Double-Diffusive Magneto Convection in a Compressible Couple-Stress Fluid Through Porous Medium

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The double-diffusive convection in a compressible couple-stress fluid layer heated and soluted from below through porous medium is considered in the presence of a uniform vertical magnetic field. Following the linearized stability theory and normal mode analysis, the dispersion relation is obtained. For stationary convection, the compressibility, stable solute gradient, magnetic field, and couple-stress postpone the onset of convection whereas medium permeability hastens the onset of convection. Graphs have been plotted by giving numerical values to the parameters to depict the stability characteristics. The stable solute gradient and magnetic field introduce oscillatory modes in the system, which were non-existent in their absence. A condition for the system to be stable is obtained by using the Rayleigh-Ritz inequality. The sufficient conditions for the non-existence of overstability are also obtained.

Key words: Double-Diffusive Convection; Compressible Couple-Stress Fluid; Magnetic Field; Porous Medium.

1. Introduction

The investigation of double-diffusive convection is motivated by its interesting complexities as a doublediffusion phenomena as well as its direct relevance to geophysics and astrophysics. The conditions under which convective motion in double-diffusive convection are important (e.g. in lower parts of the Earth's atmosphere, astrophysics, and several geophysical situations) are usually far removed from the consideration of a single component fluid and rigid boundaries and therefore it is desirable to consider a fluid acted on by a solute gradient and free boundaries.

When the fluids are compressible, the equations governing the system become quite complicated. Spiegel and Veronis [1] have simplified the set of equations governing the flow of compressible fluids under the assumptions that (a) the depth of the fluid layer is much less than the scale height, as defined by them, and (b) the fluctuations in temperature, density, and pressure, introduced due to motion, do not exceed their total static variations.

Under the above approximations, the flow equations are the same as those for incompressible fluids, except that the static temperature gradient is replaced by its excess over the adiabatic one and C_v is replaced by C_p .

With the growing importance of non-Newtonian fluids in modern technology and industry, the investigations of such fluids are desirable. The theory of couplestress fluids has been formulated by Stokes [2]. One of the applications of couple-stress fluids is the study of the mechanisms of lubrication of synovial joints. The normal synovial fluid is a viscous, non-Newtonian fluid and is generally clear or yellowish. The theory due to Stokes [2] allows for polar effects such as the presence of couple stresses and body couples. According to this theory, couple stresses are found to appear in noticeable magnitudes in fluids with very large molecules. Since the long-chain hyaluronic acid molecules are found as additives in synovial fluid, Walicki and Walicka [3] modelled the synovial fluid as couple-stress fluid in human joints. The synovial fluid is the natural lubricant of joints of the vertebrates. The detailed description of the joint lubrication has very important practical applications. Practically all diseases of joints are caused by or connected with a malfunction of the lubrication. The efficiency of the physiological joint lubrication is caused by several mechanisms. The synovial fluid is, due to its content of the hyaluronic acid, a fluid of high viscosity, near to a gel.

Thermosolutal convection in a couple-stress fluid in presence of a magnetic field and rotation, sep-

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arately, has been investigated by Kumar and Singh [4, 5]. The problem of thermal instability of a compressible, electrically conducting couple-stress fluid in the presence of a uniform magnetic field has been considered by Singh and Kumar [6]. Magnetic fields are used for clinical purposes in detection and treatment of certain diseases with the help of magnetic field devices/instruments.

In recent years, the investigations of flow of fluids through porous media have become an important topic due to the recovery of crude oil from the pores of reservoir rocks.

Keeping in mind the importance in geophysics, soil sciences, ground water hydrology, astrophysics, chemical technology, industry, and biomechanics (e. g. physiotherapy), the double-diffusive convection in a compressible couple-stress fluid in the presence of a magnetic field through porous medium has been considered in the present paper.

2. Formulation of the Problem and Perturbation Equations

Here we consider an infinite, horizontal, compressible, electrically conducting couple-stress fluid layer of thickness *d* in a porous medium, heated and soluted from below so that the temperatures, densities, and solute concentrations at the bottom surface z = 0are T_0 , ρ_0 , and C_0 , and at the upper surface z = dare T_d , ρ_d , and C_d , respectively, with the *z*-axis being taken as vertical, and that a uniform temperature gradient $\beta (= |dT/dz|)$ and a uniform solute gradient $\beta' (= |dC/dz|)$ are maintained. This layer is acted on by a uniform vertical magnetic field $\vec{H}(0,0,H)$ and the gravity field $\vec{g}(0,0,-g)$.

Assume that X_m is the constant space distribution of X, X_0 is the variation in X in the absence of motion and X'(x, y, z, t) is the fluctuation in X due to the motion of the fluid. Spiegel and Veronis [1] defined X as any of the state variables (pressure (p), density (ρ) or temperature (T)) and expressed these in the form

$$X(x, y, z, t) = X_{\rm m} + X_0(z) + X'(x, y, z, t) .$$
(1)

The initial state is, therefore, a state in which the fluid velocity, temperature, solute concentration, pressure, and density at any point in the fluid are given by

$$\vec{q} = 0, \ T = T(z), \ C = C(z), \ p = p(z), \ \rho = \rho(z), \ (2)$$

respectively, where

$$T(z) = T_0 - \beta z, \quad C(z) = C_0 - \beta' z,$$

$$p(z) = p_{\rm m} - g \int_0^z (\rho_{\rm m} + \rho_0) dz,$$

$$\rho(z) = \rho_{\rm m} [1 - \alpha_{\rm m} (T - T_{\rm m}) + \alpha'_{\rm m} (C - C_{\rm m}) + K_{\rm m} (p - p_{\rm m})],$$
(3)

and

$$\alpha_{\rm m} = -\left(\frac{1}{\rho}\frac{\partial\rho}{\partial T}\right)_{\rm m} (=\alpha, \text{ say}),$$

$$\alpha_{\rm m}' = -\left(\frac{1}{\rho}\frac{\partial\rho}{\partial C}\right)_{\rm m} (=\alpha', \text{ say}),$$

$$K_{\rm m} = \left(\frac{1}{\rho}\frac{\partial\rho}{\partial p}\right)_{\rm m}.$$

Let δp , $\delta \rho$, θ , γ , $\vec{q}(u,v,w)$ and $\vec{h}(h_x,h_y,h_z)$ denote, respectively, the perturbations in pressure p, density ρ , temperature T, solute concentration C, fluid velocity $\vec{q}(0,0,0)$ and magnetic field $\vec{H}(0,0,H)$. The linearized hydromagnetic perturbation equations, relevant to the problem, are

$$\frac{1}{\varepsilon} \frac{\partial \vec{q}}{\partial t} = -\frac{1}{\rho_{\rm m}} \nabla \delta p - \vec{g} (\alpha \theta - \alpha' \gamma) -\frac{1}{k_1} \left(\nu - \frac{\mu'}{\rho_{\rm m}} \nabla^2 \right) \vec{q} + \frac{\mu_{\rm e}}{4\pi \rho_{\rm m}} (\nabla \times \vec{h}) \times \vec{H},$$
⁽⁴⁾

$$\nabla \cdot \vec{q} = 0, \tag{5}$$

$$E\frac{\partial\theta}{\partial t} = \left(\beta - \frac{g}{C_{\rm p}}\right)w + \kappa\nabla^2\theta,\tag{6}$$

$$E'\frac{\partial\gamma}{\partial t} = \beta'w + \kappa'\nabla^2\gamma,\tag{7}$$

$$\nabla \cdot \vec{h} = 0, \tag{8}$$

$$\varepsilon \frac{\partial \vec{h}}{\partial t} = \left(\vec{H \cdot \nabla} \right) \vec{q} + \varepsilon \eta \, \nabla^2 \vec{h}. \tag{9}$$

Here $\frac{g}{C_p}$ is the adiabatic gradient; $v (= \mu/\rho_m)$, μ' , κ , κ' , ε , and k_1 stand for kinematic viscosity, couplestress viscosity, thermal diffusivity, solute diffusivity, medium porosity, and medium permeability, respectively. $E = \varepsilon + (1 - \varepsilon) (\rho_s C_s / \rho_0 C)$ is a constant and E' is a constant analogous to E but corresponding to the solute rather to the heat; ρ_s , C_s and ρ_0 , C stand for density and heat capacity of the solid (porous matrix) material and the fluid, respectively.

The equation of state is

$$\rho = \rho_{\rm m} [1 - \alpha (T - T_0) + \alpha' (C - C_0)], \qquad (10)$$

where α is the coefficient of thermal expansion and α' analogous the solute coefficient. The suffix zero refers to the values at the reference level z = 0. The change in density $\delta \rho$ caused mainly by the perturbations θ and γ in temperature and concentration, is given by

$$\delta \rho = -\rho_{\rm m} \left(\alpha \theta - \alpha' \gamma \right). \tag{11}$$

In writing (4), use has been made of (11).

Writing the scalar components of (4) and (9) and eliminating u, v, h_x , h_y , and δp by using (5) and (8), we obtain

$$\begin{bmatrix} \frac{1}{\varepsilon} \frac{\partial}{\partial t} + \frac{1}{k_1} \left(v - \frac{\mu}{\rho_{\rm m}} \nabla^2 \right) \end{bmatrix} \nabla^2 w$$

$$-g \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) (\alpha \theta - \alpha' \gamma) - \frac{\mu_{\rm e} H}{4 \pi \rho_{\rm m}} \frac{\partial}{\partial z} \nabla^2 h_z = 0,$$

$$(12)$$

$$\left(E\frac{\partial}{\partial t} - \kappa \nabla^2\right)\theta = \left(\beta - \frac{g}{C_p}\right)w,\tag{13}$$

$$\left(E'\frac{\partial}{\partial t} - \kappa'\nabla^2\right)\gamma = \beta w, \tag{14}$$

$$\varepsilon \left(\frac{\partial}{\partial t} - \eta \nabla^2\right) h_z = H \frac{\partial w}{\partial z}.$$
(15)

Considering the case in which both the boundaries are free and the temperatures, concentrations at the boundaries are kept constant, then the boundary conditions appropriate to the problem are

$$w = \frac{\partial^2 w}{\partial z^2} = 0, \quad \theta = 0,$$

$$\gamma = 0 \text{ at } z = 0 \text{ and } z = d.$$
(16)

The constitutive equations for the couple-stress fluid are

$$\tau_{ij} = (2\mu - 2\mu' \nabla^2) e_{ij};$$

$$e_{ij} = \frac{1}{2} \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right).$$
(17)

The conditions on a free surface are the vanishing of tangential stresses τ_{xz} and τ_{yz} , which yield

$$\tau_{xz} = (\mu - \mu' \nabla^2) \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right) = 0, \quad (18)$$

$$\tau_{yz} = (\mu - \mu' \nabla^2) \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right) = 0.$$
 (19)

Since w vanishes for all x and y on the bounding surface, it follows from (18) and (19) that

$$\left(\mu - \mu' \nabla^2\right) \frac{\partial u}{\partial z} = 0, \quad \left(\mu - \mu' \nabla^2\right) \frac{\partial v}{\partial z} = 0.$$
 (20)

From the equation of continuity (5) and differentiated with respect to *z*, we conclude that

$$\left[\mu - \mu' \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\right] \frac{\partial^2 w}{\partial z^2} = 0, \quad (21)$$

which on using (12) and (16) implies that

$$\frac{\partial^4 w}{\partial z^4} = 0$$
 at $z = 0$ and $z = d$. (22)

Equations (12) and (15), using (16) and (21), yield

$$\frac{\partial h_z}{\partial z} = 0$$
 at $z = 0$ and $z = d$. (23)

3. Dispersion Relation

We now analyze the disturbances into normal modes, assuming that the perturbation quantities are of the form

$$[w, \theta, \gamma, h_z] = [W(z), \Theta(z), \Gamma(z), K(z)]$$

$$\cdot \exp(ik_x x + ik_y y + nt), \qquad (24)$$

where k_x , k_y are the wave numbers along *x*- and *y*-directions, respectively, $k\left(=\sqrt{k_x^2+k_y^2}\right)$ is the resultant wave number and *n* is the growth rate which is, in general, a complex constant.

Using expression (24), (12)-(15), in non-dimensional form, they become

$$\left[\frac{\sigma}{\varepsilon} + \frac{1}{P_{l}} \left\{1 - F\left(D^{2} - a^{2}\right)\right\}\right] \left(D^{2} - a^{2}\right) W$$

$$+ \frac{ga^{2}d^{2}}{\nu} \left(\alpha\Theta - \alpha'\Gamma\right) - \frac{\mu_{e}Hd}{4\pi\rho_{0}\nu} \left(D^{2} - a^{2}\right) DK = 0,$$

$$(25)$$

$$\left(\mathsf{D}^2 - a^2 - Ep_1\sigma\right)\Theta = -\frac{d^2}{\kappa}\frac{g}{C_p}(G-1)W, \quad (26)$$

$$\left(\mathsf{D}^2 - a^2 - Eq\sigma\right)\Gamma = -\frac{\beta' d^2}{\kappa'}W,\tag{27}$$

$$\left(\mathbf{D}^2 - a^2 - p_2 \boldsymbol{\sigma}\right) K = -\frac{Hd}{\varepsilon \eta} DW,$$
(28)

where we have put a = kd, $\sigma = \frac{nd^2}{v}$, $x = x^*d$, $y = y^*d$, $z = z^*d$, and $D = \frac{d}{dz^*}$. Here $p_1 = \frac{v}{\kappa}$ is the Prandtl number, $p_2 = \frac{v}{\eta}$ is the magnetic Prandtl number, $q = \frac{v}{\kappa'}$ is the Schmidt number, $P_1 = \frac{k_1}{d^2}$ is the dimensionless permeability, $F = \frac{\mu'}{\rho_0 d^2 v}$ is the dimensionless couplestress parameter, and $G = \frac{C_p \beta}{g}$ is the dimensionless compressibility parameter. We shall suppress the star (*) for convenience hereafter. Eliminating Θ , Γ , and Kbetween (25)–(28), we obtain

$$\left(\mathbf{D}^{2} - a^{2} - Ep_{1}\boldsymbol{\sigma} \right) \left[\left\{ \frac{\boldsymbol{\sigma}}{\varepsilon} + \frac{1}{P_{1}} (1 - F\overline{\mathbf{D}^{2}} - a^{2}) \right\} \right] \cdot \left(\mathbf{D}^{2} - a^{2} \right) \left(\mathbf{D}^{2} - a^{2} - E'q\boldsymbol{\sigma} \right) \left(\mathbf{D}^{2} - a^{2} - p_{2}\boldsymbol{\sigma} \right) + \frac{Q}{\varepsilon} \mathbf{D}^{2} \cdot \left(\mathbf{D}^{2} - a^{2} \right) \left(\mathbf{D}^{2} - a^{2} - E'q\boldsymbol{\sigma} \right) + Sa^{2} \left(\mathbf{D}^{2} - a^{2} - p_{2}\boldsymbol{\sigma} \right) \right] W = Ra^{2} \frac{G-1}{G} \left(\mathbf{D}^{2} - a^{2} - p_{2}\boldsymbol{\sigma} \right) \left(\mathbf{D}^{2} - a^{2} - E'q\boldsymbol{\sigma} \right) W,$$
(29)

where $R = \frac{g\alpha\beta d^4}{\nu\kappa}$ is the Rayleigh number, $S = \frac{g\alpha'\beta' d^4}{\nu\kappa'}$ is the solute Rayleigh number, and $Q = \frac{\mu_c H^2 d^2}{4\pi\rho_0 v\eta}$ is the Chandrasekhar number.

The boundary conditions (16), (22), and (23), in non-dimensional form, using expression (24) transform to

$$W = D^2 W = 0, \quad \Theta = 0, \quad \Gamma = 0,$$

DK = 0 at z = 0 and z = 1. (30)

Using the boundary conditions (30), it can be shown with the help of (25)-(28) that all the even-order derivatives of W must vanish at z = 0 and z = 1. Hence, the proper solution of W characterizing the lowest mode is

$$W = W_0 \sin \pi z, \tag{31}$$

where W_0 is a constant. Substituting the proper solution (31) in (29), we obtain the dispersion relation

$$R_{1} = \frac{G}{G-1} \left(1 + x + Ep_{1} \frac{\sigma}{\pi^{2}} \right) \left[\left\{ \frac{\sigma}{\pi^{2} \varepsilon} + \frac{1}{P} (1 + \pi^{2}) + F\overline{1 + x} \right\} \left(1 + x \right) \left(1 + x + E'q \frac{\sigma}{\pi^{2}} \right) \left(1 + x + p_{2} \frac{\sigma}{\pi^{2}} \right) + \frac{Q_{1}}{\varepsilon} (1 + x) \left(1 + x + E'q \frac{\sigma}{\pi^{2}} \right) + S_{1}x \left(1 + x + p_{2} \frac{\sigma}{\pi^{2}} \right) \right] \cdot \left[x \left(1 + x + p_{2} \frac{\sigma}{\pi^{2}} \right) \left(1 + x + E'q \frac{\sigma}{\pi^{2}} \right) \right]^{-1}, \quad (32)$$

where $R_1 = \frac{R}{\pi^4}$, $S_1 = \frac{S}{\pi^4}$, $Q_1 = \frac{Q}{\pi^2}$, $P = \pi^2 P_1$, and $x = \frac{a^2}{\pi^2}$.

4. The Stationary Convection

When the instability sets in as stationary convection, the marginal state will be characterized by $\sigma = 0$. Putting $\sigma = 0$, the dispersion relation (32) reduces to

$$R_1 = \frac{G}{G-1} \left[\frac{1+x}{x} \left\{ \frac{1+x}{P} (1+\pi^2 F \overline{1+x}) + \frac{Q_1}{\varepsilon} \right\} + S_1 \right].$$
(33)

Equation (33) expresses the modified Rayleigh number R_1 as a function of the dimensionless wave number x and the parameters G, P, F, Q_1 , and S_1 . For fixed P, F, Q_1 , and S_1 , let G (accounting for the compressibility effects) also be kept fixed.

Then we find that

$$\bar{R}_{\rm C} = \left(\frac{G}{G-1}\right) R_{\rm C},\tag{34}$$

where $\bar{R}_{\rm C}$ and $R_{\rm C}$ denote, respectively, the critical Rayleigh numbers in the presence and absence of compressibility. G > 1 is relevant here. The cases G < 1 and G = 1 correspond to negative and infinite values of the critical Rayleigh numbers in the presence of compressibility, which are not relevant in the present study. The effect of compressibility is thus to postpone the onset of double-diffusive convection.

Equation (33) yields

$$\frac{\mathrm{d}R_1}{\mathrm{d}S_1} = \frac{G}{G-1},\tag{35}$$

$$\frac{\mathrm{d}R_1}{\mathrm{d}Q_1} = \frac{G}{G-1} \frac{(1+x)}{\varepsilon x},\tag{36}$$

$$\frac{\mathrm{d}R_1}{\mathrm{d}F} = \frac{G}{G-1} \frac{\pi^2 (1+x)^3}{Px},$$
(37)

$$\frac{\mathrm{d}R_1}{\mathrm{d}P} = -\frac{G}{G-1} \frac{(1+x)^2 (1+\pi^2 F \overline{1+x})}{x P^2}, \quad (38)$$

which imply that stable solute gradient, magnetic field, and couple-stress postpone the onset of convection whereas medium permeability hastens the onset of convection. A result derived by Singh and Kumar [6] and Kumar [8]. This is in contrast to the result derived by Kumar and Singh [5] in which couple-stress has both stabilizing and destabilizing effects. Graphs have been plotted between R_1 and x for various values of S_1 , Q_1 , F_1 , and P. It is also evident from Figures 1–4 that stable solute gradient, magnetic field, and couple-stress



Fig. 1. Variation of R_1 with x for a fixed G = 10, P = 2, $F_1 = 5$, $Q_1 = 25$, $\varepsilon = 0.5$ for different values of S_1 (= 10, 15, 20).



Fig. 2. Variation of R_1 with x for a fixed G = 10, P = 2, $F_1 = 5$, $\varepsilon = 0.5$, $S_1 = 10$ for different values of Q_1 (= 25, 50, 75).

postpone the onset of convection whereas medium permeability hastens the onset of convection in a compressible couple-stress fluid heated and soluted from below through porous medium in hydromagnetics.

5. Some Important Theorems

Theorem 1: The system is stable for G < 1.

Proof: Multiplying (25) by W^* , the complex conjugate of W, integrating over the range of z, and using (26)-(28) together with the boundary conditions (30), we obtain

$$\left(\frac{\sigma}{\varepsilon} + \frac{1}{P_{\rm l}}\right)I_{\rm l} + \frac{F}{P_{\rm l}}I_{\rm 2} - \frac{1}{G-1}\frac{C_{\rm p}\alpha\kappa a^2}{\nu}(I_3 + Ep_1\sigma^*I_4) + \frac{g\alpha'\kappa'a^2}{\nu\beta'}(I_5 + E'q\sigma^*I_6)\frac{\mu_{\rm e}\varepsilon\eta}{4\pi\rho_0\nu}(I_7 + p_2\sigma^*I_8) = 0,$$
(39)







Fig. 4. Variation of R_1 with x for a fixed G = 10, $F_1 = 5$, $Q_1 = 25$, $S_1 = 10$, $\varepsilon = 0.5$ for different values of P (= 2, 4, 6).

where σ^* is the complex conjugate of σ and the integrals $I_1 - I_8$ are all positive definite.

Putting $\sigma = \sigma_r + i\sigma_i$ in (39) and equating real and imaginary parts, we obtain

$$\sigma_{\rm r} \left(\frac{I_1}{\varepsilon} - \frac{1}{G-1} \frac{C_{\rm p} \alpha \kappa a^2}{\nu} E p_1 I_4 + \frac{g \alpha' \kappa' a^2}{\nu \beta'} E' q I_6 + \frac{\mu_{\rm e} \varepsilon \eta}{4 \pi \rho_0 \nu} p_2 I_8 \right) = - \left(\frac{I_1}{P_1} + \frac{F}{P_1} I_2 - \frac{1}{G-1} \frac{C_{\rm p} \alpha \kappa a^2}{\nu} I_3 + \frac{g \alpha' \kappa' a^2}{\nu \beta'} I_5 + \frac{\mu_{\rm e} \varepsilon \eta}{4 \pi \rho_0 \nu} I_7 \right)$$

$$(40)$$

and

$$\sigma_{i}\left(\frac{I_{1}}{\varepsilon} + \frac{1}{G-1}\frac{C_{p}\alpha\kappa a^{2}}{\nu}Ep_{1}I_{4} - \frac{g\alpha'\kappa'a^{2}}{\nu\beta'}E'qI_{6} - \frac{\mu_{e}\varepsilon\eta}{4\pi\rho_{0}\nu}p_{2}I_{8}\right) = 0.$$
(41)

It is evident from (40) that if G < 1, σ_r is negative meaning thereby the stability of the system, a result derived by Singh and Kumar [6] for non-porous medium.

Theorem 2: The modes may be oscillatory or nonoscillatory in contrast to the case of no magnetic field and in absence of stable solute gradient where modes are non-oscillatory, for G > 1.

Proof: Equation (41) yields that $\sigma_i = 0$ or $\sigma_i \neq 0$, which means that modes may be non-oscillatory or oscillatory. In the absence of stable solute gradient and magnetic field, (41) gives

$$\sigma_{\rm i}\left(\frac{I_1}{\varepsilon} + \frac{1}{G-1}\frac{C_{\rm p}\alpha\kappa a^2}{\nu}Ep_1I_4\right) = 0, \qquad (42)$$

and the terms in brackets are positive definite when G > 1. Thus $\sigma_i = 0$, which means that oscillatory modes are not allowed and the principle of exchange of stabilities is satisfied for a porous medium in compressible, couple-stress fluid in the absence of stable solute gradient and magnetic field, a result derived by Sharma and Sharma [7]. This result is true for compressible, couple-stress fluids as well as for incompressible Newtonian fluids (Chandrasekhar [9]) in the absence of a magnetic field. The presence of each, the stable solute gradient and the magnetic field, brings oscillatory modes (as σ_i may not be zero) which were non-existent in their absence.

Theorem 3: The system is stable for $\frac{1}{G-1} \frac{C_p \alpha \kappa}{v} \le \frac{4\pi^2}{P_l} \left[1 + \frac{27\pi^2 F}{16} \right]$ and under the condition $\frac{1}{G-1} \frac{C_p \alpha \kappa}{v} > \frac{4\pi^2}{P_l} \left[1 + \frac{27\pi^2 F}{16} \right]$, the system becomes unstable.

Proof: From (42) it is clear that σ_i is zero when the quantity multiplying it is not zero and arbitrary when this quantity is zero.

If $\sigma_i \neq 0$, equation (40) upon utilizing (41) and the Rayleigh-Ritz inequality gives

$$\left[\frac{4\pi^{2}}{P_{1}}\left(1+\frac{27\pi^{2}F}{16}\right)-\frac{1}{G-1}\frac{C_{p}\alpha\kappa}{\nu}\right]\int_{0}^{1}|W|^{2}dz +\frac{\pi^{2}+a^{2}}{a^{2}}\left\{\frac{g\alpha'\kappa'a^{2}}{\nu\beta'}I_{5}+\frac{\mu_{e}\varepsilon\eta}{4\pi\rho_{0}\nu}I_{7}+\frac{2\sigma_{r}}{\varepsilon}I_{1}\right\}\leq0,$$
(43)

since the minimum values of $\frac{(\pi^2+a^2)^3}{a^2}$ and $\frac{(\pi^2+a^2)^2}{a^2}$ with respect to a^2 are $\frac{27\pi^4}{4}$ and $4\pi^2$, respectively.

Now, let $\sigma_{\!r} \geq \! 0,$ we necessarily have from inequality (43) that

$$\frac{1}{G-1} \frac{C_{\rm p} \alpha \kappa}{\nu} > \frac{4\pi^2}{P_l} \left(1 + \frac{27\pi^2 F}{16} \right). \tag{44}$$

Hence, if

$$\frac{1}{G-1} \frac{C_{\rm p} \alpha \kappa}{v} \le \frac{4\pi^2}{P_{\rm l}} \left(1 + \frac{27\pi^2 F}{16} \right),\tag{45}$$

then $\sigma_r < 0$. Therefore, the system is stable.

Therefore, under condition (45), the system is stable and under condition (44) the system becomes unstable.

Theorem 4: $Ep_1 > p_2$ and $Ep_1 > E'q$, are the sufficient conditions for the non-existence of overstability.

Proof: For overstability, we put $\frac{\sigma}{\pi^2} = i\sigma_1$ where σ_1 is real, (32) can be written as

$$R_{1} = \frac{G}{G-1} (1+x+iEp_{1}\sigma_{1}) \left[\left\{ \frac{i\sigma_{1}}{\varepsilon} + \frac{1}{P} (1+\pi^{2} + \frac{1}{P}) + \frac{1}{2} (1+x)(1+x+iE'q\sigma_{1}) + \frac{Q_{1}}{\varepsilon} (1+x)(1+x+iE'q\sigma_{1}) + S_{1}x(1+x+ip_{2}\sigma_{1}) + \frac{Q_{1}}{\varepsilon} (1+x+ip_{2}\sigma_{1})(1+x+iE'q\sigma_{1}) + S_{1}x(1+x+ip_{2}\sigma_{1}) \right]^{-1} \cdot \left[x(1+x+ip_{2}\sigma_{1})(1+x+iE'q\sigma_{1}) \right]^{-1} \cdot (46)$$

Since for overstability, we wish to determine the critical Rayleigh number for the onset of instability via a state of pure oscillations, it is suffice to find conditions for which (46) will admit of solutions with σ_1 real. Equating the real and imaginary parts of (46) and eliminating R_1 between them and setting $c_1 = \sigma_1^2$, b = 1 + x, we obtain

$$A_2c_1^2 + A_1c_1 + A_0 = 0, (47)$$

where

$$A_{2} = \frac{q^{2}p_{2}^{2}E'^{2}b^{2}}{\varepsilon} + \frac{q^{2}p_{1}p_{2}^{2}EE'^{2}b}{P}(1+\pi^{2}Fb),$$

$$A_{1} = \frac{q^{2}E'^{2}b^{3}(Ep_{1}-p_{2})+p_{1}p_{2}^{2}b^{3}E}{P}(q^{2}E'^{2}+P^{2})$$

$$\cdot \left\{b^{4}\left(\frac{1}{\varepsilon}+\frac{Ep_{1}\overline{1+\pi^{2}F}}{P}\right)\right\}\frac{Q_{1}}{\varepsilon}q^{2}E'^{2}b^{2}(Ep_{1}-p_{2})$$

$$+S_{1}b(b-1)p_{2}^{2}(Ep_{1}-E'q),$$

$$A_{0} = b^{5}\left[\frac{b}{\varepsilon}+\frac{Ep_{1}\overline{1+\pi^{2}Fb}}{P}\right]+\frac{F\pi^{2}b^{5}}{P}(b-1)(E'q+p_{2})$$

$$+\frac{Q_{1}}{\varepsilon}b^{4}(Ep_{1}-p_{2})+S_{1}xb^{3}(Ep_{1}-E'q).$$
(48)

Since σ_1 is real for overstability, both the values of $c_1(=\sigma_1^2)$ are positive. Equation (47) is quadratic in c_1 and does not involve any of its roots to be positive, if

$$Ep_1 > p_2 \text{ and } Ep_1 > E'q.$$
 (49)

Thus $Ep_1 > p_2$ and $Ep_1 > E'q$, are the sufficient conditions for the non-existence of overstability, the violation of which does not necessarily imply the occurrence of overstability.

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