Identities for Eigenvalues of the Schrödinger Equation with Energy-Dependent Potential

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The present paper deals with eigenvalue problems for the Schrödinger equation with energy-dependent potential and some separated boundary conditions. Using the method of contour integration, we obtain some new regularized traces for this class of Schrödinger operators.

Key words: Schrödinger Equation; Regularized Trace Formula.
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1. Introduction

The problem of describing interactions between colliding particles is of fundamental interest in physics. One is interested in collisions of two spinless particles, and it is supposed that the s-wave scattering matrix and the s-wave binding energies are exactly known from collision experiments. With a radial static potential $V(x)$ the s-wave Schrödinger equation is written as

$$y''(x)+[E-V(E,x)]y(x)=0,$$

where $V(E,x)=2\sqrt{E}P(x)+Q(x)$.

In particular, with an additional condition $Q(x)=-P^2(x)$ the above equation reduces to the Klein–Gordon s-wave equation for a particle of zero mass and energy $\sqrt{E}$ [1].

In this paper, we consider the boundary-value problems generated by the differential equation

$$l_\lambda u(x)\overset{\text{def}}{=}u''(x)+[\lambda^2-2\lambda p(x)-q(x)]u(x)=0,\quad x\in(0,\pi), \tag{1}$$

where λ is a spectral parameter and the functions $q(x)\in W_2^1[0,\pi]$ and $p(x)\in W_2^2[0,\pi]$. Equation (1) is respectively endowed with boundary conditions

$$(\text{BC1})\quad \begin{array}{l} u(0)=0, \\ u(\pi)=0; \end{array} \tag{2}$$

$$(\text{BC2})\quad \begin{array}{l} u(0)=0, \\ u'(\pi)+Hu(\pi)=0; \end{array} \tag{3}$$

and

$$(\text{BC3})\quad \begin{array}{l} u'(0)-hu(0)=0, \\ u(\pi)=0. \end{array} \tag{4}$$

The trace identity of a differential operator deeply reveals the spectral structure of the differential operator and has important applications in the numerical calculation of eigenvalues. Here we refer to the references [2 – 11], with which the author became acquainted while doing research on the present paper. In [12], we obtained regularized trace formula for (1) with the boundary condition

$$(\text{BC4})\quad \begin{array}{l} u'(0)-hu(0)=0, \\ u'(\pi)+Hu(\pi)=0,\ h,\ H\in\mathbb{R}. \end{array} \tag{5}$$

However, the boundary condition in (5) does not include boundary conditions (2), (3), and (4). In this paper, we try to obtain some new regularized traces for this class of Schrödinger equation with boundary conditions (2), (3), and (4), respectively.

2. Results

Problem 1. Let λ_n, $n\in\mathbb{Z}\setminus\{0\}$, be the eigenvalues of (1) and (2). Then the sequence $\{\lambda_n : n=\pm1,\pm2,\dots\}$ satisfies the following asymptotic form:

$$\lambda_n=n+c_0+\frac{b_1}{n\pi}+\frac{\beta}{4n^2\pi}+o\left(\frac{1}{n^2}\right), \tag{6}$$

where

$$
\begin{aligned}
c_0 &= \frac{1}{\pi}\int_0^\pi p(x)\,\mathrm{d}x\,,\\
b_1 &= \frac{1}{2}\int_0^\pi [p^2(x)+q(x)]\,\mathrm{d}x\,, \qquad (7)\\
\beta &= p'(0)-p'(\pi)+2\int_0^\pi [p(x)-c_0][p^2(x)+q(x)]\,\mathrm{d}x\,.
\end{aligned}
$$

It is seen from (6) that the series

$$
s_1 \stackrel{\text{def}}{=} \sum_{n=1}^{\infty}\left[(\lambda_n-c_0)^2+(\lambda_{-n}-c_0)^2-2n^2-\frac{4b_1}{\pi}\right] \qquad (8)
$$

is absolutely convergent.

Problem 2. Let ζ_n, $n\in\mathbb{Z}$, be the eigenvalues of (1) and (3). We can prove that the sequence $\{\zeta_n : n = 0,\pm1,\pm2,\ldots\}$ satisfies the following asymptotic expression:

$$
\zeta_n = n+\frac{1}{2}+c_0+\frac{b_1+H}{(n+\frac{1}{2})\pi}+\frac{\gamma}{4(n+\frac{1}{2})^2\pi}+o\left(\frac{1}{n^2}\right), \qquad (9)
$$

where

$$
\gamma = p'(0)+p'(\pi)+2\int_0^\pi [p(x)-c_0][p^2(x)+q(x)]\,\mathrm{d}x +H[p(\pi)-c_0]\,.
$$

It is seen from (9) that the series

$$
\begin{aligned}
s_2 \stackrel{\text{def}}{=} &\sum_{n=0}^{\infty}\left[(\zeta_n-c_0)^2-\left(n+\frac{1}{2}\right)^2-\frac{2(b_1+H)}{\pi}\right]\\
&+\sum_{n=1}^{\infty}\left[(\zeta_{-n}-c_0)^2-\left(n-\frac{1}{2}\right)^2-\frac{2(b_1+H)}{\pi}\right]
\end{aligned} \qquad (10)
$$

is absolutely convergent.

Problem 3. Let μ_n, $n\in\mathbb{Z}$, be the eigenvalues of (1) and (4). Then the sequence $\{\mu_n : n = 0,\pm1,\pm2,\ldots\}$ satisfies the following asymptotic form:

$$
\lambda_n = n+\frac{1}{2}+c_0+\frac{b_1+h}{(n+\frac{1}{2})\pi}+\frac{\theta}{4(n+\frac{1}{2})^2\pi}+o\left(\frac{1}{n^2}\right), \qquad (11)
$$

where

$$
\theta = -p'(0)-p'(\pi)+2\int_0^\pi [p(x)-c_0][p^2(x)+q(x)]\,\mathrm{d}x +h[p(0)-c_0]\,.
$$

It is seen from (11) that the series

$$
\begin{aligned}
s_3 \stackrel{\text{def}}{=} &\sum_{n=0}^{\infty}\left[(\mu_n-c_0)^2-\left(n+\frac{1}{2}\right)^2-\frac{2(b_1+h)}{\pi}\right]\\
&+\sum_{n=1}^{\infty}\left[(\mu_{-n}-c_0)^2-\left(n-\frac{1}{2}\right)^2-\frac{2(b_1+h)}{\pi}\right]
\end{aligned} \qquad (12)
$$

is absolutely convergent.

In this work, we shall derive the sums of the series in (8), (10), and (12) in an explicit form, which are so-called regularized traces.

Theorem 2.1. *We have the trace formulae*

$$
\begin{aligned}
s_1 &= \frac{2b_1}{\pi}-p^2(\pi)-p^2(0)+[p(\pi)+p(0)]c_0 -c_0^2-\frac{q(\pi)+q(0)}{2}\,, \qquad (13)\\
s_2 &= p^2(\pi)-p^2(0)+[p(0)-p(\pi)]c_0 +\frac{q(\pi)-q(0)}{2}-H^2\,, \qquad (14)
\end{aligned}
$$

and

$$
s_3 = p^2(0)-p^2(\pi)+[p(\pi)-p(0)]c_0 +\frac{q(0)-q(\pi)}{2}-h^2\,, \qquad (15)
$$

where b_1 and c_0 are defined by (7).

Remark 2.2. For a special case $p(x)\equiv 0$ in (1), the trace formula (13) implies

$$
\sum_{n=1}^{\infty}\left[\lambda_n^2-n^2-\frac{1}{\pi}\int_0^\pi q(x)\,\mathrm{d}x\right] = -\frac{q(0)+q(\pi)}{4} +\frac{1}{2\pi}\int_0^\pi q(x)\,\mathrm{d}x\,.
$$

Here λ_n^2 are eigenvalues of the well-known Sturm–Liouville problem with the Dirichlet boundary condition of (2). This result was previously obtained in [6].

3. Solutions to the Schrödinger Equation

In this section, we recall a refined estimate for a fundamental pair of solutions to the equation $l_\lambda u(x,\lambda) = 0$, which will be used in Section 3.

Lemma 3.1. [13] *Let $\psi(x,\lambda)$ and $\varphi(x,\lambda)$ be the solutions to $l_\lambda u(x,\lambda)=0$ with the initial conditions*

$$(\psi'(0,\lambda),\psi(0,\lambda))=(0,1)=(\varphi(0,\lambda),\varphi'(0,\lambda)),$$

then the following representations hold:

$$\varphi(x,\lambda)=\frac{\sin[\lambda x-\alpha(x)]}{\lambda}-b_1(x)\frac{\cos[\lambda x-\alpha(x)]}{\lambda^2}$$
$$+a_1(x)\frac{\sin[\lambda x-\alpha(x)]}{\lambda^2}+b_2(x)\frac{\cos[\lambda x-\alpha(x)]}{\lambda^3}$$
$$+a_2(x)\frac{\sin[\lambda x-\alpha(x)]}{\lambda^3}+o\left(\frac{\mathrm{e}^{\tau x}}{\lambda^3}\right),$$
$$\psi(x,\lambda)=\cos[\lambda x-\alpha(x)]-c_1(x)\frac{\cos[\lambda x-\alpha(x)]}{\lambda}$$
$$+b_1(x)\frac{\sin[\lambda x-\alpha(x)]}{\lambda}+d_2(x)\frac{\cos[\lambda x-\alpha(x)]}{\lambda^2}$$
$$+d_1(x)\frac{\sin[\lambda x-\alpha(x)]}{\lambda^2}+o\left(\frac{\mathrm{e}^{\tau x}}{\lambda^2}\right),$$

where $\tau=|\operatorname{Im}\lambda|$, and

$$\alpha(x)=\int_0^x p(t)\,\mathrm{d}t,\ \ b_1(x)=\frac{1}{2}\int_0^x[p^2(t)+q(t)]\,\mathrm{d}t,$$
$$a_1(x)=\frac{1}{2}[p(x)+p(0)],$$
$$b_2(x)=\frac{1}{4}[p'(x)-p'(0)]-\frac{1}{2}b_1(x)[p(x)$$
$$+p(0)]-\frac{1}{2}\int_0^x p(t)[p^2(t)+q(t)]\,\mathrm{d}t,$$
$$a_2(x)=\frac{1}{8}[5p^2(x)+5p^2(0)+2p(0)p(x)]$$
$$+\frac{q(x)+q(0)}{4}-\frac{1}{2}b_1^2(x),$$
$$c_1(x)=\frac{1}{2}[p(0)-p(x)],$$
$$d_2(x)=\frac{1}{8}[5p^2(x)-2p(0)p(x)-3p^2(0)+2q(x)$$
$$-2q(0)]-\frac{1}{2}b_1^2(x),$$
$$d_1(x)=-\frac{1}{4}[p'(x)+p'(0)]+\frac{1}{2}b_1(x)[p(x)-p(0)]$$
$$+\frac{1}{2}\int_0^x p(t)[p^2(t)+q(t)]\,\mathrm{d}t.$$

It is easy to obtain asymptotic expressions of the functions $\psi'(x,\lambda)$ and $\varphi'(x,\lambda)$.

Corollary 3.2. [13] *Let $\psi(x,\lambda)$ and $\varphi(x,\lambda)$ be the solutions to $l_\lambda u(x,\lambda)=0$ with the initial conditions*

$$(\psi'(0,\lambda),\psi(0,\lambda))=(0,1)=(\varphi(0,\lambda),\varphi'(0,\lambda)),$$

then the solutions have the following representations:

$$\varphi'(x,\lambda)=\cos[\lambda x-\alpha(x)]+c_1(x)\frac{\cos[\lambda x-\alpha(x)]}{\lambda}$$
$$+b_1(x)\frac{\sin[\lambda x-\alpha(x)]}{\lambda}+c_2(x)\frac{\cos[\lambda x-\alpha(x)]}{\lambda^2}$$
$$+c_3(x)\frac{\sin[\lambda x-\alpha(x)]}{\lambda^2}+o\left(\frac{\mathrm{e}^{\tau x}}{\lambda^2}\right),$$
$$\psi'(x,\lambda)=-\lambda\sin[\lambda x-\alpha(x)]$$
$$+b_1(x)\cos[\lambda x-\alpha(x)]+a_1(x)\sin[\lambda x-\alpha(x)]$$
$$+e_1(x)\frac{\cos[\lambda x-\alpha(x)]}{\lambda}+e_2(x)\frac{\sin[\lambda x-\alpha(x)]}{\lambda}$$
$$+o\left(\frac{\mathrm{e}^{\tau x}}{\lambda}\right),$$

where

$$c_2(x)=-\frac{1}{8}[3p^2(x)+2p(0)p(x)-5p^2(0)+2q(x)$$
$$-2q(0)]-\frac{1}{2}b_1^2(x),$$
$$c_3(x)=\frac{1}{4}[p'(x)+p'(0)]-\frac{1}{2}b_1(x)[p(x)-p(0)]$$
$$+\frac{1}{2}\int_0^x p(t)[p^2(t)+q(t)]\,\mathrm{d}t,$$
$$e_1(x)=\frac{1}{4}[p'(x)-p'(0)]-\frac{1}{2}b_1(x)[p(x)+p(0)]$$
$$+\frac{1}{2}\int_0^x p(t)[p^2(t)+q(t)]\,\mathrm{d}t,$$
$$e_2(x)=\frac{1}{8}[3p^2(x)-2p(0)p(x)+3p^2(0)+2q(x)$$
$$+2q(0)]+\frac{1}{2}b_1^2(x).$$

4. Proof of Theorem 2.1

For convenience, we now set

$$a_1=a_1(\pi),\ \ a_2=a_2(\pi),\ \ b_1=b_1(\pi),\ \ b_2=b_2(\pi).$$

We only present the proof of the identity in (13). The proofs of identities (14) and (15) are similar to the proof of the identity (13), thus we omit them. First, we shall prove that (13) is true under the assumption $c_0=0$.

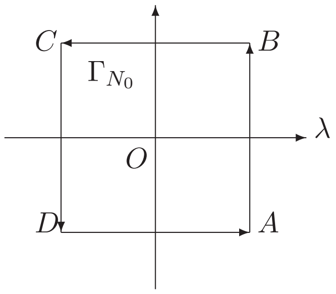

Fig. 1. Contour Γ_{N_0} in a λ-complex plane.

From Lemma 3.1 and Corollary 3.2, we see that the characteristic equation in (1) and (2) can be reduced to $\varphi(\lambda)=0$, where

$$\varphi(\lambda)=\frac{\sin(\lambda\pi)}{\lambda}-b_1\frac{\cos(\lambda\pi)}{\lambda^2}+a_1\frac{\sin(\lambda\pi)}{\lambda^2}+b_2\frac{\cos(\lambda\pi)}{\lambda^3}+a_2\frac{\sin(\lambda\pi)}{\lambda^3}+o\left(\frac{e^{\tau\pi}}{\lambda^3}\right). \tag{16}$$

Define

$$\varphi_0(\lambda)=\frac{\sin(\lambda\pi)}{\lambda},$$

and denote by $\lambda_n^0, n\in\mathbb{Z}\setminus\{0\}$, zeros (simple) of the function $\varphi_0(\lambda)$, then

$$\lambda_n^0=n.$$

Let C_n be circles of radii r (r small enough) with the centers at the points n. For an integer n, let Γ_{N_0} be the counterclockwise square contours $ABCD$ as in Figure 1 with

$$A=\left(N_0+\frac{1}{2}\right)(1-\mathrm{i}),\ B=\left(N_0+\frac{1}{2}\right)(1+\mathrm{i}),$$
$$C=\left(N_0+\frac{1}{2}\right)(-1+\mathrm{i}),\ D=\left(N_0+\frac{1}{2}\right)(-1-\mathrm{i}).$$

For N_0 large enough, on the contour Γ_{N_0}, for $t\in[0,\pi]$, there hold uniformly (see [12]: Lemma 3.1)

$$\left|\frac{\sin(\lambda t)}{\sin(\lambda\pi)}\right|\le 4 \text{ and } \left|\frac{\cos(\lambda t)}{\sin(\lambda\pi)}\right|\le 4. \tag{17}$$

Combining (16) and arranging the terms on the right-hand side in decreasing order of powers of λ, we have

$$\frac{\varphi(\lambda)}{\varphi_0(\lambda)}=1+\frac{a_1-b_1\cot(\lambda\pi)}{\lambda}+\frac{a_2+b_2\cot(\lambda\pi)}{\lambda^2}+o\left(\frac{1}{\lambda^2}\right)$$

on the contour Γ_{N_0} or C_n. Expanding $\log\frac{\varphi(\lambda)}{\varphi_0(\lambda)}$ by the Maclaurin formula, we find

$$\log\frac{\varphi(\lambda)}{\varphi_0(\lambda)}=\frac{a_1-b_1\cot(\lambda\pi)}{\lambda}+\frac{\left(a_2-\frac{1}{2}a_1^2\right)+(b_2+a_1b_1)\cot(\lambda\pi)-\frac{b_1^2}{2}\cot^2(\lambda\pi)}{\lambda^2}+o\left(\frac{1}{\lambda^2}\right) \tag{18}$$

on the contour Γ_{N_0} or C_n.

By the residue calculation [14], the following identities are true:

$$\frac{1}{2\pi\mathrm{i}}\oint_{C_n}\frac{\cot(\lambda\pi)}{\lambda}\,\mathrm{d}\lambda=\frac{1}{n\pi},$$
$$\frac{1}{2\pi\mathrm{i}}\oint_{C_n}\frac{\cot(\lambda\pi)}{\lambda^2}\,\mathrm{d}\lambda=\frac{1}{n^2\pi},$$
$$\frac{1}{2\pi\mathrm{i}}\oint_{C_n}\frac{\cot^2(\lambda\pi)}{\lambda^2}\,\mathrm{d}\lambda=-\frac{2}{n^3\pi^2}.$$

Using the residue formula

$$\lambda_n-n=-\frac{1}{2\pi\mathrm{i}}\oint_{C_n}\log\frac{\varphi(\lambda)}{\varphi_0(\lambda)}\,\mathrm{d}\lambda,$$

we obtain

$$\lambda_n=n+\frac{b_1}{n\pi}+\frac{-b_2-a_1b_1}{4n^2\pi}+o\left(\frac{1}{n^2}\right), \tag{19}$$

where

$$b_1=\frac{1}{2}\int_0^\pi[p^2(x)+q(x)]\,\mathrm{d}x,$$
$$b_2+a_1b_1=p'(\pi)-p'(0)-2\int_0^\pi p(x)[p^2(x)+q(x)]\,\mathrm{d}x.$$

Thus, we have the asymptotic formula of the eigenvalues for (1) and (2) with $c_0=0$.

The asymptotic formula (19) implies that, for all sufficiently large N_0, the numbers λ_n which are the zeros of the function $\varphi(\lambda)$, with $|n|\le N_0$, are inside Γ_{N_0} and the number λ_n, with $|n|>N_0$, are outside Γ_{N_0}. Obviously, λ_n^0, which are the zeros of the function $\varphi_0(\lambda)$, don't lie on the contour Γ_{N_0}.

By residue theorem, we obtain that

$$\begin{aligned}
&\sum_{\Gamma_{N_0}}(\lambda_n^2-n^2)\\
&=\sum_{n=1}^{N_0}\left(\lambda_n^2+\lambda_{-n}^2-2n^2\right)\\
&=\frac{1}{2\pi \mathrm{i}}\oint_{\Gamma_{N_0}}\lambda^2\left[\frac{\varphi'(\lambda)}{\varphi(\lambda)}-\frac{\varphi_0'(\lambda)}{\varphi_0(\lambda)}\right]\mathrm{d}\lambda\\
&=\frac{1}{2\pi \mathrm{i}}\oint_{\Gamma_{N_0}}\lambda^2 \mathrm{d}\log\frac{\varphi(\lambda)}{\varphi_0(\lambda)}\\
&=-\frac{1}{2\pi \mathrm{i}}\oint_{\Gamma_{N_0}}2\lambda\log\frac{\varphi(\lambda)}{\varphi_0(\lambda)}\,\mathrm{d}\lambda\,.
\end{aligned}\tag{20}$$

Using the well-known formulae

$$\cot z=\frac{1}{z}+2z\sum_{n=1}^{\infty}\frac{1}{z^2-n^2\pi^2},\quad \csc^2 z=\sum_{n=-\infty}^{\infty}\frac{1}{(z+n\pi)^2},$$

we get

$$\begin{aligned}
&\frac{1}{2\pi \mathrm{i}}\oint_{\Gamma_{N_0}}\cot(\lambda\pi)\,\mathrm{d}\lambda=\frac{2N_0+1}{\pi},\\
&\frac{1}{2\pi \mathrm{i}}\oint_{\Gamma_{N_0}}\frac{\cot(\lambda\pi)}{\lambda}\,\mathrm{d}\lambda=0,\\
&\frac{1}{2\pi \mathrm{i}}\oint_{\Gamma_{N_0}}\frac{\cot^2(\lambda\pi)}{\lambda}\,\mathrm{d}\lambda=-1+O\left(\frac{1}{N_0}\right).
\end{aligned}$$

From (18), by calculating residues, we have

$$\begin{aligned}
&-\frac{1}{2\pi \mathrm{i}}\oint_{\Gamma_{N_0}}2\lambda\log\frac{\varphi(\lambda)}{\varphi_0(\lambda)}\,\mathrm{d}\lambda\\
&=-\frac{1}{2\pi \mathrm{i}}\oint_{\Gamma_{N_0}}\Bigg[2(a_1-b_1\cot(\lambda\pi))\\
&+\frac{(2a_2-a_1^2)+2(b_2+a_1b_1)\cot(\lambda\pi)-b_1^2\cot^2(\lambda\pi)}{\lambda}\Bigg]\mathrm{d}\lambda\\
&+o(1)\\
&=2b_1\frac{1}{2\pi \mathrm{i}}\oint_{\Gamma_{N_0}}\cot(\lambda\pi)\,\mathrm{d}\lambda+a_1^2-2a_2-2(b_2+a_1b_1)\\
&\cdot\frac{1}{2\pi \mathrm{i}}\oint_{\Gamma_{N_0}}\frac{\cot(\lambda\pi)}{\lambda}\,\mathrm{d}\lambda\\
&+b_1^2\frac{1}{2\pi \mathrm{i}}\oint_{\Gamma_{N_0}}\frac{\cot^2(\lambda\pi)}{\lambda}\,\mathrm{d}\lambda+o(1)\\
&=2b_1\frac{2N_0+1}{\pi}+a_1^2-2a_2-b_1^2+o(1)\,.
\end{aligned}$$

From (20), we get

$$\begin{aligned}
&\sum_{n=1}^{N_0}\left(\lambda_n^2+\lambda_{-n}^2-2n^2\right)=2b_1\frac{2N_0+1}{\pi}\\
&+a_1^2-2a_2-b_1^2+o(1)\,.
\end{aligned}\tag{21}$$

Passing to the limit as $N_0\to\infty$ in (21), we find that

$$\begin{aligned}
&\sum_{n=1}^{\infty}\left(\lambda_n^2+\lambda_{-n}^2-2n^2-\frac{4b_1}{\pi}\right)=\frac{2b_1}{\pi}\\
&+a_1^2-2a_2-b_1^2\,.
\end{aligned}\tag{22}$$

From Lemma 3.1 and Corollary 3.2, a direct computation yields

$$\begin{aligned}
&a_1^2-2a_2-b_1^2=-p^2(\pi)-p^2(0)\\
&-\frac{q(0)+q(\pi)}{2}\,.
\end{aligned}\tag{23}$$

Substituting (23) into (22), we see that the regularized trace s_1 with $c_0=0$ has the following form:

$$\begin{aligned}
&\sum_{n=1}^{\infty}\left(\lambda_n^2+\lambda_{-n}^2-2n^2-\frac{4b_1}{\pi}\right)=\frac{2b_1}{\pi}\\
&-p^2(\pi)-p^2(0)-\frac{q(0)+q(\pi)}{2}\,.
\end{aligned}\tag{24}$$

Now we consider the case $c_0\neq 0$. By a direct calculation, we note that the equation

$$-u''(x)+[q(x)+2\lambda p(x)]u(x)=\lambda^2 u(x)$$

is equivalent to

$$\begin{aligned}
&-u''(x)+[q(x)+2pc_0-c_0^2+2(\lambda-c_0)(p(x)-c_0)]\\
&\cdot u(x)=(\lambda-c_0)^2u(x)\,.
\end{aligned}$$

Let

$$\widehat{\lambda}_n=\lambda_n-c_0,\ \ \widehat{q}(x)=q(x)+2pc_0-c_0^2$$

and

$$\widehat{p}(x)=p(x)-c_0=p(x)-\frac{1}{\pi}\int_0^{\pi}p(x)\,\mathrm{d}x\,,$$

then in this case we have $\int_0^{\pi}\widehat{p}(x)\,\mathrm{d}x=0$ and

$$\widehat{b_1}=\frac{1}{2}\int_0^{\pi}[\widehat{q}(x)+\widehat{p}^2(x)]\,\mathrm{d}x=b_1\,.$$

Substituting the expressions for $\widehat{q}(x)$, $\widehat{p}(x)$, and $\widehat{\lambda}_n$ into (19), we find that the eigenvalues λ_n satisfy the following asymptotic formula as $|n|\to\infty$:

$$\lambda_n = n + c_0 + \frac{b_1}{n\pi} + \frac{\beta}{4n^2\pi} + o\left(\frac{1}{n^2}\right),$$

where

$$c_0 = \frac{1}{\pi}\int_0^\pi p(x)\,\mathrm{d}x,$$

$$b_1 = \frac{1}{2}\int_0^\pi [p^2(x)+q(x)]\,\mathrm{d}x,$$

$$\beta = p'(0)-p'(\pi)+2\int_0^\pi [p(x)-c_0][p^2(x)+q(x)]\,\mathrm{d}x.$$

Substituting the expressions for $\widehat{q}(x)$, $\widehat{p}(x)$, and $\widehat{\lambda}_n$ into (24), we prove that (13) holds. The proof of theorem is finished.

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