# Flow around a Triaxial Ellipsoid in a Long Circular Tube 

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This paper deals with the three-dimensional analysis of viscous fluid flow in a long circular cylinder containing an ellipsoidal obstacle. The center of the ellipsoid coincides with that of the cylinder, and the flow is confined to the space between the ellipsoid and the cylinder when the fluid velocity at the large distance from the ellipsoid is uniform. The equations of the classical theory of fluid dynamics are solved in terms of an unknown function which is then shown to be the solution of a boundary integro-differential equation.

A numerical solution of the integro-differential equation is obtained and the pressure on the surface of the ellipsoid is presented in graphical forms for various values of the radius of the circular tube.

Key words: Triaxial Ellipsoid; Cylinder; Integro-Differential Equations; Reduction Formula.

## 1. Introduction

It has been long time since the study on the flow around a spherical and non-spherical object in a tube began. The investigations varied from the vortical to irrotational flow and from the inviscid to viscous flow. The problem of determining the distribution of a vector potential in a long circular cylinder containing a spherical or a spheroidal obstacle has been investigated by Smythe [1, 2]. The problem of flow around a sphere in a tube has been also investigated by others [3, 4]. However, relatively sparse attention has been paid to the solution concerning a triaxial ellipsoidal obstacle, as a special case of which the analysis on spheres or spheroids can be dealt. Motion of a viscous liquid past an ellipsoid in an unbounded space was however investigated by Venkates [5].

In more recent years, numerical studies on the motion of an ellipsoid in a circular tube have appeared. Sugihara-Seki [6] studied numerically the motions of an ellipsoidal particle in a tube flow. She used a finiteelement method to solve the Stokes equations for flow around a spheroid placed at various positions in the tube. The instantaneous velocity was used to compute the particle trajectories. Swaminathan et al. [7] have used direct numerical simulations to investigate the motion of an ellipsoid settling in an infinitely long
circular tube, under the influence of gravity, at low and intermediate Reynolds numbers. They examined the issue of damping of the oscillatory motion for different cases of particle inertia.

Information on the potential flow around an ellipsoid will be of value to the circumstances that occur in a wind tunnel, to a circular cylindrical flow with bubbles or to an electrical flow in a circular cylindrical conductor with defects that can be approximated by a triaxial ellipsoid.

Applications of the study on such flow can be made in a broad range of biological and engineering fields; examples include flow due to the motion of proteins in various biomedical applications and transport of encapsuled solid matter in pipelines.

In this paper, we derive the solution of the problem determining the distribution of the potential in a long circular cylinder containing a triaxial ellipsoid whose center coincides with that of the cylinder when the flow is uniform at a large distance from the ellipsoid. We assume that the fluid is incompressible and viscous. In more recent years, the present author has considered the same problem for the spheroid [8].

In Section 2, by the use of the field equations and employing Fourier transform, the boundary integrodifferential equation is derived in which the unknown function is subsequently solved by the Galerkin
method. In Section 3, some numerical examples are given.

## 2. Derivation of the Integro-Differential Equation

In this section we consider the Stokes problem when the fluid is viscous. Consider a circular cylinder of the radius $h$ having an ellipsoid whose semi-axes are $a, b$, and $c$. We take the center of the ellipsoid and the cylinder as the origin of the Cartesian coordinates, and the $x, y$ - and $z$-axis along the semi-axes of the ellipsoid, respectively, then the ellipsoid occupies the region $V$ which is governed by the equation $\frac{x^{2}}{a^{2}}+\frac{y^{2}}{b^{2}}+\frac{z^{2}}{c^{2}} \leq 1$. The surface of the ellipsoid is denoted by $S$. We shall also use cylindrical coordinates $(r, \theta, z)$ which are connected to the Cartesian coordinates by

$$
x=r \cos \theta, y=r \sin \theta, z=z
$$

Let the velocity of the flow at large distances from the ellipsoid be $v_{0}$.

The fluid velocity $\mathbf{v}$ and the pressure $p$ satisfy the Stokes equation and the continuity equation:

$$
\begin{align*}
& \nabla p=\mu \nabla^{2} \mathbf{v}  \tag{1}\\
& \nabla \cdot \mathbf{v}=0 \tag{2}
\end{align*}
$$

where $\mu$ is the coefficient of viscosity. Let $\mathbf{v}=$ $\left(u_{x}, u_{y}, u_{z}\right)$ be the velocity components in Cartesian coordinates. If we choose the velocity components as
$u_{x}=2 B_{x}-\frac{\partial \Phi}{\partial x}+\frac{\partial \Omega}{\partial y}, u_{y}=2 B_{y}-\frac{\partial \Phi}{\partial y}-\frac{\partial \Omega}{\partial x}$,
$u_{z}=2 B_{z}-\frac{\partial \Phi}{\partial z}$,
where $\Phi$ is defined by

$$
\Phi=B_{0}+x B_{x}+y B_{y}+z B_{z}
$$

with $B_{0}, B_{x}, B_{y}, B_{z}$, and $\Omega$ being the harmonic functions, we see that (1) and (2) are satisfied by

$$
\begin{equation*}
p=2 \mu\left(\frac{\partial B_{x}}{\partial x}+\frac{\partial B_{y}}{\partial y}+\frac{\partial B_{z}}{\partial z}\right) . \tag{4}
\end{equation*}
$$

The suitable functions for the problem are

$$
\begin{equation*}
B_{x}=\sum_{m=0}^{\infty} \int_{-\infty}^{\infty} A_{m}(\xi) I_{m+1}(\xi r) \mathrm{e}^{-\mathrm{i} \xi z} \mathrm{~d} \xi \cos (m+1) \theta \tag{5}
\end{equation*}
$$

$B_{y}=\sum_{m=0}^{\infty} \int_{-\infty}^{\infty} A_{m}(\xi) I_{m+1}(\xi r) \mathrm{e}^{-\mathrm{i} \xi z} \mathrm{~d} \xi \sin (m+1) \theta$,

$$
\begin{align*}
B_{z}= & \left\{1-\frac{1}{2}\left(x \frac{\partial}{\partial x}+y \frac{\partial}{\partial y}+z \frac{\partial}{\partial z}\right)\right\} \int_{V} \frac{a(\mathbf{u}) \mathrm{d} \mathbf{v}}{R(\mathbf{x}-\mathbf{u})},  \tag{7}\\
B_{0}= & \frac{\partial}{\partial z} \int_{V} \frac{b(\mathbf{u}) \mathrm{d} \mathbf{v}}{R(\mathbf{x}-\mathbf{u})}  \tag{8}\\
& +\sum_{m=0}^{\infty} \int_{-\infty}^{\infty} B_{m}(\xi) h I_{m}(\xi r) \mathrm{e}^{-\mathrm{i} \xi z} \mathrm{~d} \xi \cos m \theta-\frac{v_{0} z}{2}, \\
\Omega= & \sum_{m=1}^{\infty} \int_{-\infty}^{\infty} C_{m}(\xi) h I_{m}(\xi r) \mathrm{e}^{-\mathrm{i} \xi z} \mathrm{~d} \xi \sin m \theta  \tag{9}\\
& +\frac{1}{h} \int_{V} \frac{c(\mathbf{u}) \mathrm{d} \mathbf{v}}{R(\mathbf{x}-\mathbf{u})}\left(1-\delta_{m, 0}\right),
\end{align*}
$$

where

$$
R(\mathbf{x}-\mathbf{u})=\sqrt{(x-u)^{2}+(y-v)^{2}+(z-w)^{2}}
$$

and $I_{m}(x)$ is the modified Bessel function of the first kind, $\mathbf{u}=(u, v, w), \mathrm{d} \mathbf{v}$ is used for $\mathrm{d} u \mathrm{~d} v \mathrm{~d} w$, and $\delta_{m, 0}$ is a Kronecker delta.

Let $\left(u_{r}, u_{\theta}, u_{z}\right)$ be the velocity components in cylindrical coordinates. The velocity at the tube wall is zero:

$$
\begin{align*}
& u_{r}=0  \tag{10a}\\
& u_{\theta}=0  \tag{10b}\\
& u_{z}=0 \tag{10c}
\end{align*}
$$

Boundary condition (10a) can be written in an alternative form as

$$
\mathcal{F}\left[u_{r}(h, \theta, z) ; z \rightarrow \xi\right]=0,
$$

where $\mathcal{F}$ means the Fourier transform.
We make use of the known integral in Erdélyi et al. [9]

$$
\begin{align*}
& \int_{0}^{\infty} \frac{\cos (\xi z) \mathrm{d} z}{\sqrt{(x-u)^{2}+(y-v)^{2}+z^{2}}}  \tag{11}\\
& =K_{0}\left(\xi \sqrt{(x-u)^{2}+(y-v)^{2}}\right)
\end{align*}
$$

and

$$
\begin{align*}
& K_{0}\left(\xi \sqrt{(x-u)^{2}+(y-v)^{2}}\right) \\
& =K_{0}\left(\xi\left\{r^{2}+r^{\prime 2}-2 r r^{\prime} \cos \left(\theta-\theta^{\prime}\right)\right\}^{\frac{1}{2}}\right) \\
& =I_{0}\left(\xi r_{<}\right) K_{0}\left(\xi r_{>}\right)  \tag{12}\\
& \quad+2 \sum_{m=1}^{\infty} \cos \left\{m\left(\theta-\theta^{\prime}\right)\right\} I_{m}\left(\xi r_{<}\right) K_{m}\left(\xi r_{>}\right)
\end{align*}
$$

where $r_{<}=\min \left(r, r^{\prime}\right), r_{>}=\max \left(r, r^{\prime}\right), K_{0}, K_{m}$ are the modified Bessel functions of the second kind, and we have set

$$
\begin{equation*}
u=r^{\prime} \cos \theta^{\prime}, \quad v=r^{\prime} \sin \theta^{\prime} \tag{13}
\end{equation*}
$$

We obtain following relation for solving unknown $A_{m}(\xi), B_{m}(\xi)$, and $C_{m}(\xi)$ :

$$
\begin{aligned}
& \left\{I_{m+1}(\xi h)-\xi h I_{m+1}^{\prime}(\xi h)\right\} A_{m}(\xi) \\
& -\xi h I_{m}^{\prime}(\xi h) B_{m}(\xi)+m I_{m}(\xi h) C_{m}(\xi) \\
& =\frac{1}{\pi \mathrm{i} \zeta} \int_{V} a(\mathbf{u}) \mathrm{e}^{\mathrm{i} w \xi}\left[\frac { K _ { m } ( | \zeta | ) } { I _ { m } ( \zeta ) } \left\{F ( \xi , r ^ { \prime } , w ) \left[\zeta I_{m+1}(\zeta)\right.\right.\right. \\
& \left.\quad+m I_{m}(\zeta)\right]+f\left(\xi, r^{\prime}, w\right)\left[-(m+2) \zeta I_{m+1}(\zeta)\right.
\end{aligned}
$$

$$
\begin{equation*}
\left.\left.+\left(\zeta^{2}-2 m\right) I_{m}(\zeta)\right]\right\} \tag{14}
\end{equation*}
$$

The condition (10c) can be alternatively written as

$$
\left.-\frac{F\left(\xi, r^{\prime}, w\right)-(2+m) f\left(\xi, r^{\prime}, w\right)}{I_{m}(|\zeta|)}\right] \cos m \theta^{\prime} \mathrm{d} \mathbf{v} \varepsilon_{m}
$$

$$
\mathcal{F}\left[u_{z}(h, \theta, z) ; z \rightarrow \xi\right]=0
$$

$$
+\frac{1}{\pi \mathrm{i}} \xi \int_{V} b(\mathbf{u}) \mathrm{e}^{\mathrm{i} w \xi}|\xi| K_{m}^{\prime}(|\zeta|) I_{m}\left(|\xi| r^{\prime}\right) \cos m \theta^{\prime} \mathrm{d} \mathbf{v} \varepsilon_{m}
$$

from which we obtain another relation to solve un-

$$
-\frac{m}{\pi h^{2}} \int_{V} c(\mathbf{u}) \mathrm{e}^{\mathrm{i} w \xi} K_{m}(|\zeta|) I_{m}\left(|\xi| r^{\prime}\right) \sin m \theta^{\prime} \mathrm{d} \mathbf{v} \varepsilon_{m}
$$ known $A_{m}(\xi)$ and $B_{m}(\xi)$ :

$$
\begin{align*}
& \zeta I_{m+1}(\zeta) A_{m}(\xi)+\zeta I_{m}(\zeta) B_{m}(\xi) \\
& =-\frac{1}{\pi \mathrm{i}} \int_{V} a(\mathbf{u}) \mathrm{e}^{\mathrm{i} w \xi}\left[K_{m}(|\zeta|) F\left(\xi, r^{\prime}, w\right)\right. \\
& \left.+f\left(\xi, r^{\prime}, w\right)\left(\frac{I_{m+1}(\zeta)}{I_{m}(\zeta)} \zeta K_{m}(|\zeta|)-\frac{1}{I_{m}(|\zeta|)}\right)\right] \\
& \cdot \cos m \theta^{\prime} \mathrm{d} \mathbf{v} \varepsilon_{m}-\frac{1}{\pi \mathrm{i}} \xi^{2} \int_{V} b(\mathbf{u}) \mathrm{e}^{\mathrm{i} w \xi} K_{m}(|\zeta|) I_{m}\left(|\xi| r^{\prime}\right) \\
& \cdot \cos m \theta^{\prime} \mathrm{d} \mathbf{v} \varepsilon_{m}+\mathrm{i} \frac{v_{0}}{2} \delta(\xi) \tag{16}
\end{align*}
$$

where $\delta(\xi)$ is the Dirac delta function, and we have used the known relation

$$
\frac{1}{\pi} \int_{0}^{\infty} \cos \xi z \mathrm{~d} z=\delta(\xi)
$$

Therefore if we solve (14), (15), and (16) simultaneously for $A_{m}(\xi), B_{m}(\xi)$, and $C_{m}(\xi)$, we obtain follow-
ing equations:
$A_{m}(\xi)=-\frac{1}{\pi \mathrm{i}}\left[\int_{V} a(\mathbf{u})\left\{\frac{F\left(\xi, r^{\prime}, w\right)}{\Delta_{m}(\zeta) I_{m}(|\zeta|)} \zeta I_{m}(\zeta) I_{m}^{\prime}(\zeta)\right.\right.$
$+f\left(\xi, r^{\prime}, w\right)$
$\left.\cdot\left(G_{m}(\zeta)-\frac{2(m+1) \zeta I_{m}(\zeta) I_{m}^{\prime}(\zeta)}{\Delta_{m}(\zeta) I_{m}(|\zeta|)}\right)\right\} \mathrm{e}^{\mathrm{i} \xi_{w}} \cos m \theta^{\prime} \mathrm{d} \mathbf{v}$
$\left.+\frac{\xi^{2} \zeta I_{m}^{\prime}(\zeta)}{\Delta_{m}(\zeta)} \int_{V} b(\mathbf{u}) I_{m}\left(\xi^{\prime}\right) \mathrm{e}^{\mathrm{i} \xi w} \cos m \theta^{\prime} \mathrm{d} \mathbf{v}\right] \varepsilon_{m}$
$-\frac{m \zeta I_{m}(\zeta)}{\pi \Delta_{m}(\zeta) h^{2}} \int_{V} c(\mathbf{u}) I_{m}\left(\xi r^{\prime}\right) \mathrm{e}^{\mathrm{i} \xi w} \sin m \theta^{\prime} \mathrm{d} \mathbf{v} \varepsilon_{m}$
$-\frac{\delta(\xi) \zeta^{2} I_{1}^{2}(\zeta) v_{0}}{2 i \Delta_{0}(\zeta)} \delta_{m, 0}$,
where
$\Delta_{m}(\zeta)=2 m \zeta I_{m}^{2}(\zeta) I_{m+1}(\zeta)+\zeta^{2} I_{m}^{\prime}(\zeta)$
$\cdot\left\{I_{m}(\zeta) I_{m+1}(\zeta) 2(m+1)-\zeta\left(I_{m}^{2}(\zeta)-I_{m+1}^{2}(\zeta)\right)\right\}$,
$G_{m}(\zeta)=\frac{K_{m}(|\zeta|)}{I_{m}(\zeta)}-\frac{\zeta I_{m+1}(\zeta)}{\Delta_{m} I_{m}(|\zeta|)}\left\{\zeta I_{m+1}(\zeta)+m I_{m}(\zeta)\right\}$,
and
$B_{m}(\xi)=-\frac{1}{\pi \mathrm{i}}\left[\int_{V} a(\mathbf{u})\left\{\frac{1}{\zeta} G_{m}(\zeta) F\left(\xi, r^{\prime}, w\right)\right.\right.$
$-\frac{f\left(\xi, r^{\prime}, w\right)}{\Delta_{m}(\zeta) I_{m}(|\zeta|)}\left[2 m I_{m+1}(\zeta)\right.$
$\left.\left.-\zeta^{2} I_{m}^{\prime}(\zeta)\right] I_{m}(\zeta)\right\} \mathrm{e}^{\mathrm{i} \xi w} \cos m \theta^{\prime} \mathrm{d} \mathbf{v} \varepsilon_{m}$
$\left.+G_{m}(\zeta) \frac{\xi}{h} \int_{V} b(\mathbf{u}) I_{m}\left(|\xi| r^{\prime}\right) \mathrm{e}^{\mathrm{i} \xi w} \cos m \theta^{\prime} \mathrm{d} \mathbf{v} \varepsilon_{m}\right]$
$+\frac{\zeta m}{\pi h^{2}} I_{m+1}(\zeta) \int_{V} c(\mathbf{u}) I_{m}\left(\xi r^{\prime}\right) \mathrm{e}^{\mathrm{i} \xi w} \sin m \theta^{\prime} \mathrm{d} \mathbf{v} \varepsilon_{m}$
$+\frac{\delta(\xi) \zeta^{2} I_{1}(\zeta) I_{2}(\zeta) v_{0}}{2 \mathrm{i} \Delta_{0}(\zeta)} \delta_{m, 0}$,
and
$C_{m}(\xi)=-\frac{2}{\pi \mathrm{i} \zeta} \int_{V} a(\mathbf{u})\left\{\frac{F\left(\xi, r^{\prime}, w\right)}{\Delta_{m}(\zeta) I_{m}(|\zeta|)}\right.$

- $\zeta I_{m+1}(\zeta) I_{m}(\zeta)+f\left(\xi, r^{\prime}, w\right)$
$\left.\cdot\left(G_{m}(\zeta)-\frac{(m+2) \zeta I_{m+1}(\zeta) I_{m}(\zeta)}{\Delta_{m}(\zeta) I_{m}(|\zeta|)}\right)\right\} \mathrm{e}^{\mathrm{i} \xi_{w}} \cos m \theta^{\prime} \mathrm{d} \mathbf{v} \varepsilon_{m}$
$-\frac{2 \xi^{2} I_{m+1}(\zeta)}{\Delta_{m}(\zeta) \pi \mathrm{i}} \int_{V} b(\mathbf{u}) I_{m}\left(\xi r^{\prime}\right) \mathrm{e}^{\mathrm{i} \xi w} \cos m \theta^{\prime} \mathrm{d} \mathbf{v} \varepsilon_{m}$
$-\left(G_{m}(\zeta)-\frac{\zeta\left\{(m+2) I_{m+1}(\zeta)-\zeta I_{m}(\zeta)\right\} I_{m}(\zeta)}{\Delta_{m}(\zeta) I_{m}(|\zeta|)}\right) \frac{1}{\pi h^{2}}$
$\cdot \int_{V} c(\mathbf{u}) I_{m}\left(|\xi| r^{\prime}\right) \mathrm{e}^{\mathrm{i} \xi w} \sin m \theta^{\prime} \mathrm{d} \mathbf{v} \varepsilon_{m}$.

The velocity components ( $u_{x}, u_{y}, u_{z}$ ) are zero on the surface of the ellipsoid. Thus if we substitute the value of $A_{m}(\xi), B_{m}(\xi)$, and $C_{m}(\xi)$ given by (17), (18), and (19) into $B_{x}, B_{y}$, and $B_{0}$ in (5), (6), (8), and (9), we obtain following three conditions:
$u_{x}=2 \sum_{m=0}^{\infty} \mathcal{B}_{m} \cos (m+1) \theta-\frac{\partial \Phi}{\partial x}+\frac{\partial \Omega}{\partial y}=0$, $(x, y, z) \in S$,
$u_{y}=2 \sum_{m=0}^{\infty} \mathcal{B}_{m} \sin (m+1) \theta-\frac{\partial \Phi}{\partial y}-\frac{\partial \Omega}{\partial x}=0$,
$(x, y, z) \in S$,
$u_{z}=2 \mathcal{L} \int_{V} \frac{a(\mathbf{u}) \mathrm{d} \mathbf{v}}{R(\mathbf{x}-\mathbf{u})}-\frac{\partial \Phi}{\partial z}+v_{0}\left(1-\frac{r^{2}}{h^{2}}\right)=0$,

$$
\begin{equation*}
(x, y, z) \in S \tag{22}
\end{equation*}
$$

where
$\mathcal{B}_{m}=-\frac{2}{\pi} \varepsilon_{m}\left[\int_{V} a(\mathbf{u}) \cos m \theta^{\prime} \int_{0}^{\infty}\left\{\mathcal{F}_{1}\left(\xi, r^{\prime}, w, z\right)\right.\right.$

$$
\left.\left(G_{m}(\zeta)-\frac{2(m+1) \zeta I_{m}^{\prime}(\zeta)}{\Delta_{m}(\zeta)}\right)\right\} I_{m+1}(\xi r) \mathrm{d} \xi \mathrm{~d} \mathbf{v}
$$

$\left.\cdot\left(G_{m}(\zeta)-\frac{2(m+1) \zeta I_{m}^{\prime}(\zeta)}{\Delta_{m}(\zeta)}\right)\right\} I_{m+1}(\xi r) \mathrm{d} \xi \mathrm{d} \mathbf{v}$
$\cdot I_{m}\left(\xi r^{\prime}\right) I_{m+1}(\xi r) \xi^{2} \sin \xi(w-z) \mathrm{d} \xi \mathrm{d} \mathbf{v}$
$+\frac{m}{h^{2}} \int_{V} c(\mathbf{u}) \sin m \theta^{\prime} \int_{0}^{\infty} \frac{\zeta I_{m}(\zeta)}{\Delta_{m}(\zeta)}$
$\left.\cdot I_{m}\left(\xi r^{\prime}\right) I_{m+1}(\xi r) \cos \xi(w-z) \mathrm{d} \xi \mathrm{d} \mathbf{v}\right]$
with

$$
\pi^{-m}\left[J_{V} \quad J_{0}(>), \infty\right.
$$

$$
\frac{\zeta I_{m+1}(\zeta)+m I_{m}(\zeta)}{\Delta_{m}(\zeta)}+\mathcal{F}_{2}\left(\xi, r^{\prime}, w, z\right)
$$

. $\frac{\zeta I_{m+1}(\zeta)+m I_{m}(\zeta)}{\Delta_{m}(\zeta)}+\mathcal{F}_{2}\left(\xi, r^{\prime}, w, z\right)$

$$
\begin{equation*}
+\int_{V} b(\mathbf{u}) \cos m \theta^{\prime} \int_{0}^{\infty} \frac{\zeta I_{m+1}(\zeta)+m I_{m}(\zeta)}{\Delta_{m}(\zeta)} \tag{23}
\end{equation*}
$$

$$
\begin{aligned}
& \mathcal{F}_{1}\left(\xi, r^{\prime}, w, z\right)= \Re F\left(\xi, r^{\prime}, w\right) \sin \xi(w-z) \\
&+\Im F\left(\xi, r^{\prime}, w\right) \cos \xi(w-z), \\
& \mathcal{F}_{2}\left(\xi, r^{\prime}, w, z\right)= \Re f\left(\xi, r^{\prime}, w\right) \sin \xi(w-z) \\
&+\Im f\left(\xi, r^{\prime}, w\right) \cos \xi(w-z), \\
& \mathcal{L}=1-\frac{1}{2}\left(x \frac{\partial}{\partial x}+y \frac{\partial}{\partial y}+z \frac{\partial}{\partial z}\right),
\end{aligned}
$$

and
$\Phi=-\frac{2}{\pi} \sum_{m=0}^{\infty} \cos m \theta \varepsilon_{m}\left[\int_{V} a(\mathbf{u}) \cos m \theta^{\prime}\right.$
$\cdot \int_{0}^{\infty} \frac{1}{\xi}\left(\mathcal{F}_{1}\left(\xi, r^{\prime}, w, z\right)\left\{G_{m}(\zeta) I_{m}(\xi r)+\frac{\xi r I_{m+1}(\xi r)}{\Delta_{m}(\zeta)}\right.\right.$
$\left.\cdot\left(\zeta I_{m+1}(\zeta)+m I_{m}(\zeta)\right)\right\}+\mathcal{F}_{2}\left(\xi, r^{\prime}, w, z\right)$
$\cdot\left\{\left(G_{m}(\zeta)-\frac{2(m+1) \zeta I_{m}^{\prime}(\zeta)}{\Delta_{m}(\zeta)}\right) \xi r I_{m+1}(\xi r)-\frac{\zeta}{\Delta_{m}(\zeta)}\right.$
$\left.\left.\cdot\left[\left(2 m-\zeta^{2}\right) I_{m+1}(\zeta)-m \zeta I_{m}(\zeta)\right] I_{m}(\xi r)\right\}\right) \mathrm{d} \xi \mathrm{d} \mathbf{v}$
$+\int_{V} b(\mathbf{u}) \cos m \theta^{\prime} \int_{0}^{\infty}\left\{G_{m}(\zeta) I_{m}(\xi r)\right.$
$\left.+\frac{\xi r I_{m+1}(\xi r)}{\Delta_{m}(\zeta)} \zeta I_{m}^{\prime}(\zeta)\right\} I_{m}\left(\xi r^{\prime}\right) \xi \sin \xi(w-z) \mathrm{d} \xi \mathrm{d} \mathbf{v}$
$+\frac{m}{h} \int_{V} c(\mathbf{u}) \sin m \theta^{\prime} \int_{0}^{\infty}\left\{I_{m}(\zeta) \xi r I_{m+1}(\xi r)\right.$
$\left.\left.-\zeta I_{m+1}(\zeta) I_{m}(\xi r)\right\} \frac{1}{\Delta_{m}(\zeta)} I_{m}\left(\xi r^{\prime}\right) \cos \xi(w-z) \mathrm{d} \xi \mathrm{d} \mathbf{v}\right]$
$+z \mathcal{L} \int_{V} \frac{a(\mathbf{u}) \mathrm{d} \mathbf{v}}{R(\mathbf{x}-\mathbf{u})}+\frac{\partial}{\partial z} \int_{V} \frac{b(\mathbf{u}) \mathrm{d} \mathbf{v}}{R(\mathbf{x}-\mathbf{u})}$,
and
$\Omega=-\frac{2}{\pi} \sum_{m=1}^{\infty} \varepsilon_{m} \sin m \theta\left[\int_{V} a(\mathbf{u}) \cos m \theta^{\prime}\right.$
$\cdot \int_{0}^{\infty} \frac{2}{\xi}\left\{\mathcal{F}_{1}\left(\xi, r^{\prime}, w, z\right) \frac{\zeta I_{m+1}(\zeta)}{\Delta_{m}(\zeta)}+\mathcal{F}_{2}\left(\xi, r^{\prime}, w, z\right)\right.$
$\left.\cdot\left(G_{m}(\zeta)-\frac{(m+2) I_{m+1}(\zeta)}{\Delta_{m}(\zeta)}\right)\right\} I_{m}(\xi r) \mathrm{d} \xi \mathrm{d} \mathbf{v}$
$+\int_{V} b(\mathbf{u}) \cos m \theta^{\prime} \int_{0}^{\infty} \frac{2 \zeta I_{m+1}(\zeta)}{\Delta_{m}(\zeta)}$
$\cdot I_{m}\left(\xi r^{\prime}\right) I_{m}(\xi r) \xi \sin \xi(w-z) \mathrm{d} \xi \mathrm{d} \mathbf{v}+\frac{1}{h} \int_{V} c(\mathbf{u}) \sin m \theta^{\prime}$
$\cdot \int_{0}^{\infty}\left(G_{m}(\zeta)-\frac{\zeta\left\{(m+2) I_{m+1}(\zeta)-\zeta I_{m}(\zeta)\right\}}{\Delta_{m}(\zeta)}\right)$
$\left.\cdot I_{m}\left(\xi r^{\prime}\right) I_{m}(\xi r) \cos \xi(w-z) \mathrm{d} \xi \mathrm{d} \mathbf{v}\right]$
$+\frac{1}{h} \int_{V} \frac{c(\mathbf{u}) \mathrm{d} \mathbf{v}}{R(\mathbf{x}-\mathbf{u})}\left(1-\delta_{m, 0}\right)$.
Of interest is the case when the radius of the cylinder tends to infinity. All terms involving $h$ vanish, and pertinent functions for the solution are constants. So
$a(\mathbf{u})=a_{1}, b(\mathbf{u})=b_{1}$ (say).

The following formula is useful:
$\int_{V} \frac{\mathrm{~d} \mathbf{v}}{R(\mathbf{x}-\mathbf{u})}$
$=\pi a b c \int_{\lambda}^{\infty}\left(1-\frac{x^{2}}{a^{2}+s}-\frac{y^{2}}{b^{2}+s}-\frac{z^{2}}{c^{2}+s}\right) \frac{\mathrm{d} s}{\Delta(s)}$,
where $\Delta(s)=\sqrt{\left(a^{2}+s\right)\left(b^{2}+s\right)\left(c^{2}+s\right)}$ and $\lambda$ is the greatest root of

$$
1-\frac{x^{2}}{a^{2}+\lambda}-\frac{y^{2}}{b^{2}+\lambda}-\frac{z^{2}}{c^{2}+\lambda}=0
$$

Then from (26) we find that

$$
\Phi=\pi z a_{1} \chi-2 \pi z b_{1} \gamma
$$

where
$\chi=a b c \int_{\lambda}^{\infty} \frac{\mathrm{d} s}{\Delta(s)}, \gamma=a b c \int_{\lambda}^{\infty} \frac{\mathrm{d} s}{\left(c^{2}+s\right) \Delta(s)}$.
Then conditions (20) and (21) require

$$
\left[-a_{1} \frac{\mathrm{~d} \chi}{\mathrm{~d} \lambda}+2 b_{1} \frac{\mathrm{~d} \gamma}{\mathrm{~d} \lambda}\right]_{\lambda=0}=0, \text { or }-a_{1}+2 \frac{b_{1}}{c^{2}}=0
$$

With the help of this relation, the condition $u_{z}=0$ reduces to

$$
\begin{equation*}
v_{0}+\left(a_{1} \chi_{0}+2 b_{1} \gamma_{0}\right) \pi=0 \tag{28}
\end{equation*}
$$

where the suffix denotes that the lower limit in the integrals (27) is to be replaced by zero. Hence,

$$
b_{1}=\frac{1}{2} a_{1} c^{2}, a_{1}=-\frac{v_{0}}{\pi\left(\chi_{0}+\gamma_{0} c^{2}\right)} .
$$

This is in agreement with Lamb [10].
Equations (20)-(22) are solved by the Galerkin method. For this, following formulae [11] along with (26) are useful:
$\int_{V} \frac{v w \mathrm{~d} \mathbf{v}}{R(\mathbf{x}-\mathbf{u})}=b^{2} c^{2} \pi a b c y z$
$\cdot \int_{\lambda}^{\infty}\left(1-\frac{x^{2}}{a^{2}+s}-\frac{y^{2}}{b^{2}+s}-\frac{z^{2}}{c^{2}+s}\right)$
$\cdot \frac{\mathrm{d} s}{\sqrt{\left(a^{2}+s\right)\left(b^{2}+s\right)^{3}\left(c^{2}+s\right)^{3}}}$,
$\int_{V} \frac{w^{2} \mathrm{~d} \mathbf{v}}{R(\mathbf{x}-\mathbf{u})}=\pi a b c$
$\cdot \int_{\lambda}^{\infty}\left\{\frac{c^{2} s}{4\left(c^{2}+s\right)} \omega^{2}(s)-\frac{c^{4} z^{2}}{\left(c^{2}+s\right)^{2}} \omega(s)\right\} \frac{\mathrm{d} s}{\Delta(s)}$,
where

$$
\omega(s)=\frac{x^{2}}{a^{2}+s}+\frac{y^{2}}{b^{2}+s}+\frac{z^{2}}{c^{2}+s}-1 .
$$

We also need to evaluate following integral:
$I_{\ell, m, n}=\int_{0}^{\infty} \frac{1}{\left(a^{2}+s\right)^{\ell}\left(b^{2}+s\right)^{m}\left(c^{2}+s\right)^{n}} \frac{\mathrm{~d} s}{\Delta(s)}$.
To evaluate (30), we let

$$
s=\left(a^{2}-c^{2}\right) \mathrm{sn}^{-2} u
$$

and use the following identities for the Jacobian elliptic functions:

$$
k^{2} \mathrm{sn}^{2} u+\mathrm{dn}^{2} u=1, \operatorname{sn}^{2} u+\mathrm{cn}^{2} u=1
$$

and

$$
k=\sqrt{\frac{a^{2}-b^{2}}{a^{2}-c^{2}}}, k^{\prime 2}=1-k^{2}
$$

So

$$
\begin{align*}
\int_{0}^{\infty} & \frac{1}{\left(a^{2}+s\right)^{\ell}\left(b^{2}+s\right)^{m}\left(c^{2}+s\right)^{n}} \frac{\mathrm{~d} s}{\Delta(s)} \\
& =\frac{2}{\left(a^{2}-c^{2}\right)^{\ell+m+n+\frac{1}{2}}} \int_{0}^{F} \frac{\mathrm{sn}^{2 \ell+2 m+2 n} u \mathrm{~d} u}{\mathrm{dn}^{2 m} u \mathrm{cn}^{2 n} u} \tag{31}
\end{align*}
$$

where

$$
\begin{align*}
& F=\int_{0}^{\theta}\left(1-k^{2} \sin ^{2} u\right)^{-\frac{1}{2}} \mathrm{~d} u \\
& \theta=\sin ^{-1}\left(\frac{\sqrt{a^{2}-c^{2}}}{a}\right) . \tag{32}
\end{align*}
$$

The integral on the right-hand side of (31) is

$$
\begin{aligned}
L_{\ell, m, n}= & \int_{0}^{F} \frac{\mathrm{sn}^{2 \ell+2 m+2 n} u \mathrm{~d} u}{\operatorname{dn}^{2 m} u \mathrm{cn}^{2 n} u} \\
= & \frac{1}{k^{\prime 2 \ell+2 m+2 n}} \sum_{j=0}^{\ell+m+n}(-1)^{j}\binom{\ell+n+m}{j} \\
& \cdot \int_{0}^{F} \operatorname{dn}^{2 \ell+2 n-2 j} u \mathrm{nc}^{2 n-2 j} u \mathrm{~d} u,
\end{aligned}
$$

where we used

$$
\operatorname{sn}^{2} u=\frac{\operatorname{dn}^{2} u-\operatorname{cn}^{2} u}{k^{\prime 2}}, \mathrm{nc} u=\frac{1}{\operatorname{cn} u} .
$$

## Further,

$L_{\ell, m, n}=\frac{1}{k^{\prime 2 \ell+2 m+2 n}}\left[\sum_{j=0}^{\ell+n}(-1)^{j}\binom{\ell+n+m}{j}\right.$
$\cdot \int_{0}^{F}\left(k^{\prime 2}+k^{2} \mathrm{cn}^{2} u\right)^{\ell+n-j} \mathrm{nc}^{2 n-2 j} u \mathrm{~d} u$
$+\sum_{j=\ell+n+1}^{\ell+m+n}(-1)^{j}\binom{\ell+n+m}{j}$
$\left.\cdot \int_{0}^{F} \mathrm{nd}^{2 j-2 \ell-2 n} u\left(\frac{\mathrm{dn}^{2} u-k^{\prime 2}}{k^{2}}\right)^{j-n} \mathrm{~d} u\right]$,
where we used

$$
\mathrm{dn}^{2} u=k^{\prime 2}+k^{2} \mathrm{cn}^{2} u, \operatorname{nd} u=\frac{1}{\operatorname{dn} u} .
$$

Expanding the powered terms by using the binomial expansion, we find
$L_{\ell, m, n}=\frac{1}{k^{\prime 2 \ell+2 m+2 n}}\left[\sum_{j=0}^{\ell+n \ell+n-j} \sum_{i=0}(-1)^{j}\binom{\ell+n+m}{j}\right.$
$\cdot\binom{\ell+n-j}{i} k^{\prime 2 i} k^{2 \ell+2 n-2 j-2 i} C_{2 \ell-2 i}$
$+\sum_{j=\ell+n+1}^{\ell+m+n} \sum_{i=0}^{j-n}(-1)^{j+i}\binom{\ell+n+m}{j}\binom{j-n}{i}$
$\left.\cdot k^{2 i} k^{2 n-2 k} G_{2 \ell-2 i}\right]$,
where

$$
C_{2 n}=\int_{0}^{F} \mathrm{cn}^{2 n} u \mathrm{~d} u, G_{2 n}=\int_{0}^{F} \operatorname{dn}^{2 n} u \mathrm{~d} u .
$$

We find the following reduction formula for $C_{2 n}$ in Byrd and Friedman [12, p. 194]:
$C_{2 n+2}=$
$\frac{2 n\left(2 k^{2}-1\right) C_{2 n}+(2 n-1) k^{\prime 2} C_{2 n-2}+\mathrm{sn} F \mathrm{dn} F \mathrm{cn}^{2 \mathrm{n}-1} F}{(2 n+1) k^{2}}$.
If $\ell-i<0$, we find $C_{-2 n}=D_{2 n}$, where
$D_{2 n+2}=$
$\frac{(2 n-1) k^{2} D_{2 n-2}+2 n\left(1-2 k^{2}\right) D_{2 n}+\operatorname{tn} F \mathrm{dn} F \mathrm{nc}^{2 n} F}{(2 n+1) k^{\prime 2}}$.

Also we have following reduction formula for $G_{2 n}$ :
$G_{2 n+2}=$
$\frac{k^{2} \mathrm{dn}^{2 n-1} F \mathrm{sn} F \mathrm{cn} F+(1-2 n) k^{\prime 2} G_{2 n-2}+2 n\left(2-k^{2}\right) G_{2 n}}{(2 n+1)}$.
If $\ell-i<0$, we find $G_{-2 n}=I_{2 n}$, where
$I_{2 n+2}=$
$\frac{2 n\left(2-k^{2}\right) I_{2 n}+(1-2 n) I_{2 n-2}-k^{2} \mathrm{sn} F \mathrm{cn} F \mathrm{nd}^{2 n+1} F}{(2 n+1) k^{\prime 2}}$.

Thus, finally we see that one needs the following starting values for evaluating the general terms of $C_{2 n}, D_{2 n}, G_{2 n}$, and $I_{2 n}$ :

$$
\begin{aligned}
C_{0} & =D_{0}=F, C_{2}=\frac{1}{k^{2}}\left[E-k^{\prime 2} F\right], \\
D_{2} & =\frac{1}{k^{\prime 2}}\left[k^{\prime 2} F-E+\operatorname{dn} F \operatorname{tn} F\right]
\end{aligned}
$$

where $E$ is the elliptic integral of the second kind of modulus $k$ and argument $\theta$ given in (32) and

$$
G_{0}=I_{0}=F, G_{2}=E, I_{2}=\frac{1}{k^{\prime 2}}\left[E-k^{2} \operatorname{sn} F \mathrm{~cd} F\right]
$$



Fig. 1. Variation of the pressure on the surface of the spheroid; $a=b=5 \mathrm{~cm}, c=$ 3 cm .

Fig. 2. Variation of the pressure on the surface of the ellipsoid; $a=5 \mathrm{~cm}, b=4 \mathrm{~cm}$, $c=3 \mathrm{~cm}$.


Fig. 3. Variation of the pressure on the surface of the ellipsoid at $z=-3 \mathrm{~cm}$ with respect to $\mathrm{h} ; a=5 \mathrm{~cm}, b=4 \mathrm{~cm}, c=3 \mathrm{~cm}$.
and

$$
\operatorname{sn} F=\frac{\sqrt{a^{2}-c^{2}}}{a}, \operatorname{cn} F=\frac{c}{a}, \operatorname{dn} F=\frac{b}{a}
$$

## 3. Numerical Examples

In this section, we present some numerical examples. In Figure 1 we present the variation of $p / 2 \mu$ with respect to $z$ on the surface of the spheroid when $x=0$ with $a=b=5 \mathrm{~cm}$ and $c=3 \mathrm{~cm}$, and on Figure 2 the pressure on the surface of a triaxial ellipsoid when $a=$
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$5 \mathrm{~cm}, b=4 \mathrm{~cm}$, and $c=3 \mathrm{~cm}$ is computed. We choose $v_{0}=1 \mathrm{~cm} / \mathrm{s}$. On Figure 3 we present the variation of $p / 2 \mu$ on the surface of the ellipsoid with respect to $h$.

## 4. Conclusion

We have presented the solution of a viscous fluid flow around a triaxial ellipsoid in a circular tube. This solution agrees with the published accounts when the radius of the cylinder approaches to infinity. Judging from the results for the infinite medium, the solution appears to be correct.
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