Some Unsteady Flows of a Jeffrey Fluid between Two Side Walls over a Plane Wall

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In this paper, some time-dependent flows of a non-Newtonian fluid between two side walls over a plane wall are investigated. The following three problems have been studied: (i) flow due to an oscillating plate, (ii) flow due to an accelerating plate, and (iii) flow due to applied constant stress. The explicit expressions for the velocity field are determined by using the integral transforms. The solutions that have been obtained, depending on the initial and boundary conditions, are written as sum of the steady state and transient solutions. The similar solutions for second-grade and Newtonian fluids can be deduced as limiting cases of our solutions. Furthermore, in absence of the side walls they reduce to the similar solutions over an infinite plate. The effects of some important parameters due to side walls on the flow field are investigated.

Key words: Exact Solutions; Jeffrey Fluid; Side Walls.

1. Introduction

The study of non-Newtonian fluids have received great attention during the recent years, because traditional Newtonian fluids cannot precisely describe the rheological characteristics of many fluids used in industrial and engineering applications. Such fluids have a nonlinear relationship between the stress and the rate of strain at a point and exhibit some worth notice facts which are due to their elastic nature. Many materials of industrial significance, notably polymer systems (melt and solutions) and multi-phase system such as foams, emulsions, and slurries, display a range of non-Newtonian characteristic including shear thinning/shear thickening, shear-dependent viscosity, stress relaxation, normal stress difference etc. Hence, due to the practical and fundamental association of non-Newtonian fluids to industrial applications, several studies [1 – 10] of these fluids in different geometries have been carried out. Amongst non-Newtonian fluids the Jeffrey model is one of the simplest types of model to account for rheological effects of viscoelastic fluids. The Jeffrey model is a relatively simple linear model using the time derivatives instead of convected derivatives. Some recent works on Jeffrey model can be found in [11 – 14] and the references therein.

The present work has been undertaken in order to obtain the exact analytical solutions for the three unsteady flows of a Jeffrey fluid between two side walls over a plane wall. Starting solutions are developed by the Fourier sine and Laplace transform methods. The paper is organized in the following way. In Section 2, the governing equations are outlined. In the subsequent three sections, we derive the solutions for (i) flow due to an oscillating plate, (ii) flow due to an accelerating plate, and (iii) flow due to applied constant stress. The limiting case when $d \to \infty$ is presented in Section 6. Numerical results and discussions are given in Section 7, and finally conclusions are made in Section 8.

2. Equations of Motion

The constitutive equation for a Jeffrey fluid [11, 14] is given by

$$\mathbf{T} = -p\mathbf{I} + \mathbf{S},$$
$$\mathbf{S} = \frac{\mu}{1+\lambda_1}\left[\mathbf{A}_1 + \lambda_2\left(\frac{\partial \mathbf{A}_1}{\partial t} + \mathbf{V}\cdot\nabla\right)\mathbf{A}_1\right], \quad (1)$$

where $\mathbf{T}$ is the Cauchy stress tensor, $\mathbf{S}$ the extra stress tensor, μ the dynamic viscosity, and λ_i $(i = 1,2)$ the material parameters of the Jeffrey fluid; the Rivlin–Ericksen tensor $\mathbf{A}_1$ is defined through

$$\mathbf{A}_1 = \nabla\mathbf{V} + (\nabla\mathbf{V})^{\mathsf{T}}. \quad (2)$$

The fundamental equations governing the unsteady flow of an incompressible fluid are given by

$$\nabla \cdot \mathbf{V} = 0, \tag{3}$$

$$\rho \frac{\mathrm{d}\mathbf{V}}{\mathrm{d}t} = \mathrm{div}\mathbf{T}, \tag{4}$$

in which $\mathbf{V}$ is the velocity field and ρ the density.

For two-dimensional flow, we shall assume a velocity and a stress field of the form

$$\mathbf{V}(y,z,t) = u(y,z,t)\mathbf{i}, \ \ \mathbf{S} = \mathbf{S}(y,z,t), \tag{5}$$

where $\mathbf{i}$ is the unit vector along the x-direction of the Cartesian coordinate system. For these flows, the constraint of incompressibility is automatically satisfied.

Substituting (5) into (1) and having in mind the initial condition

$$\mathbf{S}(y,z,0) = \mathbf{0}, \tag{6}$$

it results that $S_{xx} = S_{yy} = S_{zz} = S_{yz} = 0$ for all time and

$$S_{xy} = \frac{\mu}{1+\lambda_1}\frac{\partial}{\partial y}\left(1+\lambda_2\frac{\partial}{\partial t}\right)u(y,z,t) \tag{7}$$

and

$$S_{xz} = \frac{\mu}{1+\lambda_1}\frac{\partial}{\partial z}\left(1+\lambda_2\frac{\partial}{\partial t}\right)u(y,z,t), \tag{8}$$

where S_{xy} and S_{xz} are the non-trivial tangential stresses.

In the absence of a pressure gradient in the flow direction the balance of linear momentum (4) along with (5), (7), and (8) yield the governing equation

$$\frac{\partial u(y,z,t)}{\partial t} = \frac{\nu}{1+\lambda_1}\left(1+\lambda_2\frac{\partial}{\partial t}\right) \cdot \left[\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right]u(y,z,t), \tag{9}$$

where $\nu = \mu/\rho$ is the kinematic viscosity of the fluid.

3. Flow Due to a Rigid Oscillating Plate

Let us consider an incompressible Jeffrey fluid at rest over an infinite flat plate along the x-axis and between two side walls situated in the planes at $y = \pm d$. At time $t > 0$ the flat plate at $z = 0$ begins to oscillate in its own plane. Due to the shear, the fluid above the plate is gradually moved. The velocity field and governing equations are given in (5) and (9), respectively.

The associated boundary and initial conditions are

$$u(\pm d,z,t) = 0 \quad \text{for } z \ge 0 \text{ and } t > 0, \tag{10}$$

$$u(y,0,t) = UH(t)\cos(\omega t) \ \text{ or } UH(t)\sin(\omega t) \quad \text{for all } t, \tag{11}$$

$$u(y,z,t) \to 0 \quad \text{as } z \to \infty \text{ and } t > 0, \tag{12}$$

$$u(y,z,0) = 0 \quad \text{for } z \ge 0 \text{ and } -d < y < d, \tag{13}$$

where U is the amplitude, ω the frequency of the velocity of the wall, and $H(t)$ the Heaviside unit step function.

The first boundary condition suggests the following form for $u(y,z,t)$:

$$u(y,z,t) = \sum_{n=0}^{\infty} u_n(z,t)\cos(b_n y) \tag{14}$$

with $b_n = (2n+1)\pi/2d$.

Introducing (14) into (9), multiplying both sides of the results by $\sin(\xi z)$, and integrating the result with respect to z from 0 to ∞, having in mind the boundary conditions (11) and (12), gives

$$\left[1+\frac{\alpha}{1+\lambda_1}(b_n^2+\xi^2)\right]\frac{\partial \bar{u}_n(\xi,t)}{\partial t} + \frac{\nu(b_n^2+\xi^2)}{1+\lambda_1}\bar{u}_n(\xi,t) = \frac{4U(-1)^n\xi}{(2n+1)\pi(1+\lambda_1)}\left(\nu+\alpha\frac{\partial}{\partial t}\right)H(t)\cos(\omega t), \tag{15}$$

respectively,

$$\left[1+\frac{\alpha}{1+\lambda_1}(b_n^2+\xi^2)\right]\frac{\partial \bar{u}_n(\xi,t)}{\partial t} + \frac{\nu(b_n^2+\xi^2)}{1+\lambda_1}\bar{u}_n(\xi,t) = \frac{4U(-1)^n\xi}{(2n+1)\pi(1+\lambda_1)}\left(\nu+\alpha\frac{\partial}{\partial t}\right)H(t)\sin(\omega t), \tag{16}$$

where $\alpha = \nu\lambda_2$.

The Fourier sine transform $\bar{u}_n(\xi,t)$ of $u_n(z,t)$ has to satisfy the condition

$$\bar{u}_n(\xi,0) = 0 \quad \text{for } \xi > 0. \tag{17}$$

Now applying the Laplace transform to (15) and (16), keeping in mind the initial condition (17), we obtain

$$\bar{U}_n(\xi,q) = \frac{4U(-1)^n\xi s_n}{(2n+1)\pi\nu(\xi^2+b_n^2)}\frac{(\nu+\alpha q)}{q+s_n} \cdot \frac{q}{q^2+\omega^2}, \tag{18}$$

respectively,

$$\bar{U}_n(\xi,q) = \frac{4U(-1)^n \xi s_n}{(2n+1)\pi\nu(\xi^2+b_n^2)} \frac{(\nu+\alpha q)}{q+s_n} \cdot \frac{\omega}{q^2+\omega^2}, \tag{19}$$

where $\bar{U}_n(\xi,q)$ is the Laplace transform of $\bar{u}_n(\xi,t)$ and $s_n = \frac{\nu(b_n^2+\xi^2)}{1+\lambda_1} / \left[1+\frac{\alpha}{1+\lambda_1}\left(b_n^2+\xi^2\right)\right]$.

In order to determine $\bar{u}_n(\xi,t)$, we first write (18) and (19) under the simpler form as

$$\bar{U}_n(\xi,q) = \frac{4U(-1)^n \xi s_n}{(2n+1)\pi\nu(\xi^2+b_n^2)} \frac{1}{q} G(\xi,q), \tag{20}$$

where

$$\begin{aligned} G(\xi,q) &= \frac{(\nu+\alpha q)q^2}{(q+s_n)(q^2+\omega^2)} \\ &= \alpha + \frac{s_n^2(\nu-\alpha s_n)}{s_n^2+\omega^2}\frac{1}{q+s_n} \\ &+ \frac{\omega^2(\nu-\alpha s_n)}{s_n^2+\omega^2}\frac{q}{q^2+\omega^2} - \frac{\omega(\alpha\omega^2+\nu s_n)}{s_n^2+\omega^2}\frac{\omega}{q^2+\omega^2}, \end{aligned} \tag{21}$$

respectively,

$$\begin{aligned} G(\xi,q) &= \frac{\omega(\nu+\alpha q)q}{(q+s_n)(q^2+\omega^2)} \\ &= -\frac{\omega s_n(\nu-\alpha s_n)}{s_n^2+\omega^2}\frac{1}{q+s_n} \\ &+ \frac{\omega(\alpha\omega^2+\nu s_n)}{s_n^2+\omega^2}\frac{q}{q^2+\omega^2} + \frac{\omega^2(\nu-\alpha s_n)}{s_n^2+\omega^2}\frac{\omega}{q^2+\omega^2}, \end{aligned} \tag{22}$$

and use the inverse Laplace transform to (20); in combination with the convolution theorem [15], we obtain

$$\begin{aligned} \bar{u}_n(\xi,t) &= \frac{4U(-1)^n \xi s_n}{(2n+1)\pi\nu(\xi^2+b_n^2)}\left[-\frac{s_n(\nu-\alpha s_n)}{s_n^2+\omega^2}\mathrm{e}^{-s_n t}\right. \\ &\left.+\frac{\omega(\nu-\alpha s_n)}{s_n^2+\omega^2}\sin(\omega t) + \frac{(\alpha\omega^2+\nu s_n)}{s_n^2+\omega^2}\cos(\omega t)\right], \end{aligned} \tag{23}$$

respectively,

$$\begin{aligned} \bar{u}_n(\xi,t) &= \frac{4U(-1)^n \xi s_n}{(2n+1)\pi\nu(\xi^2+b_n^2)}\left[\frac{\omega(\nu-\alpha s_n)}{s_n^2+\omega^2}\mathrm{e}^{-s_n t}\right. \\ &\left.+\frac{(\alpha\omega^2+\nu s_n)}{s_n^2+\omega^2}\sin(\omega t) - \frac{\omega(\nu-\alpha s_n)}{s_n^2+\omega^2}\cos(\omega t)\right]. \end{aligned} \tag{24}$$

Finally, application of the inverse Fourier sine transform to (23) and (24) gives the velocity field in the following form:

$$\begin{aligned} u_n(z,t) &= \frac{8U(-1)^n H(t)}{(2n+1)\pi^2\nu} \\ &\cdot\left\{-\int_0^\infty \frac{(\nu s_n-\alpha s_n^2)s_n\mathrm{e}^{-s_n t}}{(s_n^2+\omega^2)(\xi^2+b_n^2)}\xi\sin(\xi z)\,\mathrm{d}\xi\right. \\ &+\omega\sin(\omega t)\int_0^\infty \frac{(\nu s_n-\alpha s_n^2)}{(s_n^2+\omega^2)(\xi^2+b_n^2)}\xi\sin(\xi z)\,\mathrm{d}\xi \\ &\left.+\cos(\omega t)\int_0^\infty \frac{(\alpha\omega^2 s_n+\nu s_n^2)}{(s_n^2+\omega^2)(\xi^2+b_n^2)}\xi\sin(\xi z)\,\mathrm{d}\xi\right\}, \end{aligned} \tag{25}$$

respectively,

$$\begin{aligned} u_n(z,t) &= \frac{8U(-1)^n H(t)}{(2n+1)\pi^2\nu} \\ &\cdot\left\{\int_0^\infty \frac{(\nu\omega s_n-\alpha\omega s_n^2)\mathrm{e}^{-s_n t}}{(s_n^2+\omega^2)(\xi^2+b_n^2)}\xi\sin(\xi z)\,\mathrm{d}\xi\right. \\ &+\sin(\omega t)\int_0^\infty \frac{(\alpha\omega^2 s_n+\nu s_n^2)}{(s_n^2+\omega^2)(\xi^2+b_n^2)}\xi\sin(\xi z)\,\mathrm{d}\xi \\ &\left.-\omega\cos(\omega t)\int_0^\infty \frac{(\nu s_n-\alpha s_n^2)}{(s_n^2+\omega^2)(\xi^2+b_n^2)}\xi\sin(\xi z)\,\mathrm{d}\xi\right\}. \end{aligned} \tag{26}$$

However, for equivalent but simpler forms of the above expressions, we can write

$$\begin{aligned} &\int_0^\infty \frac{s_n\xi\sin(\xi z)}{(s_n^2+\omega^2)(\xi^2+b_n^2)}\mathrm{d}\xi = \frac{\nu\alpha}{(\alpha^2\omega^2+\nu^2)} \\ &\cdot\int_0^\infty \frac{\left[\xi^2+b_n^2+\frac{1+\lambda_1}{\alpha}\right]\xi\sin(\xi z)}{\left[\xi^2+b_n^2+\frac{\alpha\omega^2(1+\lambda_1)}{\alpha^2\omega^2+\nu^2}\right]^2+\frac{\omega^2\nu^2(1+\lambda_1)^2}{(\alpha^2\omega^2+\nu^2)^2}}\mathrm{d}\xi \end{aligned} \tag{27}$$

and

$$\begin{aligned} &\int_0^\infty \frac{s_n^2\xi\sin(\xi z)}{(s_n^2+\omega^2)(\xi^2+b_n^2)}\mathrm{d}\xi = \frac{\nu^2}{(\alpha^2\omega^2+\nu^2)} \\ &\cdot\int_0^\infty \frac{\left[\xi^2+b_n^2\right]\xi\sin(\xi z)}{\left[\xi^2+b_n^2+\frac{\alpha\omega^2(1+\lambda_1)}{\alpha^2\omega^2+\nu^2}\right]^2+\frac{\omega^2\nu^2(1+\lambda_1)^2}{(\alpha^2\omega^2+\nu^2)^2}}\mathrm{d}\xi. \end{aligned} \tag{28}$$

We know that

$$\begin{aligned} &\int_0^\infty \frac{\left[\xi^2+b_n^2+\frac{\alpha\omega^2(1+\lambda_1)}{\alpha^2\omega^2+\nu^2}\right]\xi\sin(\xi z)}{\left[\xi^2+b_n^2+\frac{\alpha\omega^2(1+\lambda_1)}{\alpha^2\omega^2+\nu^2}\right]^2+\frac{\omega^2\nu^2(1+\lambda_1)^2}{(\alpha^2\omega^2+\nu^2)^2}}\mathrm{d}\xi \\ &= \frac{\pi}{2}\mathrm{e}^{-A_n z}\cos(B_n z) \end{aligned} \tag{29}$$

and

$$\int_0^\infty \frac{\xi \sin(\xi z)}{\left[\xi^2 + b_n^2 + \frac{\alpha\omega^2(1+\lambda_1)}{\alpha^2\omega^2+\nu^2}\right]^2 + \frac{\omega^2\nu^2(1+\lambda_1)^2}{(\alpha^2\omega^2+\nu^2)^2}} \mathrm{d}\xi = \frac{\pi}{2} \frac{(\alpha^2\omega^2+\nu^2)}{\nu\omega(1+\lambda_1)} \mathrm{e}^{-A_n z} \sin(B_n z) \tag{30}$$

with

$$2A_n^2 = \left[\left(b_n^2 + \frac{\alpha\omega^2(1+\lambda_1)}{\alpha^2\omega^2+\nu^2}\right)^2 + \frac{\nu^2\omega^2(1+\lambda_1)^2}{(\alpha^2\omega^2+\nu^2)^2}\right]^{1/2} + \left(b_n^2 + \frac{\alpha\omega^2(1+\lambda_1)}{\alpha^2\omega^2+\nu^2}\right),$$

$$2B_n^2 = \left[\left(b_n^2 + \frac{\alpha\omega^2(1+\lambda_1)}{\alpha^2\omega^2+\nu^2}\right)^2 + \frac{\nu^2\omega^2(1+\lambda_1)^2}{(\alpha^2\omega^2+\nu^2)^2}\right]^{1/2} - \left(b_n^2 + \frac{\alpha\omega^2(1+\lambda_1)}{\alpha^2\omega^2+\nu^2}\right).$$

In view of (27) – (30), we find for the velocity field the expressions

$$\frac{u(y,z,t)}{U} = \sum_{n=0}^\infty \frac{8(-1)^n H(t)}{(2n+1)\pi^2} \cdot \left[-\int_0^\infty \frac{(\nu s_n - \alpha s_n^2) s_n \mathrm{e}^{-s_n t}}{\nu(s_n^2+\omega^2)(\xi^2+b_n^2)} \xi \sin(\xi z)\, \mathrm{d}\xi + \frac{\pi}{2}\mathrm{e}^{-A_n z}\cos(\omega t - B_n z)\right] \cos(b_n y), \tag{31}$$

respectively,

$$\frac{u(y,z,t)}{U} = \sum_{n=0}^\infty \frac{8(-1)^n H(t)}{(2n+1)\pi^2} \cdot \left[\int_0^\infty \frac{(\nu\omega s_n - \alpha\omega s_n^2) \mathrm{e}^{-s_n t}}{\nu(s_n^2+\omega^2)(\xi^2+b_n^2)} \xi \sin(\xi z)\, \mathrm{d}\xi + \frac{\pi}{2}\mathrm{e}^{-A_n z}\sin(\omega t - B_n z)\right] \cos(b_n y). \tag{32}$$

The starting solutions (31) and (32) are presented as sum of the steady and transient solutions. For large values of time t, these starting solutions reduce to the steady state solutions given by

$$\frac{u(y,z,t)}{U} = \sum_{n=0}^\infty \frac{4(-1)^n}{(2n+1)\pi} \mathrm{e}^{-A_n z} \cdot \cos(\omega t - B_n z)\cos(b_n y), \tag{33}$$

respectively,

$$\frac{u(y,z,t)}{U} = \sum_{n=0}^\infty \frac{4(-1)^n}{(2n+1)\pi} \mathrm{e}^{-A_n z} \cdot \sin(\omega t - B_n z)\cos(b_n y), \tag{34}$$

which are periodic in time and independent of the initial condition.

4. Flow Due to an Accelerating Plate

Now, we consider the flow problem for which the bottom plate after time $t > 0$ begins to move with a constant acceleration A in the positive x-direction and induces the motion to the fluid. The governing problem consists of (9), (10), (12), (13), and

$$u(y,0,t) = At. \tag{35}$$

Employing the methodology of the previous section, we reach at the following form for $u_n(z,t)$:

$$u_n(z,t) = \frac{8A(-1)^n}{(2n+1)\pi^2\nu} \cdot \left\{-\int_0^\infty \frac{(\alpha s_n - \nu)\mathrm{e}^{-s_n t}}{s_n(\xi^2+b_n^2)} \xi \sin(\xi z)\, \mathrm{d}\xi + \int_0^\infty \frac{(\alpha s_n - \nu)\xi\sin(\xi z)}{s_n(\xi^2+b_n^2)}\mathrm{d}\xi + \nu t\int_0^\infty \frac{\xi\sin(\xi z)}{(\xi^2+b_n^2)}\mathrm{d}\xi\right\}. \tag{36}$$

We know that

$$\int_0^\infty \frac{\xi\sin(\xi z)}{(\xi^2+b_n^2)}\mathrm{d}\xi = \frac{\pi}{2}\exp(-b_n z)$$

and

$$\int_0^\infty \frac{\xi\sin(\xi z)}{(\xi^2+b_n^2)^2}\mathrm{d}\xi = \frac{\pi z}{4b_n}\exp(-b_n z), \quad b_n \neq 0.$$

Therefore, after simplification in the above expression (see [16]), we finally get the velocity field under the form

$$u(y,z,t) = \sum_{n=0}^\infty \frac{8A(-1)^n}{(2n+1)\pi^2\nu} \cdot \left\{-\int_0^\infty \frac{(\alpha s_n - \nu)\mathrm{e}^{-s_n t}}{s_n(\xi^2+b_n^2)}\xi\sin(\xi z)\,\mathrm{d}\xi + \frac{\pi}{2}\exp(-b_n z)\left[\nu t - \frac{(1+\lambda_1)z}{2b_n}\right]\right\}\cos(b_n y), \tag{37}$$

valid for $b_n \neq 0$.

5. Flow Due to an Applied Constant Stress

Here, the flow due to the bottom plate that applies a constant stress $\sigma H(t)$ to the fluid is considered. The governing problem again consists of (9), (10), (12), (13), and

$$\frac{\mu}{1+\lambda_1}\frac{\partial}{\partial z}\left(1+\lambda_2\frac{\partial}{\partial t}\right)u\bigg|_{z=0}=-\sigma H(t). \tag{38}$$

Invoking (14) into (9), multiplying both sides of the resulting equation by $\cos(\xi z)$, integrating the result with respect to z from 0 to ∞, and taking into account the boundary condition (38), the problem reduces to the solution of the following differential equation:

$$\left[1+\frac{\alpha(\xi^2+b_n^2)}{1+\lambda_1}\right]\frac{\partial\bar{u}_n(\xi,t)}{\partial t}+\left[\frac{\nu(\xi^2+b_n^2)}{1+\lambda_1}\right]\bar{u}_n(\xi,t)=\frac{4(-1)^n\sigma H(t)}{(2n+1)\pi\rho}. \tag{39}$$

Adopting a similar procedure as before, the expression for $u_n(z,t)$ is

$$u_n(z,t)=\frac{8(-1)^n\sigma(1+\lambda_1)H(t)}{(2n+1)\pi^2\rho\nu}\cdot\left[\int_0^\infty\frac{\cos(\xi z)}{(\xi^2+b_n^2)}\mathrm{d}\xi-\int_0^\infty\frac{\mathrm{e}^{-s_nt}\cos(\xi z)}{(\xi^2+b_n^2)}\mathrm{d}\xi\right]. \tag{40}$$

With the help of following expression,

$$\int_0^\infty\frac{\cos(\xi z)}{(\xi^2+b_n^2)}\mathrm{d}\xi=\frac{\pi}{2b_n}\exp(-b_nz),\quad b_n\neq 0,$$

we finally obtain the velocity field of the form

$$u(y,z,t)=\sum_{n=0}^{\infty}\frac{8(-1)^n\sigma(1+\lambda_1)H(t)}{(2n+1)\pi^2\rho\nu}\cdot\left\{-\int_0^\infty\frac{\mathrm{e}^{-s_nt}\cos(\xi z)}{(\xi^2+b_n^2)}\mathrm{d}\xi+\frac{\pi}{2b_n}\exp(-b_nz)\right\}\cos(b_ny), \tag{41}$$

valid for $b_n\neq 0$.

In the limiting case, when $t\to\infty$, the above equation reduces to the steady-state solution given by

$$u(y,z)=\sum_{n=0}^{\infty}\frac{4(-1)^n\sigma(1+\lambda_1)}{(2n+1)\pi\nu\rho}\cdot\frac{\exp(-b_nz)}{b_n}\cos(b_ny). \tag{42}$$

6. Limiting Case $d\to\infty$ (Flow over an Infinite Plate)

In absence of the side walls, namely when d goes to infinity, the above obtained solutions reduce to the corresponding solutions for the motion over an infinite plate. Consequently, for instance, (31) and (32) take the forms

$$\frac{u(z,t)}{U}=\sum_{n=0}^{\infty}\frac{8(-1)^nH(t)}{(2n+1)\pi^2}\cdot\left[-\int_0^\infty\frac{(\nu\tilde{s}-\alpha\tilde{s}^2)\tilde{s}\mathrm{e}^{-\tilde{s}t}}{\nu(\tilde{s}^2+\omega^2)}\frac{\sin(\xi z)}{\xi}\mathrm{d}\xi+\frac{\pi}{2}\mathrm{e}^{-\tilde{A}z}\cos(\omega t-\tilde{B}z)\right], \tag{43}$$

respectively,

$$\frac{u(z,t)}{U}=\sum_{n=0}^{\infty}\frac{8(-1)^nH(t)}{(2n+1)\pi^2}\cdot\left[\int_0^\infty\frac{(\nu\omega\tilde{s}-\alpha\omega\tilde{s}^2)\mathrm{e}^{-\tilde{s}t}}{\nu(\tilde{s}^2+\omega^2)}\frac{\sin(\xi z)}{\xi}\mathrm{d}\xi+\frac{\pi}{2}\mathrm{e}^{-\tilde{A}z}\sin(\omega t-\tilde{B}z)\right], \tag{44}$$

where $\tilde{s}=\frac{\nu\xi^2}{1+\lambda_1}\Big/\left[1+\frac{\alpha}{1+\lambda_1}\xi^2\right]$ and corresponding steady-state solutions are

$$\frac{u(z,t)}{U}=\sum_{n=0}^{\infty}\frac{4(-1)^n}{(2n+1)\pi}\mathrm{e}^{-\tilde{A}z}\cos(\omega t-\tilde{B}z), \tag{45}$$

respectively,

$$\frac{u(z,t)}{U}=\sum_{n=0}^{\infty}\frac{4(-1)^n}{(2n+1)\pi}\mathrm{e}^{-\tilde{A}z}\sin(\omega t-\tilde{B}z), \tag{46}$$

in which

$$2\tilde{A}^2=\frac{\omega(1+\lambda_1)\left[\sqrt{\alpha^2\omega^2+\nu^2}+\alpha\omega\right]}{(\alpha^2\omega^2+\nu^2)}$$

and

$$2\tilde{B}^2=\frac{\omega(1+\lambda_1)\left[\sqrt{\alpha^2\omega^2+\nu^2}-\alpha\omega\right]}{(\alpha^2\omega^2+\nu^2)}.$$

7. Graphical Results and Discussion

In this section, we have investigated the behaviour of the velocity field graphically. Here, we have presented the profiles of the velocity field for the first case when the flow is due to an oscillating plate. The analysis is further concerned with both cosine and sine oscillations of the plate. To analyze and interpret the relevant physical effects of the obtained results, the graphs of $u(0,z,t)/U$ giving the velocity profiles at the middle of the channel as well as $u(z,t)/U$ giving the velocity profiles over an infinite plate are drawn. Also, a comparison amongst the profiles of Jeffrey, second-grade, and Newtonian fluids is made.

Figures 1 and 2 show the profiles of the velocity for different values of time t for both cosine and sine oscillations of the boundary by keeping other parameters fixed. It is clearly seen from these figures that the maximum displacement of fluid particles occurs near the bottom plate. These oscillations dies out far away from the bottom plate. Also, it is noted that the amplitude of oscillations in the presence of side walls is smaller as compared with the absence of side walls. Moreover, it is observed that the boundary layer thickness reduces in the presence of side walls. This is due to the increasing shearing force from the side walls. Hence, one can say that due to this increasing shear force from the side walls, the velocity dies out and ap-

(a) (b)

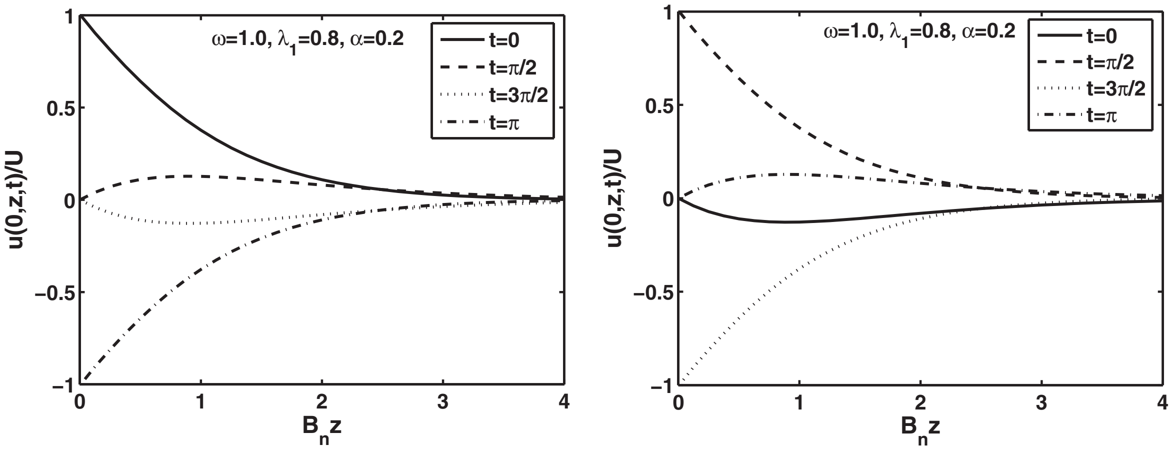

Fig. 1. Profiles of the velocity field $u(0,z,t)/U$, given by (33) [(a) cosine oscillation] and (34) [(b) sine oscillation], for various times t in the presence of side walls.

(a) (b)

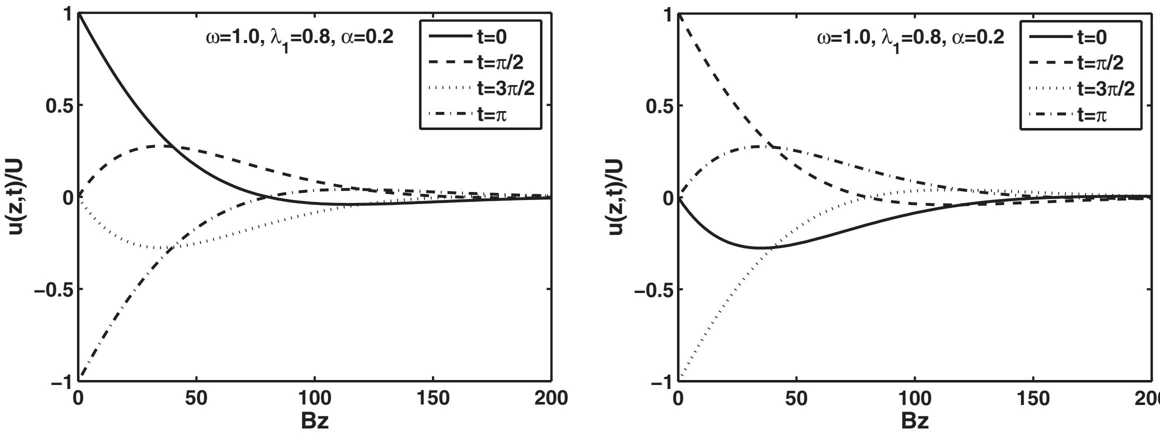

Fig. 2. Profiles of the velocity field $u(z,t)/U$, given by (45) [(a) cosine oscillation] and (46) [(b) sine oscillation], for various times t in the absence of side walls.

proaches to zero much earlier in the presence of side walls.

Figure 3 depicts the effect of the non-Newtonian parameter λ_1 of a Jeffrey fluid on the velocity field in the presence of side walls for the cosine oscillation of the boundary. To analyze the effects of the non-Newtonian parameter λ_1, the other parameters are kept constant. It appears that the velocity is a strong function of the non-Newtonian parameter λ_1. From this figure, it is observed that for a given position z, the velocity gets increased with an increase in λ_1. In other words, increasing the non-Newtonian parameter λ_1 has the effect of increasing the boundary layer thickness.

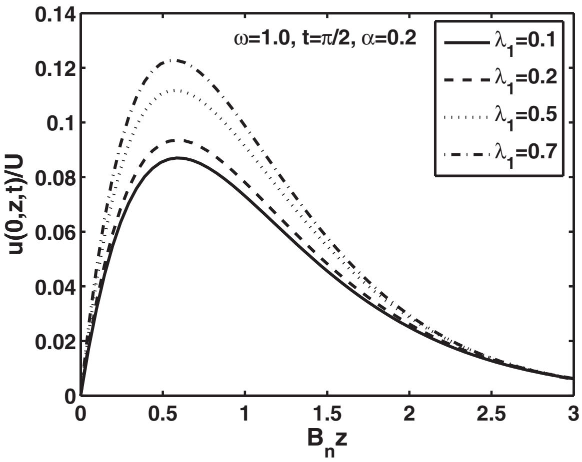

Fig. 3. Profiles of the velocity field $u(0,z,t)/U$, given by (33), for various values of λ_1 in the presence of side walls.

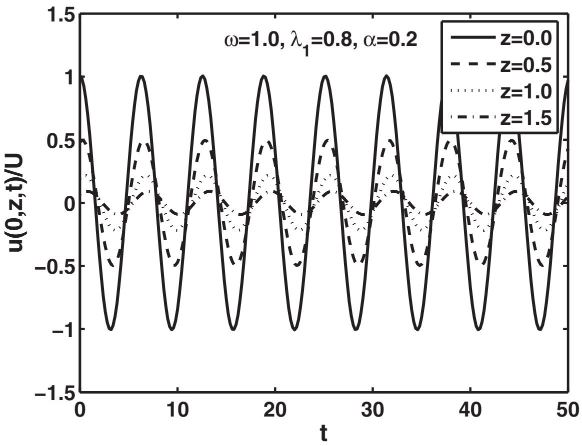

Fig. 4. Time series of the flow velocity, given by (33), for various z in the presence of the side walls.

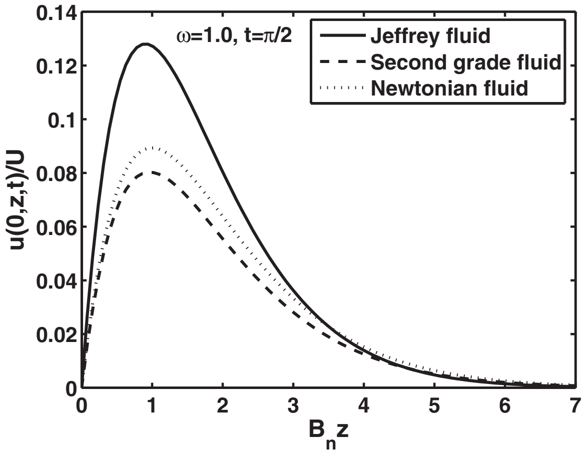

Fig. 5. Comparison of the profiles of velocity $u(0,z,t)/U$ for different fluids for cosine oscillation in the presence of side walls. (Jeffrey fluid: $\alpha = 0.2$, $\lambda_1 = 0.8$; second-grade fluid: $\alpha = 0.2$, $\lambda_1 = 0.0$; Newtonian fluid: $\alpha = 0.0$, $\lambda_1 = 0.0$).

The time series of the velocity profile for various values of z has been plotted in Figure 4. Here, we note that by increasing the parameter z lowers the amplitude of oscillation of the velocity.

Figure 5 displays a comparison amongst the profiles of Jeffrey, second-grade, and Newtonian fluids. One can see that the Jeffrey fluid is the swiftest while the second-grade fluid is the slowest.

8. Conclusions

In this paper, we have studied some unsteady flows of a Jeffrey fluid between two side walls perpendicular to a plane wall. Three time-dependent flows are considered. Analytical expressions for the velocity field are determined by means of the integral transform treatment. The final results are decomposed as a sum of steady-state and transient solutions. The effects of the various pertinent parameters on the velocity are depicted graphically. Moreover, in order to see the effects of side walls, a comparison of the velocity field corresponding to the flow over an infinite plate is made with that for flow between two side walls perpendicular to the plate. It is demonstrated that the presence of the side walls have a significant effect on the velocity field. Finally, it is worth pointing out that the corresponding solutions for the second-grade fluid, performing the same motions, are obtained as limiting cases of the general solutions.

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