# Multi-Soliton Solutions for an Inhomogeneous Nonlinear Schrödinger-Maxwell-Bloch System in the Erbium-Doped Fiber 

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Z. Naturforsch. 66a, 712-720 (2011) / DOI: 10.5560/ZNA.2011-0035

Received March 3, 2011 / revised June 30, 2011
Under investigation in this paper is an inhomogeneous nonlinear Schrödinger-Maxwell-Bloch system with variable dispersion and nonlinear effects, which describes the propagation of optical pulses in an inhomogeneous erbium-doped fiber. Under certain coefficient constraints, multi-soliton solutions are obtained by the Hirota method and symbolic computation. Evolution and interaction of the solitons are plotted, and the self-induced transparency effect caused by the doped erbium atoms is found to lead to the change of the soliton velocity and phase. Overall phase shift can be observed when the parameter accounting for the interaction between the silica and doped erbium atoms is taken as a constant.
Key words: Erbium-Doped Fiber; Inhomogeneous Nonlinear Schrödinger-Maxwell-Bloch System; Evolution and Interaction of Solitons; Multi-Soliton Solutions; Symbolic Computation.

## 1. Introduction

Propagation of the optical solitons in nonlinear fibers has its potential applications in communication systems [1]. Optical solitons, arisen as a result of the balance between the group velocity dispersion (GVD) and the nonlinear effect, have been regarded as an alternative for the next generation of ultralong distance, large capacity, and high-bit-rate communication systems [2]. The nonlinear Schrödinger (NLS) equation has been used to describe the optical-soliton propagation in homogeneous optical fibers [3], while in real fibers, there exist a number of factors which affect the generation and dynamics of the optical solitons [4]. Correspondingly, inhomogeneous NLS (INLS) equations have been thought to be more realistic [5]. Another model is the Maxwell-Bloch (MB) system, which describes the propagation of the self-induced transparency (SIT) soliton in a resonant medium [6]. SIT solitons are coherent optical pulses propagating through a resonant medium without any loss or distortion [7]. The MB system takes the form of [8]

$$
\begin{aligned}
& q_{z}=p \\
& p_{t}-2 \mathrm{i} k p=q \eta, \quad \eta_{t}=-\frac{1}{2}\left(q^{*} p+p^{*} q\right)
\end{aligned}
$$

where $t$ and $z$ are the normalized time and distance along the direction of propagation, $k$ measures the frequency shift from the resonance, and the asterisk denotes the complex conjugate, $q(z, t)$ is the slowly-varying-envelope axial field, $p(z, t)$ and $\eta(z, t)$ are respectively given by $2 v_{1} v_{2}{ }^{*}$ and $\left|v_{2}\right|^{2}-\left|v_{1}\right|^{2}$ with $v_{1}(z, t)$ and $v_{2}(z, t)$ being the wave functions of two energy levels of the resonant atoms and obeying the Zakharov-Shabat equations [3]

$$
\begin{align*}
& \frac{\partial v_{1}}{\partial t}-\mathrm{i} k v_{1}=q v_{2},  \tag{2a}\\
& \frac{\partial v_{2}}{\partial t}+\mathrm{i} k v_{2}=-q^{*} v_{1} \tag{2b}
\end{align*}
$$

which are equivalent to (1.13) and (1.14) in [8].
Some researchers have devoted their attention to the applications of fibers doped with two-level resonant atoms, such as the erbium-doped fiber, which can induce the pulse amplification [9]. In the erbium-doped fibers, the optical pulse propagation is described by the following nonlinear Schrödinger-Maxwell-Bloch (NLS-MB) system [10]:

$$
\begin{align*}
& \mathrm{i} q_{z}+\frac{1}{2} q_{t t}+|q|^{2} q=2 \mathrm{i} p  \tag{3}\\
& p_{t}-2 \mathrm{i} k p=2 q \eta, \quad \eta_{t}=-\left(q^{*} p+p^{*} q\right)
\end{align*}
$$

In such system, the SIT soliton can coexist with the NLS soliton, and this mixed state has been called the SIT-NLS soliton [11]. Since the presentation by [12], the coexistence of SIT soliton and NLS soliton in the erbium-doped fibers has attracted some interest in optical communications [10, 13].

Considering the effects of the inhomogeneities on the propagation of SIT-NLS solitons in the erbiumdoped fibers, we will investigate an INLS-MB system [14, 15],
$\mathrm{i} q_{z}+\alpha(z) q_{t t}+\beta(z)|q|^{2} q+\mathrm{i} \delta(z) q=-\mathrm{i} \gamma(z) p$,
$p_{t}-2 \mathrm{i} k p=2 \tau(z) q \eta, \quad \eta_{t}=-\tau(z)\left(q^{*} p+p^{*} q\right)$,
where $\alpha(z), \beta(z)$ are the variable dispersion and nonlinearity parameters, and $\delta(z)$ represents the gain or loss of the optical signal, $\tau(z)$ describes the interaction between the propagating field and erbium atoms, and $\gamma(z)$ is the parameter accounting for the interaction between silica and doped erbium atoms. Integrability of System (4) has been reported in [14], where certain constraints for the variable dispersion and nonlinearity parameters have been derived through the Painlevé analysis.

With $p$ and $\eta$ defined above and through the transformations

$$
\begin{equation*}
v_{1}=\mathrm{e}^{\mathrm{i} k t} a_{1}, \quad v_{2}=\mathrm{e}^{\mathrm{i} k t} a_{2} \tag{5}
\end{equation*}
$$

System (4) can be written as [16]
$\mathrm{i} q_{z}+\alpha(z) q_{t t}+\beta(z)|q|^{2} q+\mathrm{i} \delta(z) q$
$+2 \mathrm{i} \gamma(z) a_{1} a_{2}{ }^{*}=0$,
$a_{1 t}=\tau(z) q a_{2}, a_{2 t}+2 \mathrm{i} k a_{2}=-\tau(z) q^{*} a_{1}$.
Accordingly, solutions for System (4) can be given when those for System (6) are obtained. It has been shown that System (6) is Painlevé integrable and has the Lax pair [17] when $2 \alpha(z)=\beta(z)=$ constant, $\gamma(z)=$ constant, $\tau(z)=1$, and $\delta(z)=0$. To our knowledge, only one-soliton solutions for System (4) have been given [15].
This paper will be arranged as follows. In Section 2, for System (6), the bilinear form will be derived, and the $N$-soliton solutions will be deduced through the formal parameter expansion, under certain parametric constraints. $N$-soliton solutions for System (4) will be derived through the relation between System (4) and System (6). All solutions are obtained by symbolic
computation [18-20]. In Section 2, figures for System (4) will be plotted to graphically show the evolution and interaction of the SIT-NLS solitons, and the SIT effect will be found to be responsible for the change of the soliton velocity and phase. Section 4 will be our conclusion.

## 2. Bilinear Form and Soliton Solutions

In the following, we will use Hirota's bilinear method [21, 22] to construct the multi-soliton solutions for System (4).

To solve System (4), we consider the following Painlevé integrable constraints [14]:

$$
\begin{align*}
\delta(z) & =\frac{\alpha(z) \beta_{z}(z)-\beta(z) \alpha_{z}(z)}{2 \alpha(z) \beta(z)} \text { and } \\
\tau(z) & =\sqrt{\frac{\beta(z)}{2 \alpha(z)}} \tag{7}
\end{align*}
$$

and can obtain the variable-coefficient bilinear form of System (6) as

$$
\begin{align*}
& \left(\mathrm{i} D_{z}+\alpha(z) D_{t}^{2}\right)(g \cdot f)=-2 \mathrm{i} \gamma_{1}(z) m h^{*}  \tag{8a}\\
& D_{t}^{2}(f \cdot f)=2|g|^{2}  \tag{8b}\\
& D_{t}(m \cdot f)=g h,  \tag{8c}\\
& D_{t}(h \cdot f)+2 \mathrm{i} k h f=-g^{*} m, \tag{8d}
\end{align*}
$$

with the dependent variable transformations

$$
\begin{equation*}
q=\sqrt{\frac{2 \alpha(z)}{\beta(z)}} \frac{g}{f}, a_{1}=\frac{m}{f}, a_{2}=\frac{h}{f} \tag{9}
\end{equation*}
$$

where $g, h$, and $m$ are the complex functions of $z$ and $t, f$ is the real one, and $\gamma_{1}(z)=\sqrt{\frac{\beta(z)}{2 \alpha(z)}} \gamma(z) . \mathrm{D}_{z}$ and $\mathrm{D}_{t}$ are the bilinear differential operators [21] defined by

$$
\begin{align*}
& \mathrm{D}_{z}^{l} \mathrm{D}_{t}^{n}(f \cdot g)=\left(\frac{\partial}{\partial z}-\frac{\partial}{\partial z^{\prime}}\right)^{l}  \tag{10}\\
& \left.\cdot\left(\frac{\partial}{\partial t}-\frac{\partial}{\partial t^{\prime}}\right)^{n} f(z, t) g(z, t)\right|_{z^{\prime}=z, t^{\prime}=t}
\end{align*}
$$

Expand $g, h, m$, and $f$ with respect to a formal expansion parameter $\varepsilon$ as below:

$$
\begin{align*}
& g=\varepsilon g_{1}+\varepsilon^{3} g_{3}+\varepsilon^{5} g_{5}+\ldots \\
& m=1+\varepsilon^{2} m_{2}+\varepsilon^{4} m_{4}+\varepsilon^{6} m_{6}+\ldots  \tag{11}\\
& h=\varepsilon h_{1}+\varepsilon^{3} h_{3}+\varepsilon^{5} h_{5}+\ldots \\
& f=1+\varepsilon^{2} f_{2}+\varepsilon^{4} f_{4}+\varepsilon^{6} f_{6}+\ldots
\end{align*}
$$

where $g_{j}, h_{j}$, and $m_{n}(j=1,3,5, \ldots ; n=2,4,6, \ldots)$ are the complex functions of $z$ and $t$, and $f_{n}(n=2,4,6, \ldots)$ are the real ones, which will be determined.

### 2.1. One-Soliton Solutions

In order to obtain one-soliton solutions for System (6), we choose

$$
\begin{align*}
& g=\varepsilon g_{1}, \quad m=1+\varepsilon^{2} m_{2} \\
& h=\varepsilon h_{1}, \quad f=1+\varepsilon^{2} f_{2} \tag{12}
\end{align*}
$$

Substituting (12) into (8a)-(8d) and collecting the terms with the same power of $\varepsilon$, we have the following solutions:

$$
\begin{align*}
& g_{1}=\mathrm{e}^{\theta_{1}}, m_{2}=d_{1} \mathrm{e}^{\theta_{1}+\theta_{1}^{*}} \\
& h_{1}=b_{1} \mathrm{e}^{\theta_{1}^{*}}, f_{2}=c_{1} \mathrm{e}^{\theta_{1}+\theta_{1}^{*}} \tag{13}
\end{align*}
$$

with
$\theta_{1}=k_{1} t+\mathrm{i} k_{1}^{2} \int \alpha(z) \mathrm{d} z+\frac{2 \mathrm{i}}{2 k+\mathrm{i} k_{1}} \int \gamma_{1}(z) \mathrm{d} z+\xi_{1}$,
$b_{1}=\frac{\mathrm{i}}{2 k-\mathrm{i} k_{1}^{*}}, \quad c_{1}=\frac{1}{\left(k_{1}+k_{1}^{*}\right)^{2}}$,
$d_{1}=\frac{\left(2 k+\mathrm{i} k_{1}\right)}{\left(2 k-\mathrm{i} k_{1}^{*}\right)\left(k_{1}+k_{1}^{*}\right)^{2}}$,
where $\xi_{1}, k_{1}$ are all arbitrary complex constants. Onesoliton solutions for System (6) can be explicitly ex-
pressed as
$q=\sqrt{\frac{\alpha(z)}{2 \beta(z)}} \frac{1}{\sqrt{c_{1}}} \mathrm{e}^{\mathrm{i} \phi_{1}} \operatorname{sech} \chi, a_{2}=\frac{b_{1}}{2 \sqrt{c_{1}}} \mathrm{e}^{-\mathrm{i} \phi_{1}} \operatorname{sech} \chi$, $a_{1}=\frac{1}{2} \mathrm{e}^{\mathrm{i} \varphi_{1}}(1+\tanh \chi)+\frac{1}{2}(1-\tanh \chi)$,
with

$$
\begin{aligned}
& \varphi_{1}=-\mathrm{i} \ln \frac{2 k+\mathrm{i} k_{1}}{2 k-\mathrm{i} k_{1}^{*}}, \quad \chi=\frac{\theta_{1}+\theta_{1}^{*}+\ln c_{1}}{2}, \\
& \phi_{1}=\operatorname{Im}\left(\theta_{1}\right) .
\end{aligned}
$$

Then, the one-soliton expressions for System (4) are given as follows:

$$
\begin{align*}
q= & \sqrt{\frac{\alpha(z)}{2 \beta(z)}} \frac{1}{\sqrt{c_{1}}} \mathrm{e}^{\mathrm{i} \phi_{1}} \operatorname{sech} \chi  \tag{15}\\
p= & \frac{b_{1}}{2 \sqrt{c_{1}}} \mathrm{e}^{\mathrm{i} \phi_{1}} \operatorname{sech} \chi\left[\mathrm{e}^{\mathrm{i} \varphi_{1}}(1+\tanh \chi)\right.  \tag{16}\\
& +(1-\tanh \chi)] \\
\eta= & \frac{b_{1}^{2}}{4 c_{1}} \operatorname{sech}^{2} \chi-\frac{1}{4}\left[\left(\mathrm{e}^{\mathrm{i} \varphi_{1}}-1\right) \tanh \chi+\left(\mathrm{e}^{\mathrm{i} \varphi_{1}}+1\right)\right] \\
& \cdot\left[\left(\mathrm{e}^{-\mathrm{i} \varphi_{1}^{*}}-1\right) \tanh \chi+\left(\mathrm{e}^{\mathrm{i} \varphi_{1}^{*}}+1\right)\right] \tag{17}
\end{align*}
$$

### 2.2. Two-Soliton Solutions

Employing the following expansions

$$
\begin{align*}
& g=\varepsilon g_{1}+\varepsilon^{3} g_{3}, \quad m=1+\varepsilon^{2} m_{2}+\varepsilon^{4} m_{4},  \tag{18}\\
& h=\varepsilon h_{1}+\varepsilon^{3} h_{3}, f=1+\varepsilon^{2} f_{2}+\varepsilon^{4} f_{4},
\end{align*}
$$

we obtain the two-soliton solutions for System (6) as

$$
\begin{align*}
& q=\sqrt{\frac{2 \alpha(z)}{\beta(z)}} \frac{1}{1+c_{11} \mathrm{e}^{\theta_{1}+\theta_{1}^{*}}+c_{21} \mathrm{e}^{\theta_{2}+\theta_{1}^{*}}+c_{12} \mathrm{e}^{\theta_{1}+\theta_{2}^{*}}+c_{22} \mathrm{e}^{\theta_{2}+\theta_{2}^{*}}+c_{31} \mathrm{e}^{\theta_{1}+\theta_{1}+\theta_{1}^{*}+\theta_{1}^{*}+\theta_{2}+\theta_{2}^{*}}},  \tag{19}\\
& a_{1}=\frac{1+d_{11} \mathrm{e}^{\theta_{1}+\theta_{1}^{*}+\theta_{1}+\theta_{2}^{*}}+d_{21} \mathrm{e}^{\theta_{2}+\theta_{1}^{*}}+d_{12} \mathrm{e}^{\theta_{1}+\theta_{2}^{*}}+d_{22} \mathrm{e}^{\theta_{2}+\theta_{2}^{*}}+d_{4} \mathrm{e}^{\theta_{1}+\theta_{1}^{*}+\theta_{2}+\theta_{2}^{*}}}{1+c_{11} \mathrm{e}^{\theta_{1}+\theta_{1}^{*}}+c_{21} \mathrm{e}^{\theta_{2}+\theta_{1}^{*}}+c_{12} \mathrm{e}_{1}+\theta_{2}^{*}+c_{22} \mathrm{e}^{\theta_{2}+\theta_{2}^{*}}+c_{4} \mathrm{e}_{1}+\theta_{1}^{*}+\theta_{2}+\theta_{2}^{*}}  \tag{20}\\
& a_{2}=\frac{b_{1} \mathrm{e}_{1}^{\theta_{1}^{*}}+b_{2} \mathrm{e}_{2}^{\theta_{2}^{*}}+b_{31} \mathrm{e}^{\theta_{1}+\theta_{2}^{*}+\theta_{1}^{*}}+b_{32} \mathrm{e}^{\theta_{2}+\theta_{1}^{*}+\theta_{2}^{*}}}{1+c_{11} \mathrm{e}^{\theta_{1}+\theta_{1}^{*}}+c_{21} \mathrm{e}^{\theta_{2}+\theta_{1}^{*}}+c_{12} \mathrm{e}^{\theta_{1}+\theta_{2}^{*}}+c_{22} \mathrm{e}^{\theta_{2}+\theta_{2}^{*}}+c_{4} \mathrm{e}^{\theta_{1}+\theta_{1}^{*}+\theta_{2}+\theta_{2}^{*}}}, \tag{21}
\end{align*}
$$

with

$$
\begin{aligned}
& b_{j}=\frac{\mathrm{i}}{2 k-\mathrm{i} k_{j}^{*}}, \quad \theta_{j}=k_{j} t+\mathrm{i} k_{j}^{2} \int \alpha(z) \mathrm{d} z+\frac{2 \mathrm{i}}{2 k+\mathrm{i} k_{j}} \gamma_{1}(z) \mathrm{d} z+\xi_{j} \quad \text { for } j=1,2, \\
& n_{31}=\frac{\left(k_{1}-k_{2}\right)^{2}}{\left(k_{2}+k_{1}^{*}\right)^{2}\left(k_{1}+k_{1}^{*}\right)^{2}}, \quad b_{31}=\frac{\mathrm{i}\left(2 k+\mathrm{i} k_{1}\right)\left(k_{1}^{*}-k_{2}^{*}\right)^{2}}{\left(2 k-\mathrm{i} k_{1}^{*}\right)\left(2 k-\mathrm{i} k_{2}^{*}\right)\left(k_{1}+k_{2}^{*}\right)^{2}\left(k_{1}+k_{1}^{*}\right)^{2}},
\end{aligned}
$$

$$
\begin{aligned}
& n_{32}=\frac{\left(k_{1}-k_{2}\right)^{2}}{\left(k_{1}+k_{2}^{*}\right)^{2}\left(k_{2}+k_{2}^{*}\right)^{2}}, \quad b_{32}=\frac{\mathrm{i}\left(2 k+\mathrm{i} k_{2}\right)\left(k_{1}^{*}-k_{2}^{*}\right)^{2}}{\left(2 k-\mathrm{i} k_{1}^{*}\right)\left(2 k-\mathrm{i} k_{2}^{*}\right)\left(k_{2}+k_{1}^{*}\right)^{2}\left(k_{2}+k_{2}^{*}\right)^{2}}, \\
& c_{11}=\frac{1}{\left(k_{1}+k_{1}^{*}\right)^{2}}, \quad c_{12}=\frac{1}{\left(k_{1}+k_{2}^{*}\right)^{2}}, \quad c_{21}=\frac{1}{\left(k_{2}+k_{1}^{*}\right)^{2}}, \\
& c_{22}=\frac{1}{\left(k_{2}+k_{2}^{*}\right)^{2}}, \quad d_{11}=\frac{\left(2 k+\mathrm{i} k_{1}\right)}{\left(2 k-\mathrm{i} k_{1}^{*}\right)\left(k_{1}+k_{1}^{*}\right)^{2}}, \quad d_{21}=\frac{\left(2 k+\mathrm{i} k_{2}\right)}{\left(2 k-\mathrm{i} k_{1}^{*}\right)\left(k_{2}+k_{1}^{*}\right)^{2}}, \\
& d_{12}=\frac{\left(2 k+\mathrm{i} k_{1}\right)}{\left(2 k-\mathrm{i} k_{2}^{*}\right)\left(k_{1}+k_{2}^{*}\right)^{2}}, \quad c_{4}=\frac{\left|k_{1}-k_{2}\right|^{4}}{\left(k_{1}+k_{1}^{*}\right)^{2}\left(k_{2}+k_{2}^{*}\right)^{2}\left|k_{2}+k_{1}^{*}\right|^{4}}, \\
& d_{22}=\frac{\left(2 k+\mathrm{i} k_{2}\right)}{\left(2 k-\mathrm{i} k_{2}^{*}\right)\left(k_{2}+k_{2}^{*}\right)^{2}}, \quad d_{4}=\frac{\left(2 k+\mathrm{i} k_{1}\right)\left(2 k+\mathrm{i} k_{2}\right)\left|k_{1}-k_{2}\right|^{4}}{\left(2 k-\mathrm{i} k_{1}^{*}\right)\left(2 k-\mathrm{i} k_{2}^{*}\right)\left(k_{1}+k_{1}^{*}\right)^{2}\left(k_{2}+k_{2}^{*}\right)^{2}\left|k_{2}+k_{1}^{*}\right|^{4}},
\end{aligned}
$$

where $k_{1}, k_{2}, \xi_{1}, \xi_{2}$ are all arbitrary complex constants. Similarly, the two-soliton solutions for System (4) can be expressed as

$$
\begin{align*}
& q=\sqrt{\frac{2 \alpha(z)}{\beta(z)}} \frac{\mathrm{e}^{\theta_{1}}+\mathrm{e}^{\theta_{2}}+n_{31} \mathrm{e}^{\theta_{1}+\theta_{2}+\theta_{1}^{*}}+n_{32} \mathrm{e}^{\theta_{1}+\theta_{2}+\theta_{2}^{*}}}{1+c_{11} \mathrm{e}^{\theta_{1}+\theta_{1}^{*}}+c_{21} \mathrm{e}^{\theta_{2}+\theta_{1}^{*}}+c_{12} \mathrm{e}^{\theta_{1}+\theta_{2}^{*}}+c_{22} \mathrm{e}^{\theta_{2}+\theta_{2}^{*}}+c_{4} \mathrm{e}^{\theta_{1}+\theta_{1}^{*}+\theta_{2}+\theta_{2}^{*}}},  \tag{22}\\
& p=2 \frac{b_{1}^{*} \mathrm{e}^{\theta_{1}}+b_{2}^{*} \mathrm{e}^{\theta_{2}}+b_{31}^{*} \mathrm{e}^{\theta_{1}+\theta_{2}+\theta_{1}^{*}}+b_{32}^{*} \mathrm{e}^{\theta_{2}+\theta_{1}+\theta_{2}^{*}}}{1+c_{11} \mathrm{e}^{\theta_{1}+\theta_{1}^{*}}+c_{21} \mathrm{e}^{\theta_{2}+\theta_{1}^{*}}+c_{12} \mathrm{e}^{\theta_{1}+\theta_{2}^{*}}+c_{22} \mathrm{e}^{\theta_{2}+\theta_{2}^{*}}+c_{4} \mathrm{e}^{\theta_{1}+\theta_{1}^{*}+\theta_{2}+\theta_{2}^{*}}}  \tag{23}\\
& \cdot \frac{1+d_{11} \mathrm{e}^{\theta_{1}+\theta_{1}^{*}}+d_{21} \mathrm{e}^{\theta_{2}+\theta_{1}^{*}}+d_{12} \mathrm{e}^{\theta_{1}+\theta_{2}^{*}}+d_{22} \mathrm{e}^{\theta_{2}+\theta_{2}^{*}}+d_{4} \mathrm{e}^{\theta_{1}+\theta_{1}^{*}+\theta_{2}+\theta_{2}^{*}}}{1+c_{11} \mathrm{e}^{\theta_{1}+\theta_{1}^{*}}+c_{21} \mathrm{e}^{\theta_{2}+\theta_{1}^{*}}+c_{12} \mathrm{e}^{\theta_{1}+\theta_{2}^{*}}+c_{22} \mathrm{e}^{\theta_{2}+\theta_{2}^{*}}+c_{4} \mathrm{e}^{\theta_{1}+\theta_{1}^{*}+\theta_{2}+\theta_{2}^{*}}}, \\
& \eta=\frac{\left(b_{1} \mathrm{e}^{\theta_{1}^{*}}+b_{2} \mathrm{e}^{\theta_{2}^{*}}+b_{31} \mathrm{e}^{\theta_{1}+\theta_{2}^{*}+\theta_{1}^{*}}+b_{32} \mathrm{e}^{\theta_{2}+\theta_{1}^{*}+\theta_{2}^{*}}\right)\left(b_{1}^{*} \mathrm{e}^{\theta_{1}}+b_{2}^{*} \mathrm{e}^{\theta_{2}}+b_{31}^{*} \mathrm{e}^{\theta_{1}+\theta_{2}+\theta_{1}^{*}}+b_{32}^{*} \mathrm{e}^{\theta_{1}+\theta_{2}+\theta_{2}^{*}}\right)}{\left(1+c_{11} \mathrm{e}^{\theta_{1}+\theta_{1}^{*}}+c_{21} \mathrm{e}^{\theta_{2}+\theta_{1}^{*}}+c_{12} \mathrm{e}^{\theta_{1}+\theta_{2}^{*}}+c_{22} \mathrm{e}^{\theta_{2}+\theta_{2}^{*}}+c_{4} \mathrm{e}^{\theta_{1}+\theta_{1}^{*}+\theta_{2}+\theta_{2}^{*}}\right)^{2}} \\
& -\frac{1+d_{11} \mathrm{e}^{\theta_{1}+\theta_{1}^{*}}+d_{21} \mathrm{e}^{\theta_{2}+\theta_{1}^{*}}+d_{12} \mathrm{e}^{\theta_{1}+\theta_{2}^{*}}+d_{22} \mathrm{e}^{\theta_{2}+\theta_{2}^{*}}+d_{4} \mathrm{e}^{\theta_{1}+\theta_{1}^{*}+\theta_{2}+\theta_{2}^{*}}}{1+c_{11} \mathrm{e}^{\theta_{1}+\theta_{1}^{*}}+c_{21} \mathrm{e}^{\theta_{2}+\theta_{1}^{*}}+c_{12} \mathrm{e}^{\theta_{1}+\theta_{2}^{*}}+c_{22} \mathrm{e}^{\theta_{2}+\theta_{2}^{*}}+c_{4} \mathrm{e}^{\theta_{1}+\theta_{1}^{*}+\theta_{2}+\theta_{2}^{*}}}  \tag{24}\\
& \cdot \frac{1+d_{11}^{*} \mathrm{e}^{\theta_{1}+\theta_{1}^{*}}+d_{21}^{*} \mathrm{e}^{\theta_{1}+\theta_{2}^{*}}+d_{12}^{*} \mathrm{e}^{\theta_{2}+\theta_{1}^{*}}+d_{22}^{*} \mathrm{e}^{\theta_{2}+\theta_{2}^{*}}+d_{4}^{*} \mathrm{e}^{\theta_{1}+\theta_{1}^{*}+\theta_{2}+\theta_{2}^{*}}}{1+c_{11} \mathrm{e}^{\theta_{1}+\theta_{1}^{*}}+c_{21} \mathrm{e}^{\theta_{2}+\theta_{1}^{*}}+c_{12} \mathrm{e}^{\theta_{1}+\theta_{2}^{*}}+c_{22} \mathrm{e}^{\theta_{2}+\theta_{2}^{*}}+c_{4} \mathrm{e}^{\theta_{1}+\theta_{1}^{*}+\theta_{2}+\theta_{2}^{*}}} .
\end{align*}
$$

With (7), the $N$-soliton solutions for System (6) in the sense of [23] can be expressed as

$$
\begin{aligned}
& q(z, t)=\sqrt{\frac{2 \alpha(z)}{\beta(z)}} \frac{g(z, t)}{f(z, t)}, a_{1}(z, t)=\frac{m(z, t)}{f(z, t)} \\
& a_{2}(z, t)=\frac{h(z, t)}{f(z, t)}
\end{aligned}
$$

where
$f(z, t)=\sum_{\mu=0,1}{ }^{\prime} \exp \left(\sum_{l=1}^{2 N} \mu_{l} \theta_{l}+\sum_{l<j}^{2 N} \varphi(l, j) \mu_{l} \mu_{j}\right)$,
$g(z, t)=\sum_{\mu=0,1}{ }^{\prime \prime} \exp \left(\sum_{l=1}^{2 N} \mu_{l} \theta_{l}+\sum_{l<j}^{2 N} \varphi(l, j) \mu_{l} \mu_{j}\right)$,
$g^{*}(z, t)=\sum_{\mu=0,1}{ }^{\prime \prime \prime} \exp \left(\sum_{l=1}^{2 N} \mu_{l} \theta_{l}+\sum_{l<j}^{2 N} \varphi(l, j) \mu_{l} \mu_{j}\right)$,
$m(z, t)=\sum_{\mu=0,1}{ }^{\prime} \exp \left[\sum_{l=1}^{2 N} \mu_{l}\left(\theta_{l}+\psi(l)\right)\right.$

$$
\left.+\sum_{l<j}^{2 N} \varphi(l, j) \mu_{l} \mu_{j}\right]
$$

$$
m^{*}(z, t)=\sum_{\mu=0,1}{ }^{\prime} \exp \left[\sum_{l=1}^{2 N} \mu_{l}\left(\theta_{l}-\psi(l)\right)\right.
$$

$$
\left.+\sum_{l<j}^{2 N} \varphi(l, j) \mu_{l} \mu_{j}\right]
$$

$$
h(z, t)=\mathrm{i} \sum_{\mu=0,1}{ }^{\prime \prime \prime} \exp \left[\sum_{l=1}^{2 N} \mu_{l}\left(\theta_{l}+\psi(l)\right)\right.
$$

$$
\begin{aligned}
& \left.+\sum_{l<j}^{2 N} \varphi(l, j) \mu_{l} \mu_{j}\right] \\
h^{*}(z, t)= & -i \sum_{\mu=0,1}{ }^{\prime \prime} \exp \left[\sum_{l=1}^{2 N} \mu_{l}\left(\theta_{l}-\psi(l)\right)\right. \\
& \left.+\sum_{l<j}^{2 N} \varphi(l, j) \mu_{l} \mu_{j}\right]
\end{aligned}
$$

with
$\theta_{l}=k_{l} t+\mathrm{i} k_{l}^{2} \int \alpha(z) \mathrm{d} z+\frac{2 \mathrm{i}}{2 k+\mathrm{i} k_{l}} \gamma_{1}(z) \mathrm{d} z+\xi_{l}$
for $l=1,2, \ldots, 2 N$,
$\theta_{l+N}=\theta_{l}^{*}, \quad k_{l+N}=k_{l}^{*} \quad$ for $\quad l=1,2, \ldots, N$,
$\varphi(l, j)=\ln \frac{1}{\left(k_{l}+k_{j}\right)^{2}} \quad$ for $l=1,2, \ldots, N$
and $j=N+1, \ldots, 2 N$,
$\varphi(l, j)=-\ln \frac{1}{\left(k_{l}-k_{j}\right)^{2}}$ for $l=1,2, \ldots, N$
and $j=1,2, \ldots, N$,
or $l=N+1, \ldots, 2 N$ and $j=N+1, \ldots, 2 N$,
$\psi(l)=\ln \left(2 k+\mathrm{i} k_{l}\right) \quad$ for $l=1,2, \ldots, N$,
$\psi(l)=-\ln \left(2 k-\mathrm{i} k_{l}\right)$ for $l=N+1, \ldots, 2 N$,
where $k_{l}$ and $\xi_{l}$ are all complex constants related to the amplitude and phase of the $l$ th soliton, $\sum_{l<j}^{2 N}$ indicates the summation over all possible combinations taken from $2 N$ elements with the condition $l<j$, and $\sum_{\mu=0,1}^{\prime}, \sum_{\mu=0,1}^{\prime \prime}$, and $\sum_{\mu=0,1}^{\prime \prime \prime}$ indicate the summations over all possible cases of $\mu_{l}=0,1$ for $l=1,2, \ldots, N$ under the conditions

$$
\begin{aligned}
& \sum_{l=1}^{N} \mu_{l}=\sum_{l=1}^{N} \mu_{l+N}, \sum_{l=1}^{N} \mu_{l}=1+\sum_{l=1}^{N} \mu_{l+N}, \\
& 1+\sum_{l=1}^{N \prime \prime \prime} \mu_{l}=\sum_{l=1}^{N}{ }^{\prime \prime \prime} \mu_{l+N} .
\end{aligned}
$$

Then, the $N$-soliton solutions for System (4) can be given in the form of

$$
\begin{equation*}
q=q, p=2 a_{1} a_{2}^{*}, \text { and } \eta=\left|a_{2}\right|^{2}-\left|a_{1}\right|^{2} . \tag{27}
\end{equation*}
$$

## 3. Results and Discussions

Based on the solutions for System (4) involving two arbitrary functions $\alpha(z)$ and $\beta(z)$, whose different values correspond to various physical systems, we will
consider some special cases in the literature. In this paper, we choose a periodic distributed amplification with the varying GVD and nonlinear parameters in the following form [24]:

$$
\begin{align*}
& \alpha(z)=\frac{1}{D_{0}} \exp (\sigma z) \beta(z),  \tag{28}\\
& \beta(z)=R_{0}+R_{1} \sin (\rho z),
\end{align*}
$$

where $R_{0}, R_{1}$, and $\rho$ are the parameters describing Kerr nonlinearity and $D_{0}$ is the parameter related to the initial peak power in the system. For the sake of convenience, we assume the parameters as $R_{0}=0, R_{1}=1$, and $D_{0}=1$.

The soliton velocity via (15) can be given as

$$
\begin{align*}
& V=-\frac{\left|2 k+k_{1}\right|^{2}}{\mathrm{i}\left(k_{1}-k_{1}^{*}\right)\left|2 k+k_{1}\right|^{2} \alpha(z)+2 \gamma_{1}(z)} \\
& \text { with } \gamma_{1}(z)=\sqrt{\frac{\beta(z)}{2 \alpha(z)}} \gamma(z) \tag{29}
\end{align*}
$$

With suitable choice of the parameters in (15), (16), and (17), we will give Figures 1 and 2. It should be noted that in Figure 1, $\sigma=0$ corresponds to the case of the fibers without any loss or gain. In such case, the pulse does not suffer any broadening or compression except the possible phase shift induced by the SIT effect. To study the influence of the SIT effect to the solitons, we choose $\gamma(z)=0$ in Figure 2, which illustrates the soliton propagation without the SIT effect. For comparison, $k_{1}, \xi_{1}, \alpha(z)$, and $\beta(z)$ are of the same values in Figures 1 and 2. As shown in Figure 1, the solitons propagate along the $z$-axis with the periodic oscillation, as a result of periodic distributed amplification (28). However, Figure 2 illustrates that the solitons oscillate periodically in a fixed area. From the comparison between Figures 1 and 2, one can find that the SIT effect is responsible for introducing the change of soliton velocity and phase. In addition, one can observe the bright and dark two-peak solitons in Figure 1b and c, respectively, as well as in Figure 2b and c.

Above analysis and plots are based on the consideration $\gamma(z)=$ constant. Next, we analyze the optical pulse propagation for other forms of $\gamma(z)$. Without loss of generality, we assume $\gamma(z)=0.1 z$ and $\gamma(z)=\sin z$ and respectively plot Figures 3 and 4 with the same parameter values as those in Figure 1. From these two figures we can notice that the profile of the soliton changes compared with Figure 2. The corresponding evolution


Fig. 1 (colour online). One soliton represented by (15), (16), and (17) for (28). Parameters adopted here are $\rho=1, k_{1}=$ $1.3+0.6 \mathrm{i}, \xi_{1}=2+\mathrm{i}, \sigma=0, k=0.001$, and $\gamma(z)=1$.


Fig. 2 (colour online). Same as Figure 1 except for $\gamma(z)=0$.


Fig. 3 (colour online). Same as Figure 1 except for $\gamma(z)=0.1 z$.


Fig. 4 (colour online). Same as Figure 1 except for $\gamma(z)=\sin z$.


Fig. 5 (colour online). Periodic interaction of the two-soliton solutions via (22), (23), and (24) for (28). Parameters are $\rho=0.6$, $k_{1}=1+1.2 \mathrm{i}, k_{2}=1-1.2 \mathrm{i}, \xi_{1}=1, \xi_{2}=1, \gamma(z)=1, k=0.001$, and $\sigma=0$.


Fig. 6 (colour online). Same as Figure 5 except for $\gamma(z)=0$.
(a)

(b)



Fig. 7 (colour online). Same as Figure 5 except for $\rho=0.8, k_{1}=1+1.3 \mathrm{i}, k_{2}=1-1.3 \mathrm{i}$, and $\sigma=0.1$.
of one-soliton solutions for $p$ and $\eta$ are also plotted, and the bright and dark two-peak solitons can been also seen in Figures 3 and 4.

Furthermore, we will display the interaction of the two-soliton solutions for System (4). Figure 5 depicts the periodic interaction of the two solitons with equal amplitudes. Two solitons propagate with their original shapes and amplitudes, and only have a phase shift at the moment of the collision, which is one of the im-
portant properties of the solitons. A phase shift can be also observed from the comparison between Figures 5 and 6 , which illustrates the interaction without the SIT effect, and there exists the velocity change as well.

In addition, we consider the periodic interaction of the two-soliton solutions for System (4) for the case of $\sigma=0.1$ and give Figure 7. As shown in Figure 7a, the amplitudes of the two solitons increase as $t$ increases.


Fig. 8 (colour online). Same as Figure 5 except for $\rho=0.8, k_{1}=1.1+1.3 \mathrm{i}, k_{2}=1.1-1.3 \mathrm{i}$, and $\sigma=-0.1$.

This phenomenon is owing to the choice of $\sigma>0$, which corresponds to the dispersion-increasing fiber.
Similarly, the case of $\sigma=-0.1$ is also considered, and the periodic interaction is shown by Figure 8. Unlike that shown in Figure 7a, the amplitudes of the two solitons decrease while propagating in the fibers, which can be seen in Figure 8a, as a result of the negative value of $\sigma$. In this case, it represents the dispersion-decreasing fiber. Relevant issues can be seen in e.g. [25-27].

## 4. Conclusions

In this paper, we have investigated an INLS-MB system, namely System (4), which describes the optical pulse propagation in the erbium-doped fiber with the variable dispersion, nonlinearity, and gain/loss parameters. Wave functions $v_{1}$ and $v_{2}$ and transformations in (5) have been introduced, so as to generate System (6). By way of the bilinear form (8a)-(8d) for System (6) and with the constraints of (7), we have derived the one-soliton solutions (15)-(17), the twosoliton solutions (22)-(24,) and the $N$-soliton solutions (25)-(27) for System (4), through the relation between System (4) and System (6). Evolution and interaction properties of the solitons have been graphi-
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cally presented (see Figs. 1-8), under the periodic distributed amplification of (28).

Our work has shown that System (4) admits the propagation and interaction of the SIT-NLS solitons. With certain parametric choices, (16) and (17) have been found to express the bright and dark two-peak solitons, respectively (see Figs. 1 and 2). Based on (15) and the velocity expression in (29), the SIT effect caused by the doped erbium atoms has been studied, which leads to the change of the soliton velocity and phase (see Figs. 1 and 2). Interaction parameter $\gamma(z)$ has been considered, and the profile of the soliton changes with $\gamma(z)$, as seen in Figures 3 and 4. Two solitons via (22) keep their characters invariant after colliding with each other in the fiber without gain/loss (see Figs. 5 and 6). However, when the solitons propagate in the dispersion-decreasing ( $\sigma<0$ ) and dispersionincreasing $(\sigma>0)$ fibers, their amplitudes correspondingly decrease and increase (see Figs. 7 and 8, respectively).

## Acknowledgement

This work has been supported by the Fundamental Research Funds for the Central Universities of China under Grant No. 2009RC0708.
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