1. Introduction

Recently, soliton theory, one of typical topics in nonlinear science, has been widely applied in optics of nonlinear media [1], photonics [2], plasmas [3], mean-field theory of Bose–Einstein condensates [4], condensed matter physics [5], and many other fields. Among them, for describing nonlinear physical phenomenon, the nonlinear Schrödinger (NLS) equation is a fundamental model. However, the coherent structures in the higher-dimensional NLS equation are seldom involved so far. In this paper, several coherent excitations of the (2 + 1)-dimensional NLS equation with time-varying coefficients are presented through the similarity transformation.

2. Explicit Solutions through Similarity Transformation

To illustrate the above idea, we focus on the (2 + 1)-dimensional nonautonomous NLS equation

\[ i \frac{\partial \Psi}{\partial t} + \frac{\alpha(t)}{2} (\partial_x^2 + \partial_y^2) \Psi - \frac{\Omega(t)}{2} (x^2 + y^2) \Psi - g(t) |\Psi|^2 \Psi + i \gamma(t) \Psi = 0, \]

where \( \Psi(x, y, t) \) is the complex envelope of the electrical field, while \( \alpha(t), \Omega(t), \) and \( g(t) \) represent the dispersion, the potential, and the nonlinearity coefficients, respectively, and \( \gamma(t) \) is the gain (\( \gamma > 0 \)) or the loss (\( \gamma < 0 \)) coefficient [6].

We first construct the following transformation for the envelope field \( \Psi \):

\[ \Psi = (\Psi_R + i \Psi_I)e^{\eta}, \]

where the real functions \( \Psi_R \equiv \Psi_R(x, y, t), \Psi_I \equiv \Psi_I(x, y, t), \) and \( \varphi \equiv \varphi(x, y, t) \) [7–10].

For the real functions \( \Psi_R, \Psi_I, \) and the phase \( \varphi, \) utilizing the similarity transformation, we obtain

\[ \Psi_R = A + BP(\eta, \tau), \]
\[ \Psi_I = E + FQ(\eta, \tau), \]
\[ \varphi = \chi + \mu \tau, \]

(4)

where the new variables \( A \equiv A(t), B \equiv B(t), E \equiv E(t), F \equiv F(t), \tau \equiv \tau(t), \eta \equiv \eta(x, y, t), \chi \equiv \chi(x, y, t) \) and \( \mu \) is a constant) to (2), and setting the real part and the imaginary part of (1) to zero, that is

\[ \alpha(t) B \Delta \eta \eta - 2 F \Delta_2 \eta \eta - (A + BP) \Delta_3 \]
\[ - [2 \beta + (\alpha(t) (\chi_x + \chi_y) + 2 \gamma(t) + 4 \gamma(t)AE) E] Q \]
\[ - [2 \beta + (\alpha(t) (\chi_x + \chi_y) + 2 \gamma(t) + 4 \gamma(t)AE) E] 
\[ - 2 \mu (A + BP) + FQ] \tau \]
\[ + \alpha(t) B (\eta_x^2 + \eta_y^2) P \eta - 2 g(t) [(A + BP)^3 
\[ + AF] Q^2 + BP (E + FQ)^2] = 0, \]

(4)
\[ \begin{align*}
\alpha(t)F\Delta_tQ_\eta + 2B\Delta_2P_\eta &= -(E + F)\Delta_1, \\
+ [2B_1 + (2\alpha(t)(\chi_{xx} + \chi_{xy}) + 2\gamma(t) - 4g(t)AE)BP_\eta \\
+ [2A_1 + (2\alpha(t)(\chi_{xx} + \chi_{yy}) + 2\gamma(t))]A \\
+ 2(BP_\tau - \mu(E + F))\eta + \alpha(t)(\eta^2 + \eta^2)FQ_\eta \\
- 2g(t)FQ(F^2Q^2 + (A + BP)^2) \\
+ E(A^2 + B^2P^2 + E^2 + EFQ + 3F^2Q^2) &= 0.
\end{align*} \tag{5} \]

Here, \( \Delta_i(i = 1, 2, 3) \) is a symbol of the expression:
\[ \begin{align*}
\Delta_1 &= \eta_{xx} + \eta_{yy}, \\
\Delta_2 &= \eta_\tau + \alpha(t)(\chi_{xx} + \chi_{xy}), \\
\Delta_3 &= 2\chi_\tau + \alpha(t)(\chi_{xx}^2 + \chi_{xy}^2) + \Omega(t)(x^2 + y^2).
\end{align*} \tag{6} \]

The reduction equations \( \Delta_i(i = 1, 2, 3) = 0 \) and \( 2\sigma + [\alpha(t)(\chi_{xx} + \chi_{yy}) + 2\gamma(t)]\sigma = 0(\sigma = A, B, F) \) can deduce
\[ \begin{align*}
\eta &= \delta(t)x + \delta_2(t)y + \delta_0(t), \\
\Omega(t) &= -\frac{2\chi_\tau + \alpha(t)(\chi_{xx}^2 + \chi_{xy}^2)}{x^2 + y^2}.
\end{align*} \tag{9} \]

\[ \chi = -\frac{1}{\alpha(t)} \left[ \frac{\delta_1(t)}{2\delta_1(t)^2} \delta_2(t_\tau) \delta_2(t) \right] \tag{10} \]
\[ A = a_0 \exp \left( \frac{1}{2} \int_0^t \delta_1(\tau) \delta_1(\tau)^2 \delta_2(\tau) \right) \tag{11} \]

where \( a_0, b, \) and \( f \) are arbitrary constants, \( \delta_0(t), \delta_1(t), \)
\[ \delta_2(t), \text{ and } \delta_0(t) \text{ are functions of time } t, \] \( E = 0. \) When
\[ \tau = \frac{1}{2} \int_0^t \alpha(s)(\delta_1(s)^2 + \delta_2(s)^2)ds, \tag{12} \]
\[ g(t) = -\frac{\alpha(t)(\delta_1(t)^2 + \delta_2(t)^2)}{2A^2}, \tag{13} \]

the coupled system of constant coefficients from (4) and (5) are reduced to
\[ (\mu - 1)Q - bP_\tau - fQ(F^2Q^2 + b^2P^2 + 2bP) \\
- fQ_\eta \eta = 0. \]
Therefore, the first-order rational solution of (13) and (14) is
\[
P = -\frac{4}{R_1(\eta, \tau)} \cdot Q = -\frac{8\tau}{R_1(\eta, \tau)},
\]
where \( R_1(\eta, \tau) = 1 + 2\eta^2 + 4\tau^2 \), and the second-order rational solution is
\[
P = \frac{P_1(\eta, \tau)}{R_2(\eta, \tau)} \cdot Q = -\frac{Q_1(\eta, \tau)\tau}{R_2(\eta, \tau)},
\]
where \( P_1(\eta, \tau) = \frac{3}{2} - 9\tau^2 - \frac{3}{2}\eta^2 - 6\eta^2\tau^2 - 10\tau^4 - \frac{1}{2}\eta^4 \), \( Q_1(\eta, \tau) = -\frac{9}{4} + 2\tau^2 - 3\eta^2 + 4\eta^2\tau^2 + 4\tau^4 + \eta^4 \), \( R_2(\eta, \tau) = \frac{3}{2} + \frac{3}{2}\tau^2 + \frac{9}{16}\eta^2 - \frac{9}{2}\eta^2\tau^2 - \frac{3}{2}\tau^4 + \frac{1}{8}\eta^4 + \frac{7}{2}\tau^6 + \frac{7}{2}\eta^4\tau^2 + \frac{1}{12}\eta^6 \), according to the direct method developed in [8, 11, 12] (\( \mu = 1 \)).

Finally, a direct reduction solution of (1) can be driven
\[
\Psi = A(1 + bP + ifQ)e^{i(\chi + \tau)},
\]
the known functions \( A \equiv A(t), P \equiv P(\eta, \tau), Q \equiv Q(\eta, \tau), \chi \equiv \chi(x, y, t), \) and \( \tau \equiv \tau(t) \) are expressed by (11), (15), (16), (10), and (12), respectively.

3. Several Coherent Excitations of the Solutions

Now, we focus on the coherent structures of the solution for the \((2 + 1)\)-dimensional nonautonomous NLS equation (1).

According to (17), the first-order rational-like solution of (1) can be rewritten as
\[
\Psi = a_0 \left( -\frac{3 + 2\eta^2 + 4\tau^2 - 8i\tau}{1 + 2\eta^2 + 4\tau^2} \right)^{\frac{1}{2}} e^{\frac{1}{2} s_0 \left( \delta_1(t) - \delta_0(t) - 2\delta_0(t) - 2\delta_1(t) \right) g(t) + \gamma(t)},
\]
the variables \( \eta, \tau, \) and \( \chi \) are expressed by (9), (12), and (10), respectively. We can find that the above solution (18) is just a restricted combination of the coefficients \( \alpha(t), \Omega(t), g(t), \) and \( \gamma(t) \). As the time \( t \)

Fig. 2. (a) – (b) Single chirped structure with N-shape and the contour plot for the intensity \( U \equiv |\Psi|^2 \) (19) of the first-order rational-like solution (18) for \( a_0 = \delta_1(t) = 1, \delta_0(t) = 1 + e^{i + \sin(t)} \), \( \alpha(t) = \tanh(t) \), and \( \gamma(t) = 0.1 \sin(t) \). (c) – (d) Periodic structure with chirp and the contour plot for the intensity \( U \equiv |\Psi|^2 \) (19) for \( a_0 = \delta_1(t) = 1, \delta_0(t) = 1 + e^{i + \sin(t)} \), \( \alpha(t) = \tanh(\sin(t)) \), and \( \gamma(t) = 0.1 \sin(t) \).
Fig. 3. (a) – (b) Another chirped structure with N-shape and the contour plot for the intensity $U \equiv |\Psi|^2$ (19) of the first-order rational-like solution (18) for $a_0 = \delta_1(t) = 1, \delta_0(t) = 1 + e^{i + \sin(t)}, \alpha(t) = \tanh(t)$, and $\gamma(t) = 0.1 \tanh(t)$. (c) – (d) Quasi-periodic structure with chirp and the contour plot for the intensity $U \equiv |\Psi|^2$ (19) of the first-order rational-like solution (18) for $a_0 = \delta_1(t) = 1, \delta_0(t) = 1 + e^{i + \sin(t)}, \alpha(t) = \tanh(\sin(t))$, and $\gamma(t) = 0.1 \tanh(t)$.

The solutions $\delta_0(t), \delta_1(t), \delta_2(t), d_0(t), \gamma(t)$, and $\alpha(t)$ are arbitrary, several typical time-modulated excitations of the intensity

$$U \equiv |\Psi|^2 = a_0^2 \int_0^\tau \frac{\delta_1(s)\delta_0(s)^2 - \delta_2(s)\delta_0(s) - 2\delta_0(s)^2\gamma(s)}{\delta_0(s)^2} ds,$$

$$[-3 + 2(\delta_1(t)x + \delta_2(t)y + \delta_0(t))^2 + 4\tau^2]^2 + 64\tau^2$$

$$[1 + 2(\delta_1(t)x + \delta_2(t)y + \delta_0(t))^2 + 4\tau^2],$$

where $\tau = \frac{1}{2} \int_0^\tau \alpha(s)(\delta_1(s)^2 + \delta_2(s)^2) ds$, are derived.

It can be seen that the solution structure (18) is different from the solution of the generalized $(3 + 1)$-dimensional Gross–Pitaevskii (GP) equation (20) or (21) [13], where by applying a novel similarity transformation, the $(3 + 1)$-dimensional GP equation is reduced to a $(3 + 1)$-dimensional standard NLS equation, and the solution of the GP equation is thus constructed via those of the NLS equation. For simplification, we fix the parameters $a_0 = \delta_1(t) = 1$. When taking $\delta_0(t) = \sin^2(t), \alpha(t) = 0.5 \tanh(t)$, and $\gamma(t) = 0.1 \tanh(t) \tanh(t)$, a solitary structure with the twisting behaviour in a symmetric time interval $[-5, 5]$ shows oscillated variation on its propagation for the fixed $y = 0$ (Fig. 1a–b). Another case, when taking $\delta_0(t) = e^{1 + 0.1 \sin(t)}, \alpha(t) = \sin^2(0.02t)$, and $\gamma(t) = \sin^2(0.005t)$, a coherent structure with chirp in a symmetric time interval $[-28, 28]$ shows oscillated variation for the time-$t$ and space-$x$ depended solution (Fig. 1c–d).

Now, we focus on the periodic propagating wave pattern in terms of some elementary functions for (19). Considering solution (18), for analytical convenience, we restrict the attention to simple cases where the quantities $\delta_0(t), \alpha(t), \gamma(t)$, and $\alpha(t)$ can be evaluated in simple, closed forms which do not involve the complicated integrals in this paper. The selections of the triangle function sin, the hyperbolic function $\tanh/h\tanh$, and the exponential function will satisfy these requirements, and will now lead to new wave patterns for the $(2 + 1)$-dimensional nonautonomous NLS equation (1). Figure 2a–b describes a single chirped structure with N-shape and the contour plot for the intensity $U \equiv |\Psi|^2$
Fig. 4. Double structures of the intensity (21), for functions selected corresponding to Figure 1.

Fig. 5. Double structures of the intensity (21), for functions selected corresponding to Figure 2.
(19) of the first-order rational-like solution (18) (the fixed \( y = 0 \)), for \( \delta_0(t) = 1 + e^{i\sin(t)} \), \( \alpha(t) = \tanh(t) \), and \( \gamma(t) = 0.1 \sin(t) \). Figure 2c–d are the periodic structure with chirp and the contour plot for the intensity \( U \equiv |\Psi|^2 \) (19) when the dispersion coefficient \( \alpha(t) = \tanh(\sin(t)) \).

The corresponding circumstance of Figure 2 is the quasi-periodic propagation. When the coefficient \( \gamma(t) = 0.1 \sin(t) \) is substituted by the hyperbolic function \( \gamma(t) = 0.1 \sech(t) \), the quasi-periodic structure with chirp for the intensity \( U \equiv |\Psi|^2 \) (19) of the first-order rational-like solution (18) is constructed (Fig. 3a–b). Figure 3c–d depicts the pattern behaviour in a symmetric time interval \([-15, 15]\), where the chirping amplitude becomes weak as time \( t \) increases from \(-15\) to \(15\). Obviously, all these structures are different from our former works [14–17].

The second-order rational-like solution of (1) can be rewritten as

\[
\Psi = a_0 \left( 1 + \frac{P_1(\eta, \tau)}{R_2(\eta, \tau)} \frac{Q_1(\eta, \tau) \tau i}{R_2(\eta, \tau)} \right) e^{i \int t \, d\tau} e^{i(x + \tau)},
\]

its intensity

\[
U \equiv |\Psi|^2 = a_0^2 e^{2 \int t \, d\tau} \frac{\delta_1(\eta i \tan(\eta \pi), -\delta_1(\eta i \tan(\eta \pi)) -1 \sin(\eta \pi \tau \alpha))}{\delta_1(i \tau)^2} \left( \frac{P_1(\eta, \tau) + Q_1(\eta, \tau) \tau i}{R_2(\eta, \tau)} \right)^2,
\]

where \( P_1(\eta, \tau) = \frac{3}{\pi} - 9 \tau^2 - \frac{3}{2} \eta^2 - 6 \eta^2 \tau^2 - 10 \tau^4 - \frac{1}{2} \eta^4 \), \( Q_1(\eta, \tau) = \frac{-15}{\pi} + 2 \tau^2 - 3 \eta^2 + 4 \eta^2 \tau^2 + 4 \tau^4 + \eta^4 \), \( R_2(\eta, \tau) = \frac{3}{\pi} + \frac{3}{2} \tau^2 + \frac{9}{10} \eta^2 - \frac{1}{2} \eta^2 \tau^2 + \frac{9}{2} \tau^4 + \frac{1}{2} \eta^4 + \frac{3}{2} \tau^4 + \frac{1}{2} \eta^4 \tau^2 + \frac{1}{2} \eta^6 \), also, the variables \( \eta, \tau \) and \( \chi \) are expressed by (9), (12), and (10), respectively, the time \( \tau \) functions \( \delta_0(t), \delta_1(t), \delta_2(t), d_0(t), \gamma(t) \), and \( \alpha(t) \) are arbitrary, and \( \tau = \frac{1}{2} \int \alpha(s) (\delta_1(s)^2 + \delta_2(s)^2) \, ds \). Figures 4–6 show: (i) Although the same functions taken as in Figures 1–3, these excitations have double structures comparing to the former single ones. (ii) The second-order rational-like solution owns even higher amplitude, just as the description of rogue waves in the deep ocean and high intensity rogue light wave pulses in optical fibers [11].

Fig. 6. Double structures of the intensity (21), for functions selected corresponding to Figure 3.
4. Conclusion

In summary, we have obtained analytical solutions in terms of rational-like functions for the $(2+1)$-dimensional nonlinear Schrödinger equation with time-varying coefficients using the similarity transformation and direct ansatz. These obtained solutions contain several free functions of time $t$, which provide us with more choices of these functions to generate the abundant wave structures. Here, we chose three types of elementary functions to exhibit these wave propagations related to the obtained solutions. These solutions may provide more information to further study the nonlinear physical system.

Acknowledgement

The author expresses his sincere thanks for the referees for their valuable suggestions and is in debt to Profs. J.F. Zhang and C.L. Zheng and Drs. Y. Yang and Y.M. Chen for their fruitful discussions. The work was supported by the National Natural Science Foundation of China (No. 10772110) and the Natural Science Foundation of Zhejiang Province of China (Nos. Y606049 and Y6090681).