# Near Integrability in Low Dimensional Gross-Neveu Models 

Ognyan Christov

Faculty of Mathematics and Informatics, Sofia University, 5 J. Bourchier Blvd., 1164 Sofia, Bulgaria

Reprint requests to O. C.; E-mail: christov@fmi.uni-sofia.bg
Z. Naturforsch. 66a, 468-480 (2011) / DOI: 10.5560/ZNA.2011-0002

Received November 25, 2010 / revised March 19, 2011
The low-dimensional Gross-Neveu models are studied. For the systems, related to the Lie algebras so(4), so(5), sp(4), sl(3), we prove that they have Birkhoff-Gustavson normal forms which are integrable and non-degenerate in Kolmogorov-Arnold-Moser (KAM) theory sense. Unfortunately, this is not the case for systems with three degrees of freedom, related to the Lie algebras so(6) $\sim \operatorname{sl}(4)$, $\operatorname{so}(7), \operatorname{sp}(6)$; their Birkhoff-Gustavson normal forms are proven to be non-integrable in the Liouville sense. The last result can easily be extended to higher dimensions.

Key words: Normal Forms; Kolmogorov-Arnold-Moser Theory; Non-Integrability.

## 1. Introduction and Motivation

The Gross-Neveu models are Hamiltonian systems related to the root systems of simple Lie algebras:

$$
\begin{equation*}
H=\frac{1}{2}(y, y)+\sum_{\alpha} \exp [(\alpha, x)] \tag{1}
\end{equation*}
$$

where $x=\left(x_{1}, \ldots, x_{n}\right)$ and $y=\left(y_{1}, \ldots, y_{n}\right)$ are the canonical coordinates in $\mathbb{R}^{2 n},($,$) denotes the standard$ inner product, and $\alpha$ is a root of a simple Lie algebra $\mathfrak{g}$. The sum is extended over the entire root system of $\mathfrak{g}$ or over its appropriate subspace, depending on the model.

Such models are considered by Shankar [1] in his research on the Gross-Neveu model [2] in the twodimensional field theory. As a matter of fact, the physical Gross-Neveu model describing a set of fermionic fields with the local quartic interaction is related to the Lie algebra o $(2 n)$ for small $n$, but Shankar raised the question about integrability for all simple Lie algebras. So, the Hamiltonian systems (1) are known as Gross-Neveu models. To mention only such kind of systems with exponential interactions, defined by simple Lie algebras, appeared in investigations in twodimensional classical and quantum field theories and statistical physics.

The Hamiltonian functions (1), related to the root systems of the classical simple Lie algebras so $(2 n)$, $\operatorname{so}(2 n+1), \operatorname{sl}(n+1), \operatorname{sp}(2 n)$ are of kind

$$
H_{\mathfrak{g}}=\frac{1}{2} \sum_{i=1}^{N} y_{i}^{2}+V_{\mathfrak{g}}(x)
$$

where $N=n+1$ for $\operatorname{sl}(n+1)$ and $N=n$ for the remaining algebras, and the potential $V_{\mathfrak{g}}$ has the form

$$
\begin{aligned}
& V_{\mathrm{so}(2 n)}=\sum_{i, j=1, i>j}^{N}\left(\mathrm{e}^{x_{i}+x_{j}}+\mathrm{e}^{-\left(x_{i}+x_{j}\right)}\right)+\sum_{i, j=1, i \neq j}^{N} \mathrm{e}^{x_{i}-x_{j}} \\
& V_{\mathrm{so}(2 n+1)}=\sum_{i=1}^{N}\left(\mathrm{e}^{x_{i}}+\mathrm{e}^{-x_{i}}\right) \\
& +\sum_{i, j=1, i>j}^{N}\left(\mathrm{e}^{x_{i}+x_{j}}+\mathrm{e}^{-\left(x_{i}+x_{j}\right)}\right)+\sum_{i, j=1, i \neq j}^{N} \mathrm{e}^{x_{i}-x_{j}} \\
& V_{\mathrm{sl}(n+1)}=\sum_{i, j=1, i \neq j}^{N} \mathrm{e}^{x_{i}-x_{j}} \\
& V_{\mathrm{sp}(2 n)}=\sum_{i=1}^{N}\left(\mathrm{e}^{2 x_{i}}+\mathrm{e}^{-2 x_{i}}\right) \\
& +\sum_{i, j=1, i>j}^{N}\left(\mathrm{e}^{x_{i}+x_{j}}+\mathrm{e}^{-\left(x_{i}+x_{j}\right)}\right)+\sum_{i, j=1, i \neq j}^{N} \mathrm{e}^{x_{i}-x_{j}}
\end{aligned}
$$

Except the Hamiltonian $H$, we have an obvious first integral only for the case of $\operatorname{sl}(n+1)$, namely $\sum y_{i}=$ const. Hence, the Gross-Neveu model for $\mathrm{sl}(2)$ is integrable. It turns out that the model for so(4) is also integrable. The Hamiltonian systems for the remaining cases are non-integrable, more precisely the Hamiltonian systems with two and three degrees of freedom were proven to be non-integrable by Horozov [3] with a modification of Ziglin's method [4] while the rest were proven to be non-integrable by Maciejewski et al. [5] with the differential Galois theory approach.

A motivation for this work is a series of papers of Rink [6, 7] who presented the famous Fermi-PastaUlam (FPU) system as a perturbation of one integrable and KAM non-degenerate system, namely the normal form of order four in the vicinity of an equilibrium. Non-degenerate in KAM theory sense integrable system means that its frequency map is a local diffeomorphism (see Sect. 2 for more details).

Our aim is to check whether this fact is true for the Gross-Neveu models. Unfortunately, this is not the case for the Gross-Neveu models with exceptions of the two degrees of freedom cases.

Before giving the corresponding assertions, we shall remind briefly some facts about normal forms. Consider the Hamiltonian system with a Hamiltonian $H(x, y)$. In the neighbourhood of the equilibrium $(x, y)=(0,0)$, we have the following expansion of $H$ :
$H=H_{2}+H_{3}+H_{4}+\ldots, H_{2}=\sum \omega_{j}\left(x_{j}^{2}+y_{j}^{2}\right), \omega_{j}>0$.

The frequencies $\omega_{1}, \ldots, \omega_{n}$ are said to be in resonance if there exist $k_{j} \in \mathbb{Z}, j=1, \ldots, n$ such that $\sum k_{j} \omega_{j}=0, k=\sum\left|k_{j}\right|$ being the order of resonance. With the help of a near-identity canonical transformation (in fact a series of canonical transforms), $H$ is simplified. This simplified Hamiltonian in the nonresonant case is called a Birkhoff normal form. When resonances appear, the corresponding normal form is called a Birkhoff-Gustavson normal form. To avoid the problem of convergency, one can consider a Hamiltonian system which is truncated to some order normal form,

$$
\bar{H}^{\mathrm{tr}}=H_{2}+\ldots+H_{m}
$$

It is known that the truncated form to any order Birkhoff normal form of a system without resonance is integrable [8]. The truncated Birkhoff-Gustavson normal form has at least two integrals, $H_{2}$ and $\bar{H}^{\text {tr }}$. Hence, truncated normal forms of Hamiltonian systems with two degrees of freedom are integrable. It is natural to raise the question about the integrability in truncated Birkhoff-Gustavson normal forms in more degrees of freedom. The exact integrals for the normal form $\bar{H}^{\text {tr }}$, when appear, are approximate (asymptotic) integrals for the original system, i.e. if the normal form is integrable then the original system is called near integrable. More details can be found in Verhulst [9].

Our results are presented in the following theorems.
Theorem 1. The Hamiltonian systems, corresponding to the Gross-Neveu models for algebras so(4), so(5), $\mathrm{sp}(4)$, $\mathrm{sl}(3)$ have Birkhoff-Gustavson normal forms $\bar{H}^{\mathrm{tr}}=\mathrm{H}_{2}+H_{4}$ integrable and non-degenerate in KAM theory sense.

Theorem 2. The Hamiltonian systems, corresponding to the Gross-Neveu models for $\mathrm{so}(6) \sim \mathrm{sl}(4)$, $\mathrm{so}(7)$, $\mathrm{sp}(6)$ have non-integrable Birkhoff-Gustavson normal forms $\bar{H}^{\text {tr }}=H_{2}+H_{4}$.

One should note that the results are not surprising. The Hamiltonian systems for Gross-Neveu models enjoy $1: 1: \ldots: 1$ resonance, as well as many symmetries. Due to these symmetries there are no third-order resonant terms in the truncated form up to order four Hamiltonians. In the two degrees of freedom cases, where these truncations are integrable, it is natural to expect non-degeneracy. From the other side, apparently this resonance and the symmetries are not sufficient to assure integrability in the systems with more degrees of freedom as in the case of FPU chains, which is indeed very rare.

The paper is organized as follows. In Section 2 we briefly recall some definitions and results on integrability of real and complex Hamiltonian systems. The proof of Theorem 1 is presented in Section 3. The systems are naturally divided in two parts. In the first part the integrals are quadratic and in the second part integrals are quartic. Thus, we need different approaches. In Section 4 we prove Theorem 2. The proof is based on Morales-Ramis theory using Differential Galois groups of the linearized system along a particular solution. In fact, we explore only the monodromy group and prove that it is non-Abelian. As it is contained in the differential Galois group, the result follows from Morales-Ramis theorem. We study the so(6) model in details and give the main points for the other cases $\mathrm{so}(7)$ and $\mathrm{sp}(6)$.

## 2. Theory

In this section we summarize briefly some results on integrability of Hamiltonian systems in real and complex domains.

First, we consider the real case. Let $\left(M^{2 n}, \omega\right)$ be a $2 n$-dimensional symplectic manifold and let $H$ be a Hamiltonian function on $M^{2 n}$ defining the corre-
sponding Hamiltonian system

$$
\begin{equation*}
\dot{x}=X_{H}(x) . \tag{3}
\end{equation*}
$$

An Hamiltonian system is integrable if there exist $n$ independent integrals $F_{1}=H, F_{2}, \ldots, F_{n}$ in involution, namely $\left\{F_{i}, F_{j}\right\}=0$ for all $i$ and $j$, where $\{$,$\} is$ the Poison bracket [8]. On a neighbourhood $U$ of the connected compact level sets of the integrals $M_{c}=$ $\left\{F_{j}=c_{j}, j=1, \ldots, n\right\}$ by Liouville-Arnold theorem one can introduce a special set of symplectic coordinates, $I_{j}, \varphi_{j}$, called action-angle variables. Then, the integrals $F_{1}=H, F_{2}, \ldots, F_{n}$ are functions of action variables only and the flow of $X_{\mathrm{H}}$ is described by the canonical equations

$$
\begin{equation*}
\dot{I}_{j}=0, \quad \dot{\varphi}_{j}=\frac{\partial H}{\partial I_{j}}, \quad j=1, \ldots, n . \tag{4}
\end{equation*}
$$

Therefore, near $M_{c}$, the phase space is foliated with $X_{F_{i}}$ invariant tori over which the flow of $X_{\mathrm{H}}$ is quasi-periodic with frequencies $\left(\omega_{1}(I), \ldots, \omega_{n}(I)\right)=$ $\left(\frac{\partial H}{\partial I_{1}}, \ldots, \frac{\partial H}{\partial I_{n}}\right)$.

The map

$$
\begin{equation*}
\left(I_{1}, I_{2}, \ldots, I_{n}\right) \rightarrow\left(\frac{\partial H}{\partial I_{1}}, \frac{\partial H}{\partial I_{2}}, \ldots, \frac{\partial H}{\partial I_{n}}\right) \tag{5}
\end{equation*}
$$

is called frequency map.
Now, consider a small perturbation of an integrable Hamiltonian $H_{0}$

$$
H=H_{0}(I)+\varepsilon H_{1}(I, \varphi), \varepsilon \ll 1
$$

A natural question is whether this small perturbation destroy the quasi-periodic motions of the unperturbed system. KAM-theory [10-12] gives conditions for the integrable system $H_{0}$ which ensures the survival of the most of the invariant tori. One condition, usually called Kolmogorov's condition, is that the frequency map should be a local diffeomorphism, that is

$$
\begin{equation*}
\operatorname{det}\left(\frac{\partial^{2} H_{0}}{\partial I_{i} \partial I_{j}}\right) \neq 0 \tag{6}
\end{equation*}
$$

on an open and dense subset of $U$. We should note that the measure of the surviving tori decreases with the increase of both perturbation and the measure of the set where above Hessian is too close to zero.

Another condition of this type is the so called Arnold-Moser condition of isoenergetical non-degen-
eracy. Let us fix an energy level $H_{0}=h_{0}$. Define the following map:

$$
\begin{equation*}
F_{h_{0}}: I \rightarrow\left(\omega_{1}(I): \omega_{2}(I): \ldots: \omega_{n}(I)\right) \tag{7}
\end{equation*}
$$

forming the $(n-1)$-dimensional variety $H_{0}^{-1}\left(h_{0}\right)$ into the projective space $\mathbb{P}^{n-1}$. Then the system is isoenergetically non-degenerate if the map $F_{h_{0}}$ is a homeomorphism. Analytically this is equivalent to non-vanishing of the following determinant:

$$
D_{1}=\left(\begin{array}{cc}
\partial^{2} H_{0} / \partial I^{2} & \partial H_{0} / \partial I  \tag{8}\\
\partial H_{0} / \partial I & 0
\end{array}\right)
$$

Of course, again the measure of the surviving tori depends on the measure of the set where the determinant $D_{1}$ is too close to zero.

Before considering integrability in the complex setting, let us recall the notion of monodromy. Given a linear non-autonomous system

$$
\dot{x}=A(t) x, x \in \mathbb{C}^{n}
$$

with $t$ defined on some Riemann surface $\Gamma$. Continuation of the solutions along non-trivial loops on $\Gamma$ defines a linear authomorphism of the space of solutions, called the monodromy transformation. Analytically, this transformation can be presented in the following way: Let $X(t)$ be a fundamental matrix solution. The linear authomorphism $\Delta_{\gamma}$ associated with a loop $\gamma \in \pi_{1}\left(\Gamma, t_{0}\right)$ corresponds to multiplication of $X(t)$ from the right by a constant matrix $M_{\gamma}$, called monodromy matrix,

$$
\Delta_{\gamma} X(t)=X(t) M_{\gamma}
$$

The set of all these matrices form the monodromy group $\mathcal{M}$.

Now, let us consider a complex analytic symplectic manifold $\left(M^{2 n}, \omega\right)$ and a holomorphic Hamiltonian system $X_{\mathrm{H}}$ on it. Again we call such Hamiltonian system integrable in Liouville sense if there exist $n$ independent first integrals $F_{1}=H, F_{2}, \ldots, F_{n}$ in involution. It is essential to have necessary conditions for integrability or, equivalently, sufficient conditions for nonintegrability.

There are only few methods for proving nonintegrability, mainly based on a linearization of the considered system around a particular solution. Let $z=z(t)$ be a solution (not equilibrium) of the Hamiltonian system and let $\Gamma:=\{z=z(t)\}$ be its integral
curve. The variational equations (VE) corresponding to $z=z(t)$ are

$$
\dot{\eta}=\frac{\partial X_{\mathrm{H}}}{\partial x}(z(t)) \eta .
$$

Reducing VE by the first integral $\mathrm{d} H$, we get the so called normal variational equations (NVE)

$$
\begin{equation*}
\dot{\xi}=A(t) \xi \text { with dimension } 2(n-1) \tag{9}
\end{equation*}
$$

In 1982 Ziglin proved the following result for integrability of a complex-analytical Hamiltonian systems:

Theorem 3. ([4]) Suppose that a Hamiltonian system has $n$ first integrals, independent around $\Gamma$, but not necessary on $\Gamma$. Suppose that there is a non-resonant element $g$ in the monodromy group of NVE. Then every other element $g^{\prime}$ of the monodromy group transforms the set of eigendirections of $g$ into itself.

Let us remind that $g \in \operatorname{Sp}(2 n, \mathbb{C})$ (the monodromy group is a subgroup of the symplectic group) is a resonant if $l_{1}^{r_{1}} \ldots l_{n}^{r_{n}}=1$, where $r_{i}$ are non-zero integers and $l_{i}$ are the eigenvalues of $g$.

Another method for proving non-integrability is based on the differential Galois theory. The solutions of (9) define an extension $L_{1}$ of the coefficient field $L$ of NVE. This naturally defines a differential Galois group $G=\operatorname{Gal}\left(L_{1} / L\right)$. Then the following result is obtained:

Theorem 4. (Morales-Ramis [13]) Suppose that a Hamiltonian system has $n$ meromorphic first integrals in involution. Then the identity component $G^{0}$ of the Galois group $G=\operatorname{Gal}\left(L_{1} / L\right)$ is Abelian.

If once it is proven that $G^{0}$ is not Abelian, then the respective Hamiltonian system is non-integrable in the Liouville sense. However, the fact that $G^{0}$ is Abelian doesn't imply integrability. For more detailed description on differential Galois theory, as well as additional facts and technical details, see [13, 14]. One should note that by its definition the monodromy group is contained in the differential Galois group of the corresponding linear system. We will use only monodromy here.

## 3. Proof of Theorem 1

In this section, we consider the Gross-Neveu models related to Lie algebras so(4), $\mathrm{so}(5), \mathrm{sp}(4), \mathrm{sl}(3)$
referred to as low-dimensional Gross-Neveu models. They correspond to some two degrees of freedom Hamiltonian systems ( $\mathrm{sl}(3)$ after reduction). These systems near the origin can be considered as perturbations of their normal forms which are integrable and KAM-non-degenerate. These systems naturally fall in two subclasses.

For the cases $\operatorname{sl}(3)$ and $\operatorname{so}(4)$ the second integral is quadratic. This fact allows us to construct action-angle variables explicitly following [7]. Hence, the corresponding Hamiltonians of the normal forms are easily expressed via action variables, which makes the verification of Kolmogorov's condition straightforward.

For the cases so(5) and $\operatorname{sp}(4)$ the second integral is quartic. The expressions of the corresponding Hamiltonians of the normal forms via action variables are not explicit. So, we adopt Horozov's approach [15] for verification of Kolmogorov's condition in these cases. We consider $\mathrm{sl}(3)$ and so(5) in details and give the key points for the other cases.

## 3.1. $\mathrm{sl}(3)$

The Gross-Neveu model related with $\operatorname{sl}(3)$ is actually a three degrees of freedom system described with the Hamiltonian

$$
\begin{align*}
H= & \frac{1}{2} \sum_{j=1}^{3} y_{j}^{2}+\left(\mathrm{e}^{x_{1}-x_{2}}+\mathrm{e}^{-x_{1}+x_{2}}\right)+\left(\mathrm{e}^{x_{2}-x_{3}}+\mathrm{e}^{-x_{2}+x_{3}}\right) \\
& +\left(\mathrm{e}^{x_{1}-x_{3}}+\mathrm{e}^{-x_{1}+x_{3}}\right) \tag{10}
\end{align*}
$$

It is easy to check that the total momentum $y_{1}+y_{2}+$ $y_{3}$ is conserved. Hence, the motion of the mass center $\frac{1}{3} \sum_{j=1}^{3} x_{j}$ is linear and therefore unbounded.

In order to follow our aim, we reduce the Hamiltonian (10) to one with two degrees of freedom with the help of the above integral. Let us perform the following canonical transformation:
$Q_{1}=x_{1}-x_{2}, Q_{2}=x_{2}-x_{3}, Q_{3}=x_{1}+x_{2}+x_{3}$,
$P_{1}=\frac{1}{3}\left(2 y_{1}-y_{2}-y_{3}\right), \quad P_{2}=\frac{1}{3}\left(y_{1}+y_{2}-2 y_{3}\right)$,
$P_{3}=\frac{1}{3}\left(y_{1}+y_{2}+y_{3}\right)$.
In these coordinates (10) reads

$$
\begin{aligned}
H= & P_{1}^{2}-P_{1} P_{2}+P_{2}^{2}+\frac{3}{2} P_{3}^{2}+\left(\mathrm{e}^{Q_{1}}+\mathrm{e}^{-Q_{1}}\right) \\
& +\left(\mathrm{e}^{Q_{2}}+\mathrm{e}^{-Q_{2}}\right)+\left(\mathrm{e}^{Q_{1}+Q_{2}}+\mathrm{e}^{-\left(Q_{1}+Q_{2}\right)}\right) .
\end{aligned}
$$

Hence, $Q_{3}$ is cyclic and $P_{3}$ is an integral. We leave out $P_{3}$ and denote by $H_{\mathrm{R}}$ the reduced Hamiltonian

$$
\begin{align*}
H_{\mathrm{R}}= & P_{1}^{2}-P_{1} P_{2}+P_{2}^{2}+\left(\mathrm{e}^{Q_{1}}+\mathrm{e}^{-Q_{1}}\right)+\left(\mathrm{e}^{Q_{2}}+\mathrm{e}^{-Q_{2}}\right) \\
& +\left(\mathrm{e}^{Q_{1}+Q_{2}}+\mathrm{e}^{-\left(Q_{1}+Q_{2}\right)}\right) . \tag{11}
\end{align*}
$$

The corresponding equations of motion are

$$
\begin{aligned}
\dot{Q}_{1}=2 P_{1}-P_{2}, \dot{P}_{1}= & -\left[\left(\mathrm{e}^{Q_{1}}-\mathrm{e}^{-Q_{1}}\right)\right. \\
& \left.+\left(\mathrm{e}^{Q_{1}+Q_{2}}-\mathrm{e}^{-\left(Q_{1}+Q_{2}\right)}\right)\right], \\
\dot{Q}_{2}=2 P_{2}-P_{1}, \dot{P}_{2}= & -\left[\left(\mathrm{e}^{Q_{2}}-\mathrm{e}^{-Q_{2}}\right)\right. \\
& \left.+\left(\mathrm{e}^{Q_{1}+Q_{2}}-\mathrm{e}^{-\left(Q_{1}+Q_{2}\right)}\right)\right] .
\end{aligned}
$$

The point $(Q, P)=(0,0)$ is an equilibrium point. By linearization about that point one gets

$$
\begin{aligned}
& \dot{\xi}_{1}=2 \eta_{1}-\eta_{2}, \quad \dot{\eta}_{1}=-4 \xi_{1}-2 \xi_{2} \\
& \dot{\xi}_{2}=2 \eta_{2}-\eta_{1}, \quad \dot{\eta}_{2}=-2 \xi_{1}-4 \xi_{2}
\end{aligned}
$$

The eigenvalues of the linearized system are $\pm \mathrm{i} \sqrt{6}$, $\pm \mathrm{i} \sqrt{6}$ that is $1: 1$ resonance. Expanding $H_{\mathrm{R}}$ about $(Q, P)=(0,0)$ and neglecting irrelevant additive constant, we obtain

$$
\begin{aligned}
H_{\mathrm{R}}= & P_{1}^{2}-P_{1} P_{2}+P_{2}^{2}+2\left(Q_{1}^{2}+Q_{2}^{2}+Q_{1} Q_{2}\right) \\
& +\frac{1}{12}\left[Q_{1}^{4}+Q_{2}^{4}+\left(Q_{1}+Q_{2}\right)^{4}\right]+O\left(\|Q\|^{6}\right) .
\end{aligned}
$$

First, we diagonalize the quadratic part of the above Hamiltonian via coordinate change,
$\binom{Q_{1}}{Q_{2}}=\left(\begin{array}{cc}\frac{\sqrt{3}}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{\sqrt{3}}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{array}\right)\binom{q_{1}}{q_{2}}$,
$\binom{P_{1}}{P_{2}}=\left(\begin{array}{cc}\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}}\end{array}\right)\binom{p_{1}}{p_{2}}$,
and then scale $p_{j} \rightarrow \sqrt[4]{6} p_{j}, q_{j} \rightarrow q_{j} / \sqrt[4]{6}$ to obtain

$$
\begin{aligned}
H_{\mathrm{R}}= & \frac{\sqrt{6}}{2}\left(p_{1}^{2}+p_{2}^{2}+q_{1}^{2}+q_{2}^{2}\right)+\frac{1}{16}\left(q_{1}^{2}+q_{2}^{2}\right)^{2} \\
& +O\left(\|q\|^{6}\right) .
\end{aligned}
$$

Next, we put
$q_{j}=\frac{1}{2}\left(z_{j}+w_{j}\right), p_{j}=\frac{1}{2 i}\left(z_{j}-w_{j}\right), j=1,2$.
The resonant terms of order four are $z_{1}^{2} w_{1}^{2}, z_{2}^{2} w_{2}^{2}, z_{1}^{2} w_{2}^{2}$, $z_{2}^{2} w_{1}^{2}, z_{1} w_{1} z_{2} w_{2}$. Since we are interested in the normal
form truncated up to order four, we just remove the non-resonant terms and get

$$
\begin{align*}
\bar{H}_{\mathrm{R}}^{\mathrm{tr}}= & \frac{\sqrt{6}}{2}\left(z_{1} w_{1}+z_{2} w_{2}\right)+2^{-7}\left[3\left(z_{1} w_{1}+z_{2} w_{2}\right)^{2}\right.  \tag{13}\\
& \left.+\left(z_{1} w_{2}-z_{2} w_{1}\right)^{2}\right] .
\end{align*}
$$

It was mentioned earlier that the truncated normal form has two integrals $H_{2}=z_{1} w_{1}+z_{2} w_{2}$ and $\bar{H}_{\mathrm{R}}^{\text {tr }}$ or equivalently here $H_{2}$ and $B B=z_{1} w_{2}-z_{2} w_{1}$.

The Hamiltonian $\bar{H}_{\mathrm{R}}^{\mathrm{tr}}(13)$ of the truncated up to order four normal form and the quadratic integrals in cartesian coordinates $(q, p)$ take the form
$\bar{H}_{\mathrm{R}}^{\mathrm{tr}}=\frac{\sqrt{6}}{2}\left(p_{1}^{2}+p_{2}^{2}+q_{1}^{2}+q_{2}^{2}\right)$
$+\frac{1}{2^{7}}\left[3\left(p_{1}^{2}+p_{2}^{2}+q_{1}^{2}+q_{2}^{2}\right)^{2}-4\left(p_{1} q_{2}-q_{1} p_{2}\right)^{2}\right]$,
$a=\frac{1}{2}\left(p_{1}^{2}+p_{2}^{2}+q_{1}^{2}+q_{2}^{2}\right), \quad b=p_{1} q_{2}-q_{1} p_{2}$.
In order to introduce action variables, we need to find the set of regular values of the energy momentum map

$$
E M:\left(p_{1}, p_{2}, q_{1}, q_{2}\right) \rightarrow(a, b)
$$

This is already done in [7]. Denote by $U_{r}=\{(a, b) \in$ $\left.\mathbb{R}^{2}, a>0,|b|<a\right\}$. Then for all $(a, b) \in U_{r}$, the level sets of $E M^{-1}(a, b)$ are diffeomorphic to two-tori.

Let arg : $\mathbb{R}^{2} \backslash\{(0,0)\} \rightarrow \mathbb{R} / 2 \pi \mathbb{Z}$ be the argument function $\arg (r \cos \Phi, r \sin \Phi) \rightarrow \Phi$. Define the following set of variables $(a, b, \Phi, \Psi), a, b$ as above and

$$
\begin{aligned}
\Phi= & \frac{1}{2} \arg \left(p_{2}-q_{1}, p_{1}+q_{2}\right) \\
& +\frac{1}{2} \arg \left(-p_{2}-q_{1}, p_{1}-q_{2}\right), \\
\Psi= & \frac{1}{2} \arg \left(p_{2}-q_{1}, p_{1}+q_{2}\right) \\
& -\frac{1}{2} \arg \left(-p_{2}-q_{1}, p_{1}-q_{2}\right) .
\end{aligned}
$$

These functions are well defined since $(a, b) \in U_{r}$. With the formula $\operatorname{darg}(x, y)=\frac{x \mathrm{~d} y-y \mathrm{~d} x}{x^{2}+y^{2}}$ one can verify that the set $(a, b, \Phi, \Psi)$ are canonical coordinates, actually action-angle coordinates

$$
\mathrm{d} p_{1} \wedge \mathrm{~d} q_{1}+\mathrm{d} p_{2} \wedge \mathrm{~d} q_{2}=\mathrm{d} a \wedge \mathrm{~d} \Phi+\mathrm{d} b \wedge \mathrm{~d} \Psi .
$$

The truncated Hamiltonian $\bar{H}_{\mathrm{R}}^{\mathrm{tr}}$ is a function of actions $a, b$

$$
\begin{equation*}
\bar{H}_{\mathrm{R}}^{\mathrm{tr}}=\sqrt{6} a+2^{-5}\left(3 a^{2}-b^{2}\right) . \tag{14}
\end{equation*}
$$

Now, the non-degeneracy is straightforward

$$
\operatorname{det}\left(\frac{\partial^{2} \bar{H}_{\mathrm{R}}^{\mathrm{tr}}}{\partial a \partial b}\right)=\operatorname{det}\left(\begin{array}{cc}
\frac{6}{2^{5}} & 0 \\
0 & -\frac{2}{2^{5}}
\end{array}\right)=-3.2^{-8}<0 .
$$

3.2. $\operatorname{so}(4)$

The Gross-Neveu model related with so(4) is a two degrees of freedom system described with the Hamiltonian
$H=\frac{1}{2}\left(y_{1}^{2}+y_{2}^{2}\right)+\mathrm{e}^{x_{1}+x_{2}}+\mathrm{e}^{-x_{1}-x_{2}}$

$$
\begin{equation*}
+\mathrm{e}^{x_{2}-x_{1}}+\mathrm{e}^{x_{1}-x_{2}}, \tag{15}
\end{equation*}
$$

which is integrable. The second integral is $B=$ $(1 / 2)\left(y_{1}+y_{2}\right)^{2}+2\left(\exp \left(x_{1}+x_{2}\right)+\exp \left(-x_{1}-x_{2}\right)\right)$. Nevertheless, we concentrate our attention on the truncated normal form.

The corresponding equations of motion are
$\dot{x}_{1}=y_{1}, \dot{y}_{1}=-\left(\mathrm{e}^{x_{1}+x_{2}}-\mathrm{e}^{-x_{1}-x_{2}}-\mathrm{e}^{-x_{1}+x_{2}}+\mathrm{e}^{x_{1}-x_{2}}\right)$,
$\dot{x}_{2}=y_{2}, \dot{y}_{2}=-\left(\mathrm{e}^{x_{1}+x_{2}}-\mathrm{e}^{-x_{1}-x_{2}}+\mathrm{e}^{-x_{1}+x_{2}}-\mathrm{e}^{x_{1}-x_{2}}\right)$.
The point $(x, y)=(0,0)$ is an equilibrium point. By linearization about that point one gets

$$
\dot{\xi}_{i}=\eta_{i}, \quad \dot{\eta}_{i}=-4 \xi_{i}, \quad i=1,2
$$

The eigenvalues of the linearized system are $\pm \mathrm{i} 2, \pm \mathrm{i} 2$ that is $1: 1$ resonance. Expanding $H$ about $(x, y)=$ $(0,0)$ and neglecting irrelevant additive constant, we obtain

$$
\begin{aligned}
H= & \frac{1}{2}\left(y_{1}^{2}+y_{2}^{2}\right)+2\left(x_{1}^{2}+x_{2}^{2}\right)+\frac{1}{6}\left(x_{1}^{4}+x_{2}^{4}+6 x_{1}^{2} x_{2}^{2}\right) \\
& +O\left(\|x\|^{6}\right) .
\end{aligned}
$$

Further, we perform a canonical change of variables $y_{j}=\sqrt{2} p_{j}, x_{j}=q_{j} / \sqrt{2}, j=1,2$ to obtain
$\begin{aligned} H= & p_{1}^{2}+p_{2}^{2}+q_{1}^{2}+q_{2}^{2}+\frac{1}{24}\left(q_{1}^{4}+q_{2}^{4}+6 q_{1}^{2} q_{2}^{2}\right) \\ & +O\left(\|q\|^{6}\right)\end{aligned}$

$$
+O\left(\|q\|^{6}\right)
$$

Next, we put as usual (12). The resonant terms of order four are already known from the previous subsection. Since we are interested in the normal form truncated up to order four, we just remove the non-resonant terms and get

$$
\begin{align*}
\bar{H}^{\mathrm{tr}}= & z_{1} w_{1}+z_{2} w_{2}+2^{-6}\left[\left(z_{1} w_{1}+z_{2} w_{2}\right)^{2}\right. \\
& \left.+\left(z_{1} w_{2}+z_{2} w_{1}\right)^{2}\right] . \tag{16}
\end{align*}
$$

It was mentioned earlier that the truncated normal form has two integrals $H_{2}=z_{1} w_{1}+z_{2} w_{2}$ and $\bar{H}^{\text {tr }}$ or equivalently here $H_{2}$ and $B B=z_{1} w_{2}+z_{2} w_{1}$.

The Hamiltonian $\bar{H}^{\text {tr }}(16)$ of the truncated up to order four normal form and the quadratic integrals in cartesian coordinates $(q, p)$ take the form
$\begin{aligned} \bar{H}^{\mathrm{tr}}= & p_{1}^{2}+p_{2}^{2}+q_{1}^{2}+q_{2}^{2}+\frac{1}{2^{6}}\left[\left(p_{1}^{2}+p_{2}^{2}+q_{1}^{2}+q_{2}^{2}\right)^{2}\right. \\ & \left.+4\left(p_{1} p_{2}+q_{1} q_{2}\right)^{2}\right],\end{aligned}$
$a=\frac{1}{2}\left(p_{1}^{2}+p_{2}^{2}+q_{1}^{2}+q_{2}^{2}\right), \quad b=p_{1} p_{2}+q_{1} q_{2}$.
As before for $(a, b) \in U_{r}=\left\{(a, b) \in \mathbb{R}^{2}, a>0,|b|<a\right\}$ the level set of integrals is a torus and the following functions are well defined:

$$
\begin{aligned}
\Phi= & -\frac{1}{2} \arg \left(q_{1}-q_{2}, p_{1}-p_{2}\right) \\
& -\frac{1}{2} \arg \left(q_{1}+q_{2}, p_{1}+p_{2}\right) \\
\Psi= & \frac{1}{2} \arg \left(q_{1}-q_{2}, p_{1}-p_{2}\right) \\
& -\frac{1}{2} \arg \left(q_{1}+q_{2}, p_{1}+p_{2}\right) .
\end{aligned}
$$

One can verify that the set $(a, b, \Phi, \Psi)$ are canonical coordinates, actually action-angle coordinates. The truncated Hamiltonian $\bar{H}^{\text {tr }}$ is a function of actions $a, b$

$$
\begin{equation*}
\bar{H}^{\mathrm{tr}}=2 a+2^{-4}\left(a^{2}+b^{2}\right) \tag{17}
\end{equation*}
$$

Now, the non-degeneracy is immediate.

## 3.3. so(5)

The Gross-Neveu model related with so(5) is a two degrees of freedom system described with the Hamiltonian

$$
\begin{align*}
H= & \frac{1}{2}\left(y_{1}^{2}+y_{2}^{2}\right)+\mathrm{e}^{x_{1}}+\mathrm{e}^{-x_{1}}+\mathrm{e}^{x_{2}}+\mathrm{e}^{-x_{2}}+\mathrm{e}^{x_{1}-x_{2}} \\
& +\mathrm{e}^{-\left(x_{1}-x_{2}\right)}+\mathrm{e}^{x_{1}+x_{2}}+\mathrm{e}^{-\left(x_{1}+x_{2}\right)} . \tag{18}
\end{align*}
$$

The corresponding equations of motion are

$$
\begin{aligned}
\dot{x}_{1}=y_{1}, \quad \dot{y}_{1}= & -\left(\mathrm{e}^{x_{1}}-\mathrm{e}^{-x_{1}}+\mathrm{e}^{x_{1}-x_{2}}-\mathrm{e}^{-\left(x_{1}-x_{2}\right)}\right. \\
& \left.+\mathrm{e}^{x_{1}+x_{2}}-\mathrm{e}^{-\left(x_{1}+x_{2}\right)}\right) \\
\dot{x}_{2}=y_{2}, \quad \dot{y}_{2}= & -\left(\mathrm{e}^{x_{2}}-\mathrm{e}^{-x_{2}}-\mathrm{e}^{x_{1}-x_{2}}+\mathrm{e}^{-\left(x_{1}-x_{2}\right)}\right. \\
& \left.+\mathrm{e}^{x_{1}+x_{2}}-\mathrm{e}^{-\left(x_{1}+x_{2}\right)}\right) .
\end{aligned}
$$

Recall that this system is not integrable [3,5]. Clearly, $(0,0)$ is an equilibrium point. The eigenvalues of the linearized system are $\pm \mathrm{i} \sqrt{6}, \pm \mathrm{i} \sqrt{6}$, that is $1: 1$ res-
onance. Expanded around ( 0,0 ), the Hamiltonian (18) up to irrelevant constant reads

$$
\begin{aligned}
H= & \frac{1}{2}\left(y_{1}^{2}+y_{2}^{2}\right)+3\left(x_{1}^{2}+x_{2}^{2}\right)+\frac{1}{4}\left(x_{1}^{4}+x_{2}^{4}+4 x_{1}^{2} x_{2}^{2}\right) \\
& +O\left(\|x\|^{6}\right)
\end{aligned}
$$

Next, we scale $x_{j}=q_{j} / \sqrt[4]{6}, y_{j}=p_{j} \sqrt[4]{6}$, put as usual (12), remove the non-resonant terms and get the normal form up to order four:
$\bar{H}^{\mathrm{tr}}=\frac{\sqrt{6}}{2}\left(z_{1} w_{1}+z_{2} w_{2}\right)+\frac{1}{3.2^{6}}\left[3\left(z_{1} w_{1}+z_{2} w_{2}\right)^{2}\right.$
$\left.+\frac{3}{2}\left(z_{1} w_{2}+z_{2} w_{1}\right)^{2}+\frac{1}{2}\left(z_{1} w_{2}-w_{1} z_{2}\right)^{2}\right]$.
As we know, the truncated normal form is integrable and the two integrals are $H_{2}$ and $\bar{H}^{\text {tr }}$ or equivalently $H_{2}$ and $B B=3\left(z_{1} w_{2}+z_{2} w_{1}\right)^{2}+\left(z_{1} w_{2}-w_{1} z_{2}\right)^{2}$.

Next, we put

$$
\begin{equation*}
z_{j}=\sqrt{2 a_{j}} \mathrm{e}^{-\mathrm{i} \psi_{j}}, w_{j}=\sqrt{2 a_{j}} \mathrm{e}^{\mathrm{i} \psi_{j}} \tag{20}
\end{equation*}
$$

and after that, we perform the following canonical change of variables:
$J_{1}=\frac{a_{1}+a_{2}}{2}, J_{2}=\frac{a_{1}-a_{2}}{2}$,
$\chi_{1}=\psi_{1}+\psi_{2}, \chi_{2}=\psi_{1}-\psi_{2}$
to obtain

$$
\begin{align*}
\bar{H}^{\mathrm{tr}}= & 2 \sqrt{6} J_{1} \\
& +\frac{1}{24}\left[6 J_{1}^{2}+\left(J_{1}^{2}-J_{2}^{2}\right)\left(2 \cos \left(2 \chi_{2}\right)+1\right)\right] . \tag{22}
\end{align*}
$$

So, $\chi_{1}$ is a cyclic variable and $J_{1}$ is a first integral. Note that in these coordinates, the symplectic form is the exact two-form $\mathrm{d} \sigma$, where

$$
\begin{equation*}
\sigma=J_{1} \mathrm{~d} \chi_{1}+J_{2} \mathrm{~d} \chi_{2} \tag{23}
\end{equation*}
$$

In order to get rid of the linear term in $\hat{H}^{\text {tr }}$, we continue with the canonical transformation

$$
J_{j} \rightarrow J_{j}^{\prime}, \chi_{1} \rightarrow \chi_{1}^{\prime}+2 \sqrt{6} t, \chi_{2} \rightarrow \chi_{2}^{\prime}, \bar{H}^{\mathrm{tr}} \rightarrow \bar{H}^{\prime \mathrm{tr}}
$$

To simplify the notations, we drop the primes and the multiplier $1 / 24$ and reach the Hamiltonian, we will work with, as

$$
\begin{equation*}
\bar{H}^{\mathrm{tr}}=6 J_{1}^{2}+\left(J_{1}^{2}-J_{2}^{2}\right)\left(2 \cos \left(2 \chi_{2}\right)+1\right) \tag{24}
\end{equation*}
$$

which admits the integrals $\bar{H}^{\text {tr }}=h$ and $F=J_{1}=f \geq 0$.

In order to construct the action variables, we need to find the set of regular values of the energy momentum map

$$
E M:\left(J_{1}, J_{2}, \chi_{1}, \chi_{2}\right) \rightarrow\left(\bar{H}^{\mathrm{tr}}, F\right)
$$

These turn out to be

$$
\begin{equation*}
U_{r}=U_{r 1} \cup U_{r 2} \tag{25}
\end{equation*}
$$

where $U_{r 1}=\left\{(h, f) \in \mathbb{R}^{2}, f>0,6 f^{2}>h>5 f^{2}\right\}$ and $U_{r 2}=\left\{(h, f) \in \mathbb{R}^{2}, f>0,9 f^{2}>h>6 f^{2}\right\}$. Moreover, for each $(h, f) \in U_{r}$, the level set $E M^{-1}(h, f)$ is a twotorus $T_{h, f}$.

Choose a basis $\gamma_{1}, \gamma_{2}$ of the homology group $H_{1}\left(T_{h, f}, \mathbb{Z}\right)$ with the following representatives. For $\gamma_{1}$ we take the circle on $T_{h, f}$ defined by fixing $\chi_{2}, J_{1}$ and $J_{2}$ and letting $\chi_{1}$ run through $[0,2 \pi]$. For $\gamma_{2}$ we fix $\chi_{1}$ and let $J_{2}, \chi_{2}$ make one circle on the curve given by the equation

$$
6 f^{2}+\left(f^{2}-J_{2}^{2}\right)\left(2 \cos 2 \chi_{2}+1\right)=h
$$

The corresponding action variables $I_{j}=\int_{\gamma_{j}} \sigma$, where $\sigma$ is the one-form (23), have the following form:
$I_{1}=2 \pi f$,
$I_{2}=2 \int_{\chi_{2}^{-}}^{\chi_{2}^{+}} \sqrt{\frac{f^{2}\left(2 \cos 2 \chi_{2}+7\right)-h}{2 \cos 2 \chi_{2}+1}} \mathrm{~d} \chi_{2}$,
where $\chi_{2}^{-}<\chi_{2}^{+}$are the two roots of the equation

$$
f^{2}-\frac{h-6 f^{2}}{2 \cos 2 \chi_{2}+1}=0 \text { in }(0, \pi)
$$

Put $z=\cos 2 \chi_{2},|z| \leq 1, y^{2}=(2 z+1)\left(1-z^{2}\right)$ $\left(f^{2}(2 z+7)-h\right)$ and denote by $\gamma$ an oval of the curve

$$
\begin{aligned}
\Gamma_{h, f}= & \left\{(y, z) \in \mathbb{C}^{2}: y^{2}=(2 z+1)\left(1-z^{2}\right)\right. \\
& \left.\cdot\left(f^{2}(2 z+7)-h\right)\right\}
\end{aligned}
$$

Then we have

$$
\begin{equation*}
\psi(h, f) \stackrel{\text { def }}{=} I_{2}=\int_{\gamma} \frac{y \mathrm{~d} z}{(2 z+1)\left(1-z^{2}\right)} . \tag{27}
\end{equation*}
$$

Denote by $H\left(I_{1}, I_{2}\right)$ the Hamiltonian of the truncated normal form (24) expressed in action variables. Earlier in [15] it was proven that
$(2 \pi)^{2}\left(\psi_{h}\right)^{4} \operatorname{det}\left(\frac{\partial^{2} H}{\partial I_{i} \partial I_{j}}\right)=\operatorname{det}\left(\begin{array}{ll}\psi_{h h} & \psi_{h f} \\ \psi_{f h} & \psi_{f f}\end{array}\right)$.

Since

$$
\psi_{h}=-\frac{1}{2} \int_{\gamma} \frac{\mathrm{d} z}{y} \neq 0 \text { in } U_{r}
$$

one can see that Kolmogorov's condition is equivalent to the condition that $D=\psi_{h h} \psi_{f f}-\left(\psi_{h f}\right)^{2} \neq 0$. In the following we will express $D$ in terms of Abelian integrals. Since we can homotope the curve $\gamma$ to another without changing $\psi$, it follows that we can take partial derivatives under the integral sign. Denoting by $E$ the integral $E=\int_{\gamma} \frac{(2 z+1)(2 z+7)\left(1-z^{2}\right)}{y^{3}} \mathrm{~d} z$ we get successively the following expressions for the derivatives of $\psi$ :
$\psi_{h h}=-\frac{1}{4} \int_{\gamma} \frac{(2 z+1)\left(1-z^{2}\right)}{y^{3}} \mathrm{~d} z, \quad \psi_{h f}=\frac{f}{2} E$,
$\psi_{f f}=-h E$.
From here $D$ becomes
$D=\frac{1}{4} E$
$\cdot \int_{\gamma} \frac{h(2 z+1)\left(1-z^{2}\right)-f^{2}(2 z+1)(2 z+7)\left(1-z^{2}\right)}{y^{3}} \mathrm{~d} z$
$=\frac{1}{2} \psi_{h} E$,
that is, $D \neq 0 \leftrightarrow E \neq 0$ in $U_{r}$. Note that

$$
\begin{equation*}
E=\frac{2}{f} \psi_{h f}=\frac{2}{f} \frac{\partial}{\partial f} \psi_{h}=-\frac{1}{f} \frac{\partial}{\partial f}\left(\int_{\gamma} \frac{\mathrm{d} z}{y}\right) . \tag{29}
\end{equation*}
$$

To show that $E \neq 0$, we first consider $(h, f) \in U_{r 1}$. Then $z_{3}=\frac{1}{2}\left(\frac{h}{f^{2}}-7\right) \in(-1,-1 / 2)$ and
$\int_{\gamma} \frac{\mathrm{d} z}{y}=2 \int_{-1}^{z_{3}} \frac{\mathrm{~d} z}{\sqrt{4 f^{2}(z+1 / 2)\left(1-z^{2}\right)\left(z-z_{3}\right)}}$
$=\frac{4}{f \sqrt{2\left(1-z_{3}\right)}} K\left(\sqrt{\frac{3\left(z_{3}+1\right)}{1-z_{3}}}\right)$,
where $K(k)=\int_{0}^{1} \frac{\mathrm{~d} z}{\sqrt{\left(1-z^{2}\right)\left(1-k^{2} z^{2}\right)}}$ is the complete elliptic integral of first kind. By putting $k=\sqrt{\frac{3\left(z_{3}+1\right)}{1-z_{3}}}, k \in$ $(0,1)$, we obtain that $f=\sqrt{\frac{h}{3}} \sqrt{\frac{k^{2}+3}{3 k^{2}+5}}$. Then (30) becomes
$\int_{\gamma} \frac{\mathrm{d} z}{y}=\frac{2}{\sqrt{h}} \sqrt{3 k^{2}+5} K(k)$.
Therefore,
$E=-\frac{2}{f \sqrt{h}} \frac{1}{f^{\prime}(k)} \frac{\partial}{\partial k}\left(\sqrt{3 k^{2}+5} K(k)\right) \neq 0$,
since $K(k)$ is an increasing function in $k$.

Next, consider $(h, f) \in U_{r 2}$. In this case $z_{3} \in$ $(-1 / 2,1)$ and
$\int_{\gamma} \frac{\mathrm{d} z}{y}=2 \int_{-1}^{-1 / 2} \frac{\mathrm{~d} z}{\sqrt{4 f^{2}(z+1 / 2)\left(1-z^{2}\right)\left(z-z_{3}\right)}}$
$=\frac{4}{f \sqrt{6\left(1+z_{3}\right)}} K\left(\sqrt{\frac{1-z_{3}}{3\left(1+z_{3}\right)}}\right)$.
Put $k=\sqrt{\frac{1-z_{3}}{3\left(1+z_{3}\right)}}, k \in(0,1)$. Then, we obtain that $f=$ $\sqrt{\frac{h}{3}} \sqrt{\frac{3 k^{2}+1}{5 k^{2}+3}}$. Thus, (31) reads

$$
\int_{\gamma} \frac{\mathrm{d} z}{y}=\frac{2}{\sqrt{h}} \sqrt{5 k^{2}+3} K(k)
$$

From this we get

$$
E=-\frac{2}{f \sqrt{h}} \frac{1}{f^{\prime}(k)} \frac{\partial}{\partial k}\left(\sqrt{5 k^{2}+3} K(k)\right) \neq 0
$$

due to above mentioned arguments.

## 3.4. $\operatorname{sp}(4)$

The Gross-Neveu model related with $\operatorname{sp}(4)$ is a two degrees of freedom system described with the Hamiltonian

$$
\begin{align*}
H= & \frac{1}{2}\left(y_{1}^{2}+y_{2}^{2}\right)+\mathrm{e}^{2 x_{1}}+\mathrm{e}^{-2 x_{1}}+\mathrm{e}^{2 x_{2}}+\mathrm{e}^{-2 x_{2}}  \tag{32}\\
& +\mathrm{e}^{x_{1}-x_{2}}+\mathrm{e}^{-\left(x_{1}-x_{2}\right)}+\mathrm{e}^{x_{1}+x_{2}}+\mathrm{e}^{-\left(x_{1}+x_{2}\right)} .
\end{align*}
$$

The equations of motion can be written in the standard way and one can obtain the eigenvalues of the linearized equations about the equilibrium as $\pm \mathrm{i} \sqrt{12}$, $\pm \mathrm{i} \sqrt{12}$ that is they are in $1: 1$ resonance. After similar transformations as in the previous cases, the truncated up to order four normal form is

$$
\begin{align*}
\bar{H}^{\mathrm{tr}}= & \sqrt{3}\left(z_{1} w_{1}+z_{2} w_{2}\right)+\frac{1}{3.2^{6}}\left[9\left(z_{1} w_{1}+z_{2} w_{2}\right)^{2}\right.  \tag{33}\\
& \left.-3\left(z_{1} w_{2}+z_{2} w_{1}\right)^{2}+4\left(z_{1} w_{2}-w_{1} z_{2}\right)^{2}\right] .
\end{align*}
$$

We know that the truncated normal form is integrable and the two integrals are $H_{2}$ and $\bar{H}^{\text {tr }}$ or equivalently $H_{2}$ and $B B=-3\left(z_{1} w_{2}+z_{2} w_{1}\right)^{2}+4\left(z_{1} w_{2}-w_{1} z_{2}\right)^{2}$.

Performing consequently the changes of variables (20), (21) and removing the linear term, we reduce $\bar{H}^{\text {tr }}$ to a Hamiltonian with a cyclic variable

$$
\begin{equation*}
\bar{H}^{\mathrm{tr}}=18 J_{1}^{2}+\left(J_{1}^{2}-J_{2}^{2}\right)\left(\cos \left(2 \chi_{2}\right)-7\right) \tag{34}
\end{equation*}
$$

which admits the integrals $\bar{H}^{\text {tr }}=h$ and $F=J_{1}=f \geq 0$.

The regular values of the energy momentum mapping here are

$$
U_{r}=\left\{(h, f) \in \mathbb{R}^{2}, f>0,10 f^{2}<h<12 f^{2}\right\} .
$$

Then the corresponding action variables are
$I_{1}=2 \pi f, \quad I_{2}=2 \int_{\chi_{2}^{-}}^{\chi_{2}^{+}} \sqrt{\frac{h-f^{2}\left(\cos 2 \chi_{2}+11\right)}{7-\cos 2 \chi_{2}}} \mathrm{~d} \chi_{2}$,
where $\chi_{2}^{-}<\chi_{2}^{+}$are the two roots of the equation

$$
h-f^{2}\left(\cos \left(2 \chi_{2}\right)+11\right)=0 \text { in }(0, \pi) .
$$

Now, we put $z=\cos 2 \chi_{2},|z| \leq 1, y^{2}=(7-z)(1-$ $\left.z^{2}\right)\left(h-f^{2}(z+11)\right)$ and denote the oval of the curve by $\gamma$ :

$$
\begin{aligned}
\Gamma_{h, f}= & \left\{(y, z) \in \mathbb{C}^{2}: y^{2}=(7-z)\left(1-z^{2}\right)\right. \\
& \left.\cdot\left(h-f^{2}(z+11)\right)\right\}
\end{aligned}
$$

Then we have

$$
\begin{equation*}
\psi(h, f) \stackrel{\text { def }}{=} I_{2}=\int_{\gamma} \frac{y \mathrm{~d} z}{(7-z)\left(1-z^{2}\right)} . \tag{36}
\end{equation*}
$$

Since $\psi_{h}=\frac{1}{2} \int_{\gamma} \frac{\mathrm{d} z}{y} \neq 0$ in $U_{r}$ from (28) it is seen that in order to verify Kolmogorov's condition, one needs to show that the Hessian of the function $\psi$ $D=\psi_{h h} \psi_{f f}-\left(\psi_{h f}\right)^{2}$ is nonzero. We again express the entries of $D$ via Abelian integrals. Denote this time $E=\int_{\gamma} \frac{(z+11)(7-z)\left(1-z^{2}\right)}{y^{3}} \mathrm{~d} z$. Then

$$
\begin{aligned}
\psi_{h h} & =-\frac{1}{4} \int_{\gamma} \frac{(7-z)\left(1-z^{2}\right)}{y^{3}} \mathrm{~d} z, \\
\psi_{h f} & =\frac{f}{2} E, \quad \psi_{f f}=-h E .
\end{aligned}
$$

Hence,
$D=\frac{1}{4} E$
$\cdot \int_{\gamma} \frac{h(7-z)\left(1-z^{2}\right)-f^{2}(z+1)(7-z)\left(1-z^{2}\right)}{y^{3}} \mathrm{~d} z$
$=\frac{1}{4} E \int_{\gamma} \frac{\mathrm{d} z}{y}=\frac{1}{2} \psi_{h} E$
and $D \neq 0 \leftrightarrow E \neq 0$. As before, $E$ can be presented in the following way

$$
E=\frac{2}{f} \psi_{h f}=\frac{1}{f} \frac{\partial}{\partial f} \int_{\gamma} \frac{\mathrm{d} z}{y} .
$$

To show that $E \neq 0$, we consider $(h, f) \in U_{r}$. Then $z_{2}=$ $\frac{h}{f^{2}}-11 \in(-1,1)$ and
$\int_{\gamma} \frac{\mathrm{d} z}{y}=2 \int_{-1}^{z_{2}} \frac{\mathrm{~d} z}{\sqrt{f^{2}(7-z)\left(1-z^{2}\right)\left(z_{2}-z\right)}}$
$=\frac{2 \sqrt{2}}{f \sqrt{7-z_{2}}} K\left(\sqrt{\frac{3\left(z_{3}+1\right)}{7-z_{2}}}\right)$.
By putting $k=\sqrt{\frac{3\left(z_{2}+1\right)}{7-z_{2}}}, k \in(0,1)$ we obtain $f=$ $\sqrt{\frac{h}{6}} \sqrt{\frac{k^{2}+3}{3 k^{2}+5}}$. Then

$$
E=\frac{\sqrt{2}}{f \sqrt{h}} \frac{1}{f^{\prime}(k)} \frac{\partial}{\partial k}\left(\sqrt{3 k^{3}+5} K(k)\right) \neq 0 .
$$

This completes the proof of Theorem 1.
Remark 1. The variables
$W_{0}=\frac{1}{2}\left(z_{1} w_{1}+z_{2} w_{2}\right), W_{1}=\frac{\mathrm{i}}{2}\left(z_{1} w_{2}-z_{2} w_{1}\right)$,
$W_{2}=\frac{1}{2}\left(z_{1} w_{2}+z_{2} w_{1}\right), W_{3}=\frac{1}{2}\left(z_{2} w_{2}-z_{1} w_{1}\right)$
are known as Hopf variables. They satisfy the relation $W_{1}^{2}+W_{2}^{2}+W_{3}^{2}=W_{0}^{2}$. In fact, every truncated normalized Hamiltonian with two equal frequencies can be written as a function of these variables [16]. See also [17] for a nice geometrical treatment of some classical integrable systems using these variables.

Remark 2. Kummer [16], along his studies on periodic solutions of Hamiltonians with two equal frequencies, verifies Arnold-Moser's condition. Let us show how the condition (7) can be treated in these particular cases. For the cases $\operatorname{sl}(3)$ (reduced) and so(4) one can obtain that the determinant $D_{1}$ is not zero from the Hamiltonians (14) and (17), respectively, since $a, b$ are the action variables. For the cases so(5) and $\operatorname{sp}(4)$ we will show that the map (7) $F_{h}, h=$ const. is regular in $U_{r}$. Note that $f$ can be taken as a coordinate on the set $h=$ const. in $U_{r}$, so $F_{h}=F_{h}(f)$. One can infer from [18] that

$$
F_{h}(f)=-\frac{1}{2 \pi} \psi_{f}
$$

Hence, $F_{h}$ is regular if $\psi_{f f} \neq 0$ in $U_{r}$. But this is indeed the case because $\psi_{f f}=-h E \neq 0$.

## 4. Proof of Theorem 2

In this section we consider the Hamiltonian systems with three degrees of freedom, describing the Gross-Neveu models, corresponding to Lie algebras so(6) $\sim \operatorname{sl}(4), \operatorname{sp}(6)$, and $\operatorname{so}(7)$. As it was mentioned above, they are non-integrable. Here, we will show that truncated normal forms up to order four are also nonintegrable in the Liouville sense. The proof is based on the Morales-Ramis method using the differential Galois theory.

Having the truncated up to order four Hamiltonian, we bring it to the truncated normal form with the nearidentity symplectic transformation, which preserves integrability [19]. This means that the truncated Hamiltonian and the truncated normal form are simultaneously integrable or non-integrable. In this case it is more convenient to prove the non-integrability for the truncated Hamiltonians, corresponding to the above algebras. We consider the so(6) case in details and give the key points for the other cases.

## 4.1. $\operatorname{so}(6)$

Let us recall the Hamiltonian for the so (6) GrossNeveu model,
$H=\frac{1}{2} \sum_{j=1}^{3} y_{j}^{2}+\sum_{3 \geq j>k \geq 1} \mathrm{e}^{x_{j}+x_{k}}+\mathrm{e}^{-x_{j}-x_{k}}+\sum_{j \neq k} \mathrm{e}^{x_{j}-x_{k}}$.
The corresponding equations of motion are
$\dot{x}_{j}=y_{j}$,
$\dot{y}_{j}=-\sum_{k \neq j}\left(\mathrm{e}^{x_{j}+x_{k}}-\mathrm{e}^{-x_{j}-x_{k}}\right)-\sum_{k \neq j}\left(\mathrm{e}^{x_{j}-x_{k}}-\mathrm{e}^{x_{k}-x_{j}}\right)$.
After linearization about the stationary point $(x, y)=$ $(0,0)$, we obtain

$$
\dot{\xi}_{j}=\eta_{j}, \quad \dot{\eta}_{j}=-8 \xi_{j}, \quad j=1,2,3
$$

The eigenvalues of this system are $\pm 2 \sqrt{2} i, \pm 2 \sqrt{2} i$, $\pm 2 \sqrt{2} \mathrm{i}$, and thus in $1: 1: 1$ resonance.

Next, we expand the Hamiltonian $H$ around $(x, y)=$ $(0,0)$ and truncate it to order four to obtain

$$
\begin{aligned}
H^{\mathrm{tr}}= & \frac{1}{2} \sum_{j=1}^{3}\left(y_{j}^{2}+8 x_{j}^{2}\right)+\frac{1}{3}\left(x_{1}^{4}+x_{2}^{4}+x_{3}^{4}\right) \\
& +\left(x_{1}^{2} x_{2}^{2}+x_{1}^{2} x_{3}^{2}+x_{2}^{2} x_{2}^{2}\right) .
\end{aligned}
$$

In what follows, we consider the complexified system with the Hamiltonian $H^{\mathrm{tr}}$, that is $\left(x_{j}(t), y_{j}(t)\right) \in \mathbb{C}^{6}$, $t \in \mathbb{C}$. The corresponding equations read

$$
\begin{equation*}
\dot{x}_{j}=y_{j}, \dot{y}_{j}=-8 x_{j}-\frac{4}{3} x_{j}^{3}-2 x_{j}\left(\sum_{k \neq j} x_{k}^{2}\right) . \tag{41}
\end{equation*}
$$

It is easy to be seen that the equations (41) have a family of phase curves

$$
\begin{align*}
& \Gamma(h): y_{1}^{2}=2 h-8 x_{1}^{2}-\frac{2}{3} x_{1}^{4}  \tag{42}\\
& x_{2}=x_{3}=y_{2}=y_{3}=0 .
\end{align*}
$$

These curves can be parameterized as follows:

$$
\begin{align*}
& x_{1}=\sqrt{\lambda_{1}} \operatorname{dn}\left(\sqrt{\frac{2}{3} \lambda_{1}} t, k\right), y_{1}=\dot{x}_{1}  \tag{43}\\
& x_{2}=x_{3}=y_{2}=y_{3}=0
\end{align*}
$$

where dn is the Jacobi elliptic function [20] and $\lambda_{1}, \lambda_{2}$ are the roots of $\frac{2}{3} \lambda^{2}+8 \lambda-2 h=0,\left|\lambda_{1}\right|>\left|\lambda_{2}\right|, k^{\prime}=$ $\sqrt{\frac{\lambda_{2}}{\lambda_{1}}}, k^{\prime}=\sqrt{1-k^{2}}$. So, we have a particular solution (43).

The function $\operatorname{dn}(\tau, k)$ has two periods $T_{1}=\frac{2 K}{\sqrt{\frac{2}{3} \lambda_{1}}}$, $T_{2}=\frac{4 \mathrm{i} K^{\prime}}{\sqrt{\frac{2}{3} \lambda_{1}}}\left(K^{\prime}(k)=K\left(k^{\prime}\right)\right)$ and two simple poles $t_{0}=$ $\frac{\mathrm{i} K^{\prime}}{\sqrt{\frac{2}{3} \lambda_{1}}}, t_{1}=\frac{3 \mathrm{i} K^{\prime}}{\sqrt{\frac{2}{3} \lambda_{1}}}$ in the parallelogram of periods. Geometrically, the curves $\Gamma(h)$ are complex tori with two points removed.

In order to reduce the domain of the solution (43), we consider the involution
$R:\left(x_{1}, y_{1}, x_{2}, y_{2}, x_{3}, y_{3}\right) \rightarrow\left(-x_{1},-y_{1}, x_{2}, y_{2}, x_{3}, y_{3}\right)$,
which leaves the system (41) invariant, it maps the phase curves $\Gamma(h)$ onto themselves and it interchanges the places of the two missing points. Then, $\hat{\Gamma}(h)=$ $\Gamma(h) / R$ are tori with one point removed. Let $F_{\mathrm{R}}$ be the set of the fixed points of the involution $R$ i.e. $F_{\mathrm{R}}:=$ $\left(0,0, x_{2}, y_{2}, x_{3}, y_{3}\right)$. Then we can factorize $M / F_{\mathrm{R}}$ by $R$ to obtain a symplectic manifold $\hat{M}$. The Hamiltonian $H^{\mathrm{tr}}(40)$ is naturally mapped into a Hamiltonian function $\hat{H}^{\text {tr }}$ on $\hat{M}$. From [4] we know that if the system (41) has three independent integrals, then the system defined by $\hat{H}^{\text {tr }}$ has also three independent integrals.

Next, we need the normal variational equations (NVEs) along $\hat{\Gamma}(h)$. It is straightforward that the NVEs
have the form

$$
\begin{equation*}
\dot{\xi}_{j}=\eta_{j}, \quad \dot{\eta}_{j}=-8 \xi_{j}-2 x_{1}^{2}(t) \xi_{j}, \quad j=2,3 . \tag{44}
\end{equation*}
$$

Each NVE splits into two equal subsystems each of them can be written as a second order linear differential equation $\ddot{\xi}_{j}+\left(8+2 x_{1}^{2}(t)\right) \xi_{j}=0, j=2,3$. In order to prove non-integrability, we need to show that the Galois group $G_{j}$ corresponding to at least one of them is non-Abelian. Since the NVEs are equal, we consider one of them and drop the index

$$
\begin{equation*}
\ddot{\xi}+f(t) \xi=0 \tag{45}
\end{equation*}
$$

where $f(t)=8+2 \lambda_{1} \mathrm{dn}^{2}\left(\sqrt{\frac{2}{3} \lambda_{1}} t, k\right)$.
The function $f(t)$ has periods $T_{1}, T_{2} / 2$ and the parallelogram of these periods has only one pole $-t_{0}$. Equation (45) is of Fuchsian type. It is known that in this case the monodromy group topologically generates the Galois group [13, 14]. The differential Galois group of (45) is an algebraic subgroup of $S L(2, L)$ which is connected. Here $L$ is the field of all elliptic functions. Now, we shall study the monodromy group $\mathcal{M}$ of (45).

Let $\alpha_{1}$ be a path over $\hat{\Gamma}(h)$ which corresponds to adding of period $T_{1}$, and $\alpha_{2}$ be a path over $\hat{\Gamma}(h)$ which corresponds to adding of period $T_{2} / 2$. Let $g_{1}:=$ $g\left(\alpha_{1}\right)$ and $g_{2}:=g\left(\alpha_{2}\right)$ be the monodromy transformations which correspond to the closed paths $\alpha_{1}$ and $\alpha_{2}$ on $\hat{\Gamma}(h)$, respectively. The commutator $\left[g_{1}, g_{2}\right]=$ $g_{1} g_{2} g_{1}^{-1} g_{2}^{-1}$ is the transformation which corresponds to one winding around the regular singular point $t_{0}$ of (45).

It is known [21] that the eigenvalues of the commutator are given by $\exp \left(2 \pi \mathrm{i} \rho_{1,2}\right)$, where $\rho_{1,2}$ are the roots of the indicial equation

$$
\rho(\rho-1)+f_{0}=0
$$

and where $f_{0}$ is the coefficient of the term $\left(t-t_{0}\right)^{-2}$ in the Laurent expansion of $f(t)$. Since $f(t)=8+$ $2 \lambda_{1}\left(\frac{-\mathrm{i}}{\sqrt{\frac{2}{3} \lambda_{1}\left(t-t_{0}\right)}}\right)^{2}+\ldots=-\frac{3}{\left(t-t_{0}\right)^{2}}+\ldots$, we have $f_{0}=$ -3 . Then, the commutator has eigenvalues $\exp (\pi \mathrm{i}(1 \pm$ $\sqrt{13})$ ) that is $\left[g_{1}, g_{2}\right] \neq \mathrm{i} d$, so $\mathcal{M}$ is not Abelian and hence $G$ is not Abelian too. According to MoralesRamis theorem the truncated to order four Hamiltonian form (also the truncated normal form) for the Lie algebra so(6) is non-integrable.

## 4.2. $\operatorname{so}(7)$

The Hamiltonian for the so(7) Gross-Neveu model is

$$
\begin{align*}
H= & \frac{1}{2} \sum_{j=1}^{3} y_{j}^{2}+\sum_{j=1}^{3}\left(\mathrm{e}^{x_{j}}+\mathrm{e}^{-x_{j}}\right)  \tag{46}\\
& +\sum_{3 \geq j>k \geq 1} \mathrm{e}^{\left(x_{j}+x_{k}\right)}+\mathrm{e}^{-\left(x_{j}+x_{k}\right)}+\sum_{j \neq k} \mathrm{e}^{\left(x_{j}-x_{k}\right)} .
\end{align*}
$$

The corresponding equations of motion are
$\dot{x}_{j}=y_{j}, \quad \dot{y}_{j}=-\left(\mathrm{e}^{x_{j}}-\mathrm{e}^{-x_{j}}\right)$
$-\sum_{k \neq j}\left(\mathrm{e}^{\left(x_{j}+x_{k}\right)}-\mathrm{e}^{-\left(x_{j}+x_{k}\right)}\right)-\sum_{k \neq j}\left(\mathrm{e}^{\left(x_{j}-x_{k}\right)}-\mathrm{e}^{\left(x_{k}-x_{j}\right)}\right)$.
After linearization about the stationary point $(x, y)=$ $(0,0)$, we obtain

$$
\dot{\xi}_{j}=\eta_{j}, \quad \dot{\eta}_{j}=-10 \xi_{j}, \quad j=1,2,3
$$

The eigenvalues of this system are $\pm \sqrt{10} \mathrm{i}, \pm \sqrt{10} \mathrm{i}$, $\pm \sqrt{10} \mathrm{i}$, and thus in $1: 1: 1$ resonance.

Next, we expand the Hamiltonian $H$ around $(x, y)=$ $(0,0)$ and truncate it to order four to obtain

$$
\begin{align*}
H^{\mathrm{tr}}= & \frac{1}{2} \sum_{j=1}^{3}\left(y_{j}^{2}+5 x_{j}^{2}\right)+\frac{5}{12}\left(x_{1}^{4}+x_{2}^{4}+x_{3}^{4}\right)  \tag{48}\\
& +\left(x_{1}^{2} x_{2}^{2}+x_{1}^{2} x_{3}^{2}+x_{2}^{2} x_{2}^{2}\right)
\end{align*}
$$

The equations of motion of the truncated Hamiltonian read
$\dot{x}_{j}=y_{j}, \dot{y}_{j}=-10 x_{j}-\frac{5}{3} x_{j}^{3}-2 x_{j}\left(\sum_{k \neq j} x_{k}^{2}\right)$.
The curves $\Gamma(h)$ are given by the equations

$$
\begin{align*}
& \Gamma(h): y_{1}^{2}=2 h-10 x_{1}^{2}-\frac{5}{6} x_{1}^{4}  \tag{50}\\
& x_{2}=x_{3}=y_{2}=y_{3}=0
\end{align*}
$$

and correspond to the solutions

$$
\begin{align*}
& x_{1}=\sqrt{\lambda_{1}} \operatorname{dn}\left(\sqrt{\frac{5}{6} \lambda_{1}} t, k\right), y_{1}=\dot{x}_{1}  \tag{51}\\
& x_{2}=x_{3}=y_{2}=y_{3}=0
\end{align*}
$$

The corresponding Fuchsian equation is (45) with $f(t)=10+2 \lambda_{1} \operatorname{dn}^{2}\left(\sqrt{\frac{5}{6}} \lambda_{1} t, k\right)$. The eigenvalues of the commutator are $\exp (\pi \mathrm{i}(1 \pm \sqrt{53 / 5}))$ that is $\left[g_{1}, g_{2}\right] \neq$ $\mathrm{i} d$, so $\mathcal{M}$ is not Abelian and hence $G$ is not Abelian too. According to Morales-Ramis theorem the truncated to order four Hamiltonian form (also the truncated normal form) for the Lie algebra so(7) is non-integrable.

## 4.3. $\operatorname{sp}(6)$

The Hamiltonian for the $\operatorname{sp}(6)$ Gross-Neveu model is

$$
\begin{align*}
H= & \frac{1}{2} \sum_{j=1}^{3} y_{j}^{2}+\sum_{j=1}^{3}\left(\mathrm{e}^{2 x_{j}}+\mathrm{e}^{-2 x_{j}}\right)  \tag{52}\\
& +\sum_{3 \geq j>k \geq 1} \mathrm{e}^{\left(x_{j}+x_{k}\right)}+\mathrm{e}^{-\left(x_{j}+x_{k}\right)}+\sum_{j \neq k} \mathrm{e}^{\left(x_{j}-x_{k}\right)} .
\end{align*}
$$

The corresponding equations of motion are
$\dot{x}_{j}=y_{j}$,
$\dot{y}_{j}=-2\left(\mathrm{e}^{2 x_{j}}-\mathrm{e}^{-2 x_{j}}\right)-\sum_{k \neq j}\left(\mathrm{e}^{\left(x_{j}+x_{k}\right)}\right.$
$\left.-\mathrm{e}^{-\left(x_{j}+x_{k}\right)}\right)-\sum_{k \neq j}\left(\mathrm{e}^{\left(x_{j}-x_{k}\right)}-\mathrm{e}^{\left(x_{k}-x_{j}\right)}\right)$.
After linearization about the stationary point $(x, y)=$ $(0,0)$, we obtain

$$
\dot{\xi}_{j}=\eta_{j}, \quad \dot{\eta}_{j}=-16 \xi_{j}, \quad j=1,2,3
$$

The eigenvalues of this system are $\pm 4 \mathrm{i}, \pm 4 \mathrm{i}, \pm 4 \mathrm{i}$, and thus in $1: 1: 1$ resonance.

Next, we expand the Hamiltonian $H$ around $(x, y)=$ $(0,0)$ and truncate it to order four to obtain

$$
\begin{align*}
H^{\mathrm{tr}}= & \frac{1}{2} \sum_{j=1}^{3}\left(y_{j}^{2}+16 x_{j}^{2}\right)+\frac{5}{3}\left(x_{1}^{4}+x_{2}^{4}+x_{3}^{4}\right)  \tag{54}\\
& +\left(x_{1}^{2} x_{2}^{2}+x_{1}^{2} x_{3}^{2}+x_{2}^{2} x_{2}^{2}\right)
\end{align*}
$$

The equations of motion of the truncated Hamiltonian read
$\dot{x}_{j}=y_{j}, \quad \dot{y}_{j}=-16 x_{j}-\frac{10}{3} x_{j}^{3}-2 x_{j}\left(\sum_{k \neq j} x_{k}^{2}\right)$.
The curves $\Gamma(h)$ are given by the equations

$$
\begin{align*}
& \Gamma(h): y_{1}^{2}=2 h-16 x_{1}^{2}-\frac{10}{3} x_{1}^{4}  \tag{56}\\
& x_{2}=x_{3}=y_{2}=y_{3}=0
\end{align*}
$$

and correspond to the solutions

$$
\begin{align*}
& x_{1}=\sqrt{\lambda_{1}} \operatorname{dn}\left(\sqrt{\frac{10}{3} \lambda_{1}} t, k\right), y_{1}=\dot{x}_{1}  \tag{57}\\
& x_{2}=x_{3}=y_{2}=y_{3}=0
\end{align*}
$$

## Acknowledgement

This work is partially supported by grant 169/2010 of the Sofia University.
[1] R. Shankar, Phys. Lett. B 92, 333 (1980); 102, 257 (1981).
[2] D. Gross and A. Neveu, Phys. Rev. D 10, 3235 (1974).
[3] E. Horozov, Ann. of Phys. 174, 430 (1987).
[4] S. Ziglin, Func. Anal. Appl. 16, 181 (1982); 17, 6 (1983).
[5] A. Maciejewski, M. Przybylska, and T. Stachowiak, Physica D 201, 249 (2005).
[6] B. Rink and F. Verhulst, Phys. A 285, 467 (2000)
[7] B. Rink, Commun. Math. Phys. 218, 665 (2001).
[8] V. Arnold, Mathematical Methods of Classical Mechanics, Springer, Berlin 1978.
[9] F. Verhulst, in: Symmetry and Perturbation Theory (Eds. D. Bambusi and G. Gaeta), Quaderni GNFM, Firenze 1998, p. 245.
[10] A. Kolmogorov, Dokl. Akad. Nauk SSSR 98, 527 (1954).
[11] V. Arnold, Uspehi Mat. Nauk 18, 13 (1963) (Russian).
[12] J. Moser, Math. Ann. 169, 136 (1967).
[13] J. J. Morales-Ruiz, Differential Galois Theory and Non-Integrability of Hamiltonian Systems, Birkhäuser, Basel 1999.
[14] F. Beukers, in: From Number Theory to Physics (Eds. M. Waldschmidt and J.-M. Itzykson), Springer, Berlin 1992, p. 413
[15] E. Horozov, J. Reine Angew. Math. 408, 114 (1990).
[16] M. Kummer, Commun. Math. Phys. 48, 53 (1976); 58, 85 (1978).
[17] R. Cushman and L. Bates, Global Aspects of Classical Integrable Systems, Birkhäuser, Basel 1997.
[18] E. Horozov, Phys. Lett. A 173, 279 (1993).
[19] J. Hietarinta, Phys. Rep. 147, 87 (1987).
[20] E. Wittaker and G. Watson, A Course of Modern Analysis, Cambridge U. P., Cambridge 1927.
[21] E. Coddington and N. Levinson, Theory of Ordinary Differential Equations, McGraw Hill, New York 1965.

