Variational Approach to Impulsive Differential Equations Using the Semi-Inverse Method

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The semi-inverse method is used to establish a variational principle for the Dirichlet boundary value problem with impulses. All the boundary conditions can be obtained as natural conditions by making the variational principle stationary.

Key words: Impulsive; Variational Principle; Semi-Inverse Method.

1. Introduction

Many dynamical systems have an impulsive dynamical behaviour due to abrupt changes at certain instants during the evolution process [1 – 3]; in this paper, we will consider the following Dirichlet impulsive problem:

$$-u''(t)+\lambda u(t)=\sigma(t),\ \ t\in[0,T], \tag{1}$$

$$\Delta u'(t_j)=d_j\ \ j=1,2,3,\dots,p, \tag{2}$$

$$u(0)=u(T)=0, \tag{3}$$

where $0<t_1<t_2<\ldots<t_p<t_{p+1}=T$ and $\Delta u'(t_j)$ is defined as

$$\Delta u'(t_j)=u'(t_j^+)-u'(t_j^-). \tag{4}$$

Nieto and his colleagues established variational principles for various impulsive problems [1 – 3]; in this paper we suggest an alternative approach to the establishment of the variational formulation for the above problem.

2. Semi-Inverse Method

The semi-inverse method [4] is a powerful tool to establish a variational formulation directly from governing equations and boundary/initial conditions. The basic idea of the semi-inverse method is to construct a trial-functional with an unknown function. For the present problem, we can construct a trial-functional in the form

$$J(u)=\int_0^T\left\{\frac{1}{2}u'^2+F(u)\right\}\mathrm{d}t, \tag{5}$$

where F is an unknown function of u.

There are alternative approaches to construct trial-functionals, see [5 – 10].

Making the functional (5) stationary with respect to u, we have the following stationary condition (Euler–Lagrange equation):

$$-u''+\frac{\partial F}{\partial u}=0. \tag{6}$$

Equation (6) should be equivalent to (1); to this end, we set

$$\frac{\partial F}{\partial u}=\lambda u(t)-\sigma(t). \tag{7}$$

From (7), the unknown function F can be identified as

$$F=\frac{1}{2}\lambda u^2-\sigma u. \tag{8}$$

We, therefore, obtain the following functional:

$$J(u)=\int_0^T\left\{\frac{1}{2}u'^2+\frac{1}{2}\lambda u^2-\sigma u\right\}\mathrm{d}t. \tag{9}$$

In order to incorporate the impulsive condition (2) and the boundary condition (3) into the above variational formulation, we construct a trial-functional in the form

$$J(u)=\int_0^T\left\{\frac{1}{2}u'^2+\frac{1}{2}\lambda u^2-\sigma u\right\}\mathrm{d}t+\sum_{j=1}^{p}B_j\Big|_{t_j^-}^{t_j^+} + B_0|_{t=0}+B_T|_{t=T}\,, \tag{10}$$

where B_j $(j=0,1,2,3,\ldots,p,p+1)$ is an unknown continuous function.

Making (10) stationary, we have

$$\begin{aligned}\delta J(u)&=\int_0^T\{u'\delta u'+\lambda u\delta u-\sigma\delta u\}\,\mathrm{d}t\\&\quad+\sum_{j=1}^{p}\frac{\partial B_j}{\partial u}\delta u\Big|_{t_j^-}^{t_j^+}+\frac{\partial B_0}{\partial u}\delta u\Big|_{t=0}\\&\quad+\frac{\partial B_0}{\partial u'}\delta u'\Big|_{t=0}+\frac{\partial B_T}{\partial u}\delta u\Big|_{t=T}+\frac{\partial B_T}{\partial u'}\delta u'\Big|_{t=T}\\&=\int_0^T\{-u''+\lambda u-\sigma\}\,\delta u\,\mathrm{d}t+\sum_{j=1}^{p}u'\delta u\Big|_{t_j^-}^{t_j^+}\\&\quad+u'\delta u\Big|_0^T+\sum_{j=1}^{p}\frac{\partial B_j}{\partial u}\delta u\Big|_{t_j^-}^{t_j^+}+\frac{\partial B_0}{\partial u}\delta u\Big|_{t=0}\\&\quad+\frac{\partial B_0}{\partial u'}\delta u'\Big|_{t=0}+\frac{\partial B_T}{\partial u}\delta u\Big|_{t=T}+\frac{\partial B_T}{\partial u'}\delta u'\Big|_{t=T}\\&=\int_0^T\{-u''+\lambda u-\sigma\}\,\delta u\,\mathrm{d}t\\&\quad+\sum_{j=1}^{p}\left(u'+\frac{\partial B_j}{\partial u}\right)\delta u\Big|_{t_j^-}^{t_j^+}\\&\quad+\left(-u'+\frac{\partial B_0}{\partial u}\right)\delta u\Big|_{t=0}+\left(u'+\frac{\partial B_T}{\partial u}\right)\delta u\Big|_{t=T}\\&\quad+\frac{\partial B_0}{\partial u'}\delta u'\Big|_{t=0}+\frac{\partial B_T}{\partial u'}\delta u'\Big|_{t=T}=0.\end{aligned} \tag{11}$$

For any arbitrary δu, we have (1) as Euler–Lagrange equation, and the following natural boundary/initial conditions:

at $t=t_0=0$:

$$-u'(0)+\frac{\partial B_0}{\partial u}=0, \tag{12a}$$

$$\frac{\partial B_0}{\partial u'}=0; \tag{12b}$$

at $t=t_j$:

$$u'(t_j^+)-u'(t_j^-)+\frac{\partial B_j(t_j)}{\partial u}=0; \tag{13}$$

at $t=t_{p+1}=T$:

$$u'(T)+\frac{\partial B_T}{\partial u}=0, \tag{14a}$$

$$\frac{\partial B_e}{\partial u'}=0. \tag{14b}$$

In (13), we set

$$\frac{\partial B_j(t_j)}{\partial u}=d_j \tag{15}$$

so that it turns out to be (2). From (15), we can identify B_j as follows:

$$B_j(t_j)=\int_0^{u(t_j)}d_j\,\mathrm{d}t. \tag{16}$$

Equations (12) and (14) should satisfy the boundary condition (3); to this end, we set

$$B_0=u'(0)u(0) \tag{17}$$

and

$$B_T=-u'(T)u(T). \tag{18}$$

Please note in above derivation we have used the property $\int_0^T=\sum_{j=0}^{p+1}\int_{T_i}^{T_{j+1}}$, where $T_0=0$ and $T_{p+1}=T$.

We, therefore, obtain the following needed variational principle:

$$J(u)=\int_0^T\left\{\frac{1}{2}u'^2+\frac{1}{2}\lambda u^2-\sigma u\right\}\mathrm{d}t+\sum_{j=1}^{p}\int_0^{u(t_j)}d_j\,\mathrm{d}t+u'(0)u(0)-u'(T)u(T). \tag{19}$$

It is easy to prove that the stationary conditions of the above functional satisfy (1)–(3).

3. Conclusions

In this paper the semi-inverse method is applied to establish a variational formulation for the Dirichlet boundary value problem with impulses. The method can be extended to other impulsive problems with ease.

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