist and converge at $p = 1$, we have following homotopy series solution

$$w_n(\xi_n) = w^*(\xi_n) + \sum_{m=1}^{+\infty} w_{m,n}(\xi_n), \tag{20}$$

where

$$w_{m,n}(\xi_n) = \frac{1}{m!}\frac{\partial^m \phi_n(\xi_n;p)}{\partial p^m}\Big|_{p=0}. \tag{21}$$

Let G denote a function of $p \in [0,1]$ and define the so-called mth-order homotopy derivative [11]:

$$D_m[G] = \frac{1}{m!}\frac{\partial^m G}{\partial p^m}\Big|_{p=0}. \tag{22}$$

Taking above operator on both sides of the zeroth-order deformation equation (17) and the boundary conditions (18), we have following mth-order deformation equation:

$$\begin{aligned} &\mathcal{L}\left[w_{m,n}(\xi_n) - \chi_m \sum_{l=1}^{m-1} \sigma_{m-l}(c_2) w_{l,n}(\xi_n)\right] \\ &= c_0 \sum_{l=0}^{m-1} \nu_{m-l}(c_1) R_l(\xi_n), \end{aligned} \tag{23}$$

subjects to the boundary conditions

$$w_{m,n}(0) = w_{m,n}(\infty) = w'_{m,n}(\infty) = 0, \tag{24}$$

where

$$R_l(\xi_n) = k\sum_{i=0}^{l} w'_{i,n} - a\sum_{i=0}^{l} w_{l-i,n}(w_{i,n+1} - w_{i,n-1}) \tag{25}$$

and

$$\chi_m = \begin{cases} 0 & m = 1, \\ 1 & m > 1. \end{cases} \tag{26}$$

Let $w^*_m(\xi_n)$ denote a special solution of (23) and $\mathcal{L}^{-1}$ the inverse operator of $\mathcal{L}$, respectively. Then we have

$$\begin{aligned} w^*_{m,n}(\xi_n) = {} & \chi_m \sum_{l=1}^{m-1} \sigma_{m-l}(c_2) w_{l,n}(\xi_n) \\ & + c_0 \sum_{l=0}^{m-1} \nu_{m-l}(c_1) \mathcal{L}^{-1}(R_l(\xi_n)). \end{aligned} \tag{27}$$

So the common solution of (23) reads

$$w_{m,n}(\xi_n) = w^*_{m,n}(\xi_n) + C_1\, \mathrm{e}^{-\xi_n/2} + C_2. \tag{28}$$

Under the rule of solution expression (7), $C_1 = C_2 = 0$. Then we just need to identify a special solution of (23).

In this way, we can derive $w_{m,n}(\xi_n)$ for $m = 0, 1, 2, 3, \ldots$ successively. Then from (4) and (20), we can obtain the travlling-wave solution of the Volterra lattice. At the Mth-order approximation, we have the analytic solution of (3), namely

$$\begin{aligned} u_n(x,t) &= a w_n(\xi_n) \approx a W_{M,n}(\xi_n) \\ &= a \sum_{m=0}^{M} w_{m,n}(\xi_n). \end{aligned} \tag{29}$$

In usual HAM [11], there is only one unknown convergence-control parameter c_0. By the so called c_0-curve, we can determine the possible valid region of c_0, but unfortunately it can not tell us the optimal value of c_0 which gives the fastest convergent series. To find the possible optimal values of convergence-control parameters, it usually needs to minimize the exact square residual error [23]. But it is a pity that the calculation needs too much CPU time even if the order of approximation is not very high, and thus is often useless in practice. Moreover, in the expression of the obtained solution, there are three unknown convergence-control parameters c_0, c_1, and c_2 to make sure the convergence of the solutions. So in this paper, as in [23], we just determine the possible optimal values of convergence-control parameters by minimizing the averaged residual error

$$E_M = \frac{1}{L}\sum_{j=0}^{L} [\mathcal{N}(W_M(j\Delta x), K_M)]^2, \tag{30}$$

where we usually choose $\Delta x = 1/10$, $L = 20$, and $M = 10$ in this paper.

3. Comparisons of Different Approaches

In this section, we will give optimal homotopy analysis approaches with different numbers of unknown convergence-control parameters, and compare them in details. For ease of comparison, we suppose $a = t = 1$, $k = 0.5$, $q = 0.9$, and take three different cases of unknown convergence-control parameters as in [23].

3.1. Optimal c_0 in Case of $c_1 = c_2 = 0$

In this case, the method proposed above degenerates into the usual HAM and there is only one unknown convergence-control parameter c_0. In usual HAM, we can investigate the influence of c_0 on the series of $w_n(\xi_n)$ by means of the so-called c_0-curves. As pointed by Liao [11], the valid region of c_0 is a horizontal line segment. Thus, the valid region of c_0 in this example as shown in Figure 2 is $-0.4 < c_0 < 2$. So we can just determine the possible valid region of c_0. However, usually the c_0-curves can not tell us the optimal value of c_0 which gives the fastest convergent series, and it is a pity that the exact square residual error needs too much CPU time to calculate even if the order of approximation is not very high, and thus is often useless in practice.

To overcome this shortcoming, in [23], Liao advised to determine the possible optimal value of c_0 by the minimum of averaged residual error E_{10}, corresponding to the nonlinear algebraic equation $E'_{10} = 0$. And as shown in [23, 24], the averaged residual error can greatly decrease the CPU time and also give good enough approximation of the optimal convergence-control parameter. Hence, using the symbolic computation software Maple, by minimizing the averaged residual error (30), we can directly get the optimal convergence-control parameter $c_0 = 0.8658$. According to Table 1, by means of $c_0 = 0.8658$, the value of residual error converges much faster to 0 than the

m, order of approximation	Optimal value of c_0	Minimum value of E_m	Value of E_m when $c_0 = -1$
5	0.7452	$0.8693 \cdot 10^{-7}$	$0.6283 \cdot 10^{-6}$
10	0.8658	$0.3804 \cdot 10^{-12}$	$0.2126 \cdot 10^{-9}$

Table 1. Comparison of averaged residual error given by different c_0 in case of $c_1 = c_2 = 0$.

m, order of approximation	Optimal value of $c_1 = c_2$	Minimum value of E_m	Value of E_m when $c_1 = c_2 = 0$
5	0.482	$0.784 \cdot 10^{-7}$	$0.6283 \cdot 10^{-6}$
10	0.501	$0.471 \cdot 10^{-12}$	$0.2126 \cdot 10^{-9}$

Table 2. Comparison of averaged residual error given by different $c_1 = c_2$ in case of $c_0 = -1$.

m, order of approximation	Optimal value of $c_1 \neq c_2$	Minimum value of E_m	Value of E_m when $c_1 = c_2 = 0$
5	$c_1 = -0.387$, $c_2 = 0.648$	$0.1375 \cdot 10^{-8}$	$0.6283 \cdot 10^{-6}$
10	$c_1 = -0.3072$, $c_2 = 0.7758$	$0.3832 \cdot 10^{-12}$	$0.2126 \cdot 10^{-9}$

Table 3. Comparison of averaged residual error given by different $c_1 \neq c_2$ in case of $c_0 = -1$.

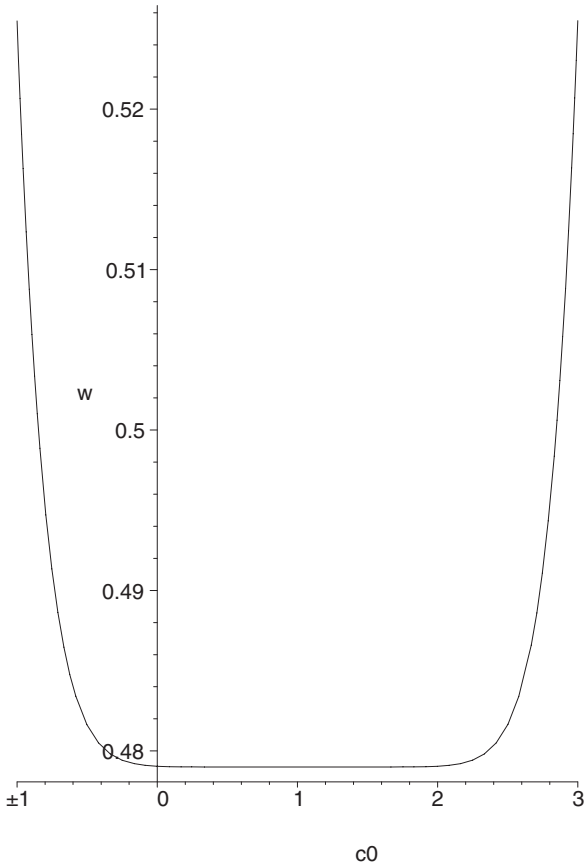

Fig. 2. c_0-curves for the solution $w_n(\xi_n)$: 10-order approximation with $n = 10$, $a = k = 1$, $q = 0.9$, and $t = 1$.

corresponding homotopy series solution given by the usual HAM in case of $c_0 = -1$ and $c_1 = c_2 = 0$, which proves the conclusion drawn by Liao [23], Abbasbandy [15] and many other researchers that $c_0 = -1$ may not be the best value for the usual HAM. So, even the one-parameter optimal HAM can give much better approximations.

3.2. *Optimal $c_1 = c_2$ in Case of $c_0 = -1$*

Here, we investigate another one-parameter optimal approach in case $c_0 = -1$ with the unknown $c_1 = c_2$. Using the symbolic computation software Maple too, we can directly get the optimal convergence-control parameter $c_1 = c_2 = 0.501$. It is found that the homotopy approximations given by $c_0 = -1$ and $c_1 = c_2 = 0.501$ converge much faster than those given by the usual HAM in case of $c_0 = -1$ and $c_1 = c_2 = 0$, as shown in Table 2. This further illustrates that the second one-parameter optimal HAM is as good as the first one mentioned above.

3.3. *Optimal $c_1 \neq c_2$ in Case of $c_0 = -1$*

Here, we investigate the two-parameter optimal approach in the case $c_0 = -1$ with the unknown $c_1 \neq c_2$. According to above section, we can directly get the optimal convergence-control parameter $c_1 = 0.63904$ and $c_2 = -0.66415$. As shown in Table 3, it is found that the homotopy approximations given by $c_0 = -1$, $c_1 = 0.63904$, and $c_2 = -0.66415$ converge much faster than those given by the usual HAM in case of $c_0 = -1$ and $c_1 = c_2 = 0$, too. This further proves that the two-parameter optimal homotopy analysis approach is efficient, too.

4. Conclusions

In this paper, the optimal HAM is extended to construct the travelling-wave solution of the Volterra lat-

tice. The obtained results show that the optimal HAM is also effective for DDEs. Unlike the usual HAM, the optimal HAM used three convergence-control parameters to guarantee the convergence of the homotopy series solution. As shown in this paper, by minimizing the averaged residual error, we can get the possible optimal value of the convergence-control parameters which may give the fastest convergent series. Note that the linear operator $\mathcal{L}$ and the nonlinear operator $\mathcal{N}$ in (17) are rather general so that the above mentioned optimal HAM can be employed to find the travelling-wave solutions with more fast convergence for different types of equations with strong nonlinearity, such as fractional differential equations, supersymmetric equations, stochastic differential equations, which we will consider in following works.

Acknowledgements

This work was supported by Leading Academic Discipline Program, 211 Project for Shanghai University of Finance and Economics (the 3rd phase). The author would like to thank the City University of Hong Kong for warm hospitality.

[1] E. Fermi, J. Pasta, and S. Ulam, Collected Papers of Enrico Fermi II, Univ. of Chicago Press, Chicago 1965.
[2] M. J. Ablowitz and P. A. Clarkson, Soliton, Nonlinear Evolution Equations and Inverse Scattering, Cambridge University Press, New York 1991.
[3] M. Wadati and M. Toda, J. Phys. Soc. Jpn. **39**, 1196 (1975).
[4] C. Rogers and W. K. Schief, Bäcklund and Darboux Transformations, Geometry and Modern Applications in Soliton Theory, Cambridge University Press, Cambridge 2002.
[5] C. H. Gu, H. S. Hu, and Z. X. Zhou, Darboux transformations in integrable systems: theory and their applications to geometry, Springer, Berlin 1991.
[6] P. J. Olver, Applications of Lie Groups to Differential Equations, Springer, New York 1993.
[7] R. Hirota, Direct Methods in Soliton Theory, Cambridge University Press, Cambridge 1980.
[8] S. J. Liao, Ph.D. Thesis, Shanghai Jiao Tong University 1992.
[9] S. J. Liao, Int. J. Nonlin. Mech. **30**, 371 (1995).
[10] S. J. Liao, Int. J. Nonlin. Mech. **32**, 815 (1997).
[11] S. J. Liao, Beyond Perturbation: Introduction to the Homotopy Analysis Method, Chapman & Hall/CRC Press, Boca Raton 2003.
[12] S. Abbasbandy, Phys. Lett. A **360**, 109 (2006).
[13] S. Abbasbandy, Phys. Lett. A **361**, 478 (2007).
[14] S. Abbasbandy, T. Hayat, R. Ellahi, and S. Asghar, Z. Naturforsch. **64a**, 59 (2009).
[15] S. Abbasbandy, Nonlin. Anal.: Real World Appl. **11**, 307 (2010).
[16] L. Zou, Z. Zong, Z. Wang, and L. He, Phys. Lett. A **370**, 287 (2007).
[17] Q. Wang, Appl. Math. Comput. **217**, 4122 (2010).
[18] D. Levi and R. I. Yamilov, J. Math. Phys. **38**, 6648 (1997).
[19] S. I. Svinolupov and R. I. Yamilov, Phys. Lett. A **160**, 548 (1991).
[20] R. I. Yamilov, J. Phys. A: Math. Gen. **27**, 6839 (1994).
[21] V. E. Adler, S. I. Svinolupov, and R. I. Yamilov, Phys. Lett. A **254**, 24 (1999).
[22] A. B. Shabat and R. I. Yamilov, Phys. Lett. A **227**, 15 (1997).
[23] S. J. Liao, Commun. Nonlin. Sci. Numer. Simul. **15**, 2003 (2010).
[24] Q. Wang, Nonlin. Anal.: Real World Appl. **12**, 1555 (2011).
[25] M. J. Ablowitz and J. F. Ladik, Stud. Appl. Math. **57**, 1 (1977).
[26] R. Hirota and M. Iwao, CRM Proc. Lect. Notes **25**, 217 (2000).
[27] Y. B. Suris, The Problem of Integrable Discretization: Hamiltonian Approach, Birkhäuser Verlag, Basel 2003.
[28] D. Baldwin, Ü. Göktaş, and W. Hereman, Comput. Phys. Commun. **162**, 203 (2004).