# Group Invariant Solution for a Liquid Film on the Surface of a Sphere 

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When a circular jet of liquid strikes a sphere normal to the tangent plane at a point, a liquid film is formed on the surface of the sphere. This is a new problem. The flow in the liquid film is studied by means of boundary layer theory for laminar flow. The governing equations are Prandtl's momentum boundary layer equation and the continuity equation. To solve the problem completely a conserved quantity is required as well as boundary conditions. The conserved quantity for the film of liquid on the sphere is derived with the help of a conserved vector. Two conservation laws for the system have been obtained by Naeem and Naz (Int. J. Nonlin. Sci. 7, 149 (2009)), and one of these is used to derive the conserved quantity for the liquid film. A stream function is introduced which reduces the system to a single third-order partial differential equation. The group invariant solution for this partial differential equation is constructed by considering a linear combination of its Lie point symmetries. The velocity profile of the liquid film on the sphere is investigated.

Key words: Liquid Film; Blasius Boundary Layer; Group Invariant; Solution; Conserved Quantity; Free Surface.

## 1. Introduction

A new problem for liquid film flow on a body of revolution is formulated in this paper. The sphere is the body of revolution and a film of liquid is formed on its surface. Riley [1] considered the wall jet on a hemi-spherical shell and derived the similarity solution. Naeem and Naz [2] constructed the group invariant solution for the same problem and they showed that the similarity solution derived by Riley is the group invariant solution. An axisymmetric liquid film on the surface of a sphere is formed when a circular jet of liquid strikes the sphere normal to the tangent plane at that point and spreads over the surface. The difference between the liquid film and the wall jet is that the liquid film impacts directly on the surface of the sphere while the wall jet impacts on a layer of fluid at rest on the surface. Some of this fluid is entrained or carried along with the wall jet by viscous drag at the outer edge of the jet.

Prandtl [3] introduced the concept of a boundary layer in large Reynolds number flow in 1904, and he also showed how the Navier-Stokes equation could
be simplified to yield approximate solutions. Prandtl's boundary layer equations consist of the momentum balance equation which can be expressed in terms of a stream function and the continuity equation. The Navier-Stokes equation in cylindrical polar coordinates is used to derive the boundary layer equations for radial and axisymmetric jets. The radial jet is obtained if the primary motion of the jet is in the radial direction. When the primary motion of the jet is axially directed then an axisymmetric jet is formed. The boundary layer equations in cylinderical polar coordinates will be used in this paper.

Glauert [4] studied the problem of two-dimensional and radial wall jets. In [5], Watson studied the problem of two-dimensional and radial liquid jets. The problems of axisymmetric free and wall jets were attempted in [6] and [7]. In all these jet problems a conserved quantity was required in the solution process. The conserved quantity is used to determine the unknown exponent in the similarity solution which cannot be obtained from the boundary conditions because they are homogeneous. The conserved quantities for laminar jets
were established either from physical arguments or by integrating Prandtl's momentum boundary layer equation across the jet and using the boundary conditions and the continuity equation. Recently Naz et al. [8] presented a new method of constructing the conserved quantities for jet flows by using conservation laws. Conserved quantities for two-dimensional and radial jets were rederived.

The concept of the liquid jet was introduced by Watson [5]. He derived the similarity solution for twodimensional and radial liquid jets for the system consisting of the momentum and continuity equations. Riley [9] derived the similarity solution for the radial liquid jet by transforming the system to a single third-order partial differential equation for the stream function. The third-order partial differential equation was transformed to a third-order ordinary differential equation in terms of the similarity variables. The analytical solution of the third-order ordinary differential equation was constructed in [9]. Later, the symmetry solution for the third-order ordinary differential equation was derived [10]. By using certain transformations for the two-dimensional liquid jet the same third-order ordinary differential equation can be obtained as for the radial liquid jet.
The liquid film on the surface of a sphere studied here also falls into the category of flows which need a conserved quantity to complete the solution. The conserved quantity for the liquid film will be derived by using a conserved vector. The boundary layer equations for the liquid film are the same as derived by Riley [1] for the wall jet on a hemi-spherical shell. The conservation laws for these boundary layer equations were derived in [2] by utilizing the variational derivative approach [11-14]. Two conservation laws were obtained for the system of equations for the velocity components and one of these is used here to construct the conserved quantity for the liquid film on the sphere. The stream function is introduced to transform the system of equations for the velocity components to a single third-order partial differential equation for the stream function. The Lie-point symmetry generators for this third-order partial differential equation were constructed in [2]. Using the approach introduced by Kara and Mahomed [15], we find the symmetry associated with the conserved vector and this is used to derive the conserved quantity for the liquid jet. This symmetry also generates the group invariant solution [16]. It is interesting that the third-order partial differential equation derived here for the liquid film on the sphere transforms to the
same third-order ordinary differential equation which arises in two-dimensional and radial laminar liquid jets. Therefore, the velocity profile plotted against the similarity variable for a film of liquid on a sphere is the same as the velocity profile for two-dimensional and radial liquid jets.
The detailed outline of the paper is as follows: In Section 2, the mathematical formulation for a liquid film on the surface of a sphere is presented. The conserved quantity for the liquid film is derived with the help of a conservation law. The group invariant solution for the liquid film is derived in Section 3. In Section 4, the analysis of the results is presented. Finally the conclusions are summarized in Section 5.

## 2. Mathematical Formulation

An axisymmetric circular jet of liquid strikes a sphere normal to the tangent plane at point $O$ on the surface and spreads out over the surface as shown in Figure 1. The point $O$ is a stagnation point. It is assumed that the Reynolds number of the impinging jet is sufficiently high that the stagnation region in the neighbourhood of $O$ is inviscid. A Blasius boundary layer forms in the stagnation region $O A$ in Figure 1 and is matched to the inviscid flow in the impinging jet. The thickness of the boundary layer grows downstream from $O$ until it fills the whole of the liquid film and the entire flow is of boundary layer type. This occurs at point $A$ in Figure 1. Downstream of $A$ the outer edge of the boundary layer is a free surface. The surrounding fluid is a gas and the shearing stress vanishes at the free surface. The liquid film we are concerned with in this paper is the region downstream of the point $A$.
Consider cylindrical polar coordinates $(x, \theta, y)$ with origin at the stagnation point $O$ where $x$ and $y$ are measured along and normal to the surface of the sphere. All fluid variables are independent of $\theta$ and $x=0$ is the axis of symmetry. The surface of the sphere is at $y=0$ and the free surface is at $y=\phi(x)$. The fluid in the film is viscous and incompressible and the flow is steady. The radius of the sphere is $a$. Surface tension and gravity are neglected. Prandtl's boundary layer equations for a steady, incompressible, viscous fluid film on the surface of a sphere of radius $a$ are [1]

$$
\begin{align*}
& u u_{x}+v u_{y}=v u_{y y}  \tag{1}\\
& {\left[a \sin \left(\frac{x}{a}\right) u\right]_{x}+\left[a \sin \left(\frac{x}{a}\right) v\right]_{y}=0,} \tag{2}
\end{align*}
$$



Fig. 1. Impinging circular jet at point $O$ on the surface of a sphere of radius $a$. The Blasius-type boundary layer is the region $O A$ and the liquid film is the region downstream from the point $A$. (Adapted from Middleman [17].)
where $u(x, y)$ and $v(x, y)$ are the velocity components in the $x$ - and $y$-directions, respectively, and $v$ is the kinematic viscosity of the fluid. Here $a \sin (x / a)$ is the perpendicular distance of a point on the surface of the sphere from the axis of symmetry. The velocity component $v(x, \phi(x))$ is

$$
\begin{equation*}
v(x, \phi(x))=\frac{\mathrm{D}}{\mathrm{D} t}[\phi(x)]=u(x, \phi(x)) \frac{\mathrm{d} \phi(x)}{\mathrm{d} x}, \tag{3}
\end{equation*}
$$

where

$$
\begin{equation*}
\frac{\mathrm{D}}{\mathrm{D} t}=\frac{\partial}{\partial t}+u(x, \phi(x)) \frac{\partial}{\partial x}+v(x, \phi(x)) \frac{\partial}{\partial y} \tag{4}
\end{equation*}
$$

is the material time derivative.
Since the liquid film is viscous there is no slip at the surface of the sphere and therefore $u(x, 0)=0$. There is no suction or blowing of fluid at the solid boundary $y=0$, thus $v(x, 0)=0$. The boundary condition on the free surface of the film, $y=\phi(x)$, is that there is no shear stress along the free surface. In the boundary layer approximation this boundary condition
is $u_{y}(x, \phi(x))=0$. Thus the boundary conditions for the liquid film are

$$
\begin{equation*}
u(x, 0)=0, v(x, 0)=0, u_{y}(x, \phi(x))=0 . \tag{5}
\end{equation*}
$$

The three boundary conditions in (5) are homogeneous. This is an indication that a further condition is required to determine all the unknowns in the group invariant solution. In addition, the strength of the fluid flow in the liquid film has yet to be specified which can be done through a conserved quantity. We will derive the conserved quantity for the liquid film by the new method introduced in [8].

### 2.1. Conserved Quantity for a Liquid Film on a Sphere

The conserved vectors for the system of differential equations (1)-(2) derived in [2] are
$T^{1}=a \sin \left(\frac{x}{a}\right) u, T^{2}=a \sin \left(\frac{x}{a}\right) v$,
$T^{1}=a \sin \left(\frac{x}{a}\right) u^{2}, T^{2}=a \sin \left(\frac{x}{a}\right)\left[u v-v u_{y}\right]$.

The conserved vector (6) will give the conserved quantity for a liquid film on a sphere. For the conserved vector (6), we have

$$
\begin{equation*}
\mathrm{D}_{x} T^{1}+\mathrm{D}_{y} T^{2}=\frac{\partial T^{1}}{\partial x}+\frac{\partial T^{2}}{\partial y} . \tag{8}
\end{equation*}
$$

But $\mathrm{D}_{x} T^{1}+\mathrm{D}_{y} T^{2}=0$ and therefore (8) yields

$$
\begin{equation*}
\frac{\partial T^{1}}{\partial x}+\frac{\partial T^{2}}{\partial y}=0 \tag{9}
\end{equation*}
$$

The conserved quantity for the liquid film is obtained by integrating (9) with respect to $y$ from $y=0$ to $y=\phi(x)$ keeping $x$ fixed. For the conserved vector (6), we obtain

$$
\begin{align*}
\int_{0}^{\phi(x)} & {\left[\frac{\partial}{\partial x}\left(a \sin \left(\frac{x}{a}\right) u(x, y)\right)\right.} \\
& \left.+\frac{\partial}{\partial y}\left(a \sin \left(\frac{x}{a}\right) v(x, y)\right)\right] \mathrm{d} y=0 . \tag{10}
\end{align*}
$$

Using the formula for differentiation under the integral sign [18], we have

$$
\begin{align*}
& \frac{\mathrm{d}}{\mathrm{~d} x} \int_{0}^{\phi(x)}\left[a \sin \left(\frac{x}{a}\right) u(x, y)\right] \mathrm{d} y-a \sin \left(\frac{x}{a}\right) \\
& \cdot u(x, \phi(x)) \frac{\mathrm{d} \phi(x)}{\mathrm{d} x}+\left[a \sin \left(\frac{x}{a}\right) v(x, y)\right]_{0}^{\phi(x)}=0 \tag{11}
\end{align*}
$$

The boundary condition (5) for $v(x, 0)$ and expression (3) for $v(x, \phi(x))$, reduce (11) to
$\int_{0}^{\phi(x)} a \sin \left(\frac{x}{a}\right) u(x, y) \mathrm{d} y=$ const. independent of $x$.

Thus the total volume flux, $2 \pi F$, where

$$
\begin{equation*}
F=\int_{0}^{\phi(x)} a \sin \left(\frac{x}{a}\right) u(x, y) \mathrm{d} y \tag{13}
\end{equation*}
$$

is constant along the film. It is the conserved quantity for the liquid film on the sphere. The constant $F$ is given and describes the strength of the liquid film.

Introducing the stream function $\psi(x, y)$ defined by

$$
\begin{equation*}
u=\frac{1}{a \sin \left(\frac{x}{a}\right)} \psi_{y}, v=-\frac{1}{a \sin \left(\frac{x}{a}\right)} \psi_{x} \tag{14}
\end{equation*}
$$

we see that (2) is identically satisfied while (1) becomes
$\frac{1}{a \sin \left(\frac{x}{a}\right)} \psi_{y} \psi_{x y}-\frac{\cos \left(\frac{x}{a}\right)}{\left[a \sin \left(\frac{x}{a}\right)\right]^{2}} \psi_{y}^{2}$
$-\frac{1}{a \sin \left(\frac{x}{a}\right)} \psi_{x} \psi_{y y}-v \psi_{y y y}=0$.
The boundary conditions (5) and conserved quantity (13) in terms of the stream function become
$\psi_{x}(x, 0)=0, \psi_{y}(x, 0)=0, \psi_{y y}(x, \phi(x))=0$,
and
$F=\int_{0}^{\phi(x)} \psi_{y}(x, y) \mathrm{d} y$.
Since $\psi_{x}(x, 0)=0$, it follows that $\psi(x, 0)=\psi_{0}$ where $\psi_{0}$ is a constant which we choose to be zero. Thus
$\psi(x, 0)=0$,
and the conserved quantity (16) becomes
$F=\psi_{y}(x, \phi(x))=$ const. independent of $x$.
The free surface $y=\phi(x)$ is a streamline and (19) is the well known result that the stream function is constant along a streamline.

## 3. Group Invariant Solution for Film of Liquid on a Sphere

The Lie point symmetry generator for the third-order partial differential equation (15) is [2]
$X=\left[\frac{c_{1}}{\sin ^{2}\left(\frac{x}{a}\right)}\left(\frac{x}{2}-\frac{a}{4} \sin \left(\frac{2 x}{a}\right)\right)+\frac{c_{2}}{\sin ^{2}\left(\frac{x}{a}\right)}\right] \frac{\partial}{\partial x}$
$+\left[y\left(-c_{3}-\frac{c_{2} \cos \left(\frac{x}{a}\right)}{a \sin ^{3}\left(\frac{x}{a}\right)}+\frac{c_{1}}{2 a \sin ^{3}\left(\frac{x}{a}\right)}\right.\right.$
$\left.\cdot\left(-x \cos \left(\frac{x}{a}\right)+a \sin \left(\frac{x}{a}\right)+a \sin ^{3}\left(\frac{x}{a}\right)\right)\right)$
$+k(x)] \frac{\partial}{\partial y}+\left[c_{3} \psi+c_{4}\right] \frac{\partial}{\partial \psi}$,
where $c_{1}, c_{2}, c_{3}$, and $c_{4}$, are constants, and $k(x)$ is an arbitrary function. The symmetry associated with the conserved vector (6) will give the group invariant solution. The symmetries associated with a known conserved vector can be determined by using the result due to Kara and Mahomed [15]

$$
\begin{equation*}
X^{[m]}\left(T^{i}\right)+\mathrm{D}_{k}\left(\xi^{k}\right) T^{i}-\mathrm{D}_{k}\left(\xi^{i}\right) T^{k}=0 \tag{21}
\end{equation*}
$$

where $X^{[m]}$ is the $m$ th prolongation of $X$ if the components $T^{i}$ depend upon $m$ th derivatives. Equation (21) results in the following two equations:

$$
\begin{align*}
& X^{[2]}\left(T^{1}\right)+T^{1} \mathrm{D}_{y}\left(\xi^{2}\right)-T^{2} \mathrm{D}_{y}\left(\xi^{1}\right)=0,  \tag{22}\\
& X^{[2]}\left(T^{2}\right)+T^{2} \mathrm{D}_{x}\left(\xi^{1}\right)-T^{1} \mathrm{D}_{x}\left(\xi^{2}\right)=0, \tag{23}
\end{align*}
$$

where $\mathrm{D}_{x}$ and $\mathrm{D}_{y}$ are the total derivative operators defined by
$\mathrm{D}_{x}=\frac{\partial}{\partial x}+\psi_{x} \frac{\partial}{\partial \psi}+\psi_{x x} \frac{\partial}{\partial \psi_{x}}+\psi_{x y} \frac{\partial}{\partial \psi_{y}}+\ldots$,
$\mathrm{D}_{y}=\frac{\partial}{\partial y}+\psi_{y} \frac{\partial}{\partial \psi}+\psi_{y y} \frac{\partial}{\partial \psi_{y}}+\psi_{y x} \frac{\partial}{\partial \psi_{x}}+\ldots$.
The conserved vector (6) in terms of the stream function becomes

$$
\begin{equation*}
T^{1}=\psi_{y}, T^{2}=-\psi_{x} \tag{26}
\end{equation*}
$$

We will find the symmetries associated with the conserved vector (26). Equations (22) and (23) yield

$$
\begin{equation*}
c_{3} T^{1}=0, c_{3} T^{2}=0 \tag{27}
\end{equation*}
$$

which is satisfied if and only if $c_{3}=0$. Thus,
$X=\left[\frac{c_{1}}{\sin ^{2}\left(\frac{x}{a}\right)}\left(\frac{x}{2}-\frac{a}{4} \sin \left(\frac{2 x}{a}\right)\right)+\frac{c_{2}}{\sin ^{2}\left(\frac{x}{a}\right)}\right] \frac{\partial}{\partial x}$
$+\left[y\left(-\frac{c_{2} \cos \left(\frac{x}{a}\right)}{a \sin ^{3}\left(\frac{x}{a}\right)}+\frac{c_{1}}{2 a \sin ^{3}\left(\frac{x}{a}\right)}\left(-x \cos \left(\frac{x}{a}\right)\right.\right.\right.$
$\left.\left.\left.+a \sin \left(\frac{x}{a}\right)+a \sin ^{3}\left(\frac{x}{a}\right)\right)\right)+k(x)\right] \frac{\partial}{\partial y}+c_{4} \frac{\partial}{\partial \psi}$
is the symmetry associated with the conserved vector (26) and will be used to derive the group invariant solution.

Now, $\psi=\Psi(x, y)$ is a group invariant solution of the third-order partial differential equation (15) if

$$
\begin{equation*}
\left.X(\psi-\Psi(x, y))\right|_{\psi=\Psi=0} \tag{29}
\end{equation*}
$$

where the operator $X$ is given by (28). Equation (29) becomes

$$
\begin{align*}
& {\left[\frac{c_{1}}{\sin ^{2}\left(\frac{x}{a}\right)}\left(\frac{x}{2}-\frac{a}{4} \sin \left(\frac{2 x}{a}\right)\right)+\frac{c_{2}}{\sin ^{2}\left(\frac{x}{a}\right)}\right] \Psi_{x}}  \tag{30}\\
& +\left[\frac { y } { a \operatorname { s i n } ^ { 3 } ( \frac { x } { a } ) } \left(-c_{2} \cos \left(\frac{x}{a}\right)+\frac{c_{1}}{2}\left(-x \cos \left(\frac{x}{a}\right)\right.\right.\right.  \tag{39}\\
& \left.\left.\left.+a \sin \left(\frac{x}{a}\right)+a \sin ^{3}\left(\frac{x}{a}\right)\right)\right)+k(x)\right] \Psi_{y}=c_{4} \Psi
\end{align*}
$$

Letting $\eta=\frac{4 A}{3 a v} \chi$ and $g=A f$ in (38), we obtain

$$
f^{\prime \prime \prime}+3 f^{\prime 2}=0
$$

where the prime denotes differentiation with respect to $\eta$ and $A$ is an arbitrary constant which is fixed later.

The boundary conditions (16) and conserved quantity (19) become
$f(0)=0, f^{\prime}(0)=0, f^{\prime \prime}(c(x))=0$,
$F=A f(c(x))=$ const. independent of $x$,
where
$c(x)=\frac{4 A}{3 a^{2} v} \frac{\phi(x) \sin \left(\frac{x}{a}\right)}{\left[\frac{2 x}{a}-\sin \left(\frac{2 x}{a}\right)+\frac{4 c_{2}}{a c_{1}}\right]}$.
Since $f(\eta)$ is not a constant function it follows from (41) that $c(x)$ must be a constant, which we take to be unity. Equations (36) and (37) yield

$$
\begin{align*}
& \psi(x, y)=A f(\eta),  \tag{43}\\
& \eta=\frac{4 A}{3 a^{2} v} \frac{y \sin \left(\frac{x}{a}\right)}{\left[\frac{2 x}{a}-\sin \left(\frac{2 x}{a}\right)+\frac{4 c_{2}}{a c_{1}}\right]} \tag{44}
\end{align*}
$$

and the conserved quantity (41) becomes

$$
\begin{equation*}
F=A f(1), \tag{45}
\end{equation*}
$$

where $F$ is a given constant and the boundary conditions (40) simplify to

$$
\begin{equation*}
f(0)=0, f^{\prime}(0)=0, f^{\prime \prime}(1)=0 \tag{46}
\end{equation*}
$$

The equation of free surface from (42) is

$$
\begin{equation*}
\phi(x)=\frac{3 a^{2} v}{4 A} \frac{\left[\frac{2 x}{a}-\sin \left(\frac{2 x}{a}\right)+\frac{4 c_{2}}{a c_{1}}\right]}{\sin \left(\frac{x}{a}\right)} . \tag{47}
\end{equation*}
$$

In $[9,10]$, (39) was solved subject to conditions (46), (45), and the condition $f^{\prime}(1)=1$ which fixes the arbitrary constant $A$. Equation (39) yields (see [9, 10])

$$
\begin{equation*}
f^{\prime \prime}=\left[2\left(k_{1}-f^{\prime 3}\right)\right]^{\frac{1}{2}} \tag{48}
\end{equation*}
$$

The boundary condition $f^{\prime \prime}(1)=0$ and the chosen condition $f^{\prime}(1)=1$ give $k_{1}=1$. Defining $t=f^{\prime}$, (48) becomes

$$
\begin{equation*}
\frac{\mathrm{d} t}{\mathrm{~d} \eta}=\left[2\left(1-t^{3}\right)\right]^{\frac{1}{2}} \tag{49}
\end{equation*}
$$

The solution of (49) is [10]

$$
\begin{equation*}
-\frac{2}{3}\left(1-t^{3}\right)^{\frac{1}{2}} \times{ }_{2} F_{1}\left[\frac{1}{2}, \frac{2}{3}, \frac{3}{2}, 1-t^{3}\right]=\sqrt{2} \eta+k_{2} \tag{50}
\end{equation*}
$$

where ${ }_{2} F_{1}$ is the hypergeometric function of the first kind and $k_{2}$ is an arbitrary constant. The boundary condition $f^{\prime}(0)=0=t(0)$ gives the constant $k_{2}$, and we obtain

$$
\begin{align*}
\eta= & \frac{2}{3 \sqrt{2}}\left({ }_{2} F_{1}\left[\frac{1}{2}, \frac{2}{3}, \frac{3}{2}, 1\right]-\left(1-t^{3}\right)^{\frac{1}{2}}\right. \\
& \left.\times{ }_{2} F_{1}\left[\frac{1}{2}, \frac{2}{3}, \frac{3}{2}, 1-t^{3}\right]\right) . \tag{51}
\end{align*}
$$

Since $f^{\prime}=t$, (51) gives the solution for $f^{\prime}(\eta)$ in parametric form with parameter $t$. The velocity component $u(x, y)$ is proportional to $f^{\prime}(\eta)$.

The constant $A$ is given in terms of $F$ by (45). Now, using (49) [5]

$$
\begin{align*}
f(1) & =\int_{0}^{1} f^{\prime}(\eta) \mathrm{d} \eta=\frac{1}{\sqrt{2}} \int_{0}^{1} t\left(1-t^{3}\right)^{-1 / 2} \mathrm{~d} t  \tag{52}\\
& =\frac{\pi}{3 \sqrt{3}},
\end{align*}
$$

and hence

$$
\begin{equation*}
A=\frac{3 \sqrt{3} F}{\pi} \tag{53}
\end{equation*}
$$

The final form of the group invariant solution is

$$
\begin{equation*}
\psi=\frac{3 \sqrt{3} F}{\pi} f(\eta) \tag{54}
\end{equation*}
$$

where

$$
\begin{equation*}
\eta=\frac{4 \sqrt{3} F}{\pi v a^{2}}\left[\frac{\sin \left(\frac{x}{a}\right)}{\left(\frac{2 x}{a}-\sin \left(\frac{2 x}{a}\right)+\frac{4 c_{2}}{a c_{1}}\right)} y\right] . \tag{55}
\end{equation*}
$$

From (47) the equation of free surface is

$$
\begin{equation*}
\phi(x)=\frac{\pi v a^{2}}{4 \sqrt{3} F} \frac{\left[\frac{2 x}{a}-\sin \left(\frac{2 x}{a}\right)+\frac{4 c_{2}}{a c_{1}}\right]}{\sin \left(\frac{x}{a}\right)}, \tag{56}
\end{equation*}
$$

and from (14),
$u(x, y)=\frac{36 F^{2}}{v a^{3} \pi^{2}\left(\frac{2 x}{a}-\sin \left(\frac{2 x}{a}\right)+\frac{4 c_{2}}{a c_{1}}\right)} f^{\prime}(\eta)$.

## 4. Analysis of the Results

The fluid velocity, given by (57), is proportional to $f^{\prime}(\eta)$. From (51) we may tabulate the values of $\eta$ for given values of the parameter $t=f^{\prime}$. The behaviour of the velocity function $f^{\prime}$ is shown in Figure 2 and is


Fig. 2. Graph of $f^{\prime}(\eta)$ against $\eta$.
the same as for the two-dimensional and radial liquid jets $[9,10]$. There are two cases to consider, $\frac{c_{2}}{c_{1}}=0$ and $\frac{c_{2}}{c_{1}}>0$.

Case i: $\frac{c_{2}}{c_{1}}=0$.
When $\frac{c_{2}}{c_{1}}=0$, the equation of the free surface (56) and the velocity (57) reduce to

$$
\begin{equation*}
\phi(x)=\frac{\pi v a^{2}}{4 \sqrt{3} F}\left[\frac{\frac{2 x}{a}-\sin \left(\frac{2 x}{a}\right)}{\sin \left(\frac{x}{a}\right)}\right] \tag{58}
\end{equation*}
$$

and

$$
\begin{equation*}
u(x, y)=\frac{36 F^{2}}{v a^{3} \pi^{2}\left(\frac{2 x}{a}-\sin \left(\frac{2 x}{a}\right)\right)} f^{\prime}(\eta) \tag{59}
\end{equation*}
$$

The range of $x$ is $0 \leq \frac{x}{a} \leq \frac{\pi}{2}$ because the film of liquid will separate from the sphere for $\frac{x}{a}>\frac{\pi}{2}$. There is no pressure gradient to cause separation for $\frac{x}{a} \leq \frac{\pi}{2}$. Using L' Hopital's rule it follows from (58) and (59) that as $x$ tends to zero the thickness of the liquid film also tends to zero and the fluid velocity $u(x, y)$ tends to infinity


Fig. 3. Graph of the free surface of the liquid film when $\frac{c_{2}}{c_{1}}=0$.
in such a way that the flux of fluid remains finite and the conserved quantity is satisfied. It is readily verified that $\phi(x)$ is an increasing function of $x$ for $0 \leq \frac{x}{a} \leq \frac{\pi}{2}$. The thickness of the liquid film increases steadily from zero at $x=0$ to

$$
\begin{equation*}
\phi\left(\frac{\pi}{2}\right)=\frac{\pi^{2} v a^{2}}{4 \sqrt{3} F} \tag{60}
\end{equation*}
$$

at $\frac{x}{a}=\frac{\pi}{2}$. The graph of the free surface of a liquid film is shown in Figure 3. The thickness of the liquid film is directly proportional to the kinematic viscosity $v$ which causes diffusion of vorticity from the surface of the sphere and is inversely proportional to the strength of the film, $F$, which opposes the diffusion from the surface.

This solution does not describe the flow illustrated in Figure 1 but it may describe the flow due to a point source at $O$.

Case ii: $\frac{c_{2}}{c_{1}}>0$.
We will follow the approach of Watson [5] who related $c_{2} / c_{1}$ to a length for a radial liquid jet on a horizontal plane.

Let the radius of the incident jet be $b$ and the speed at which it strikes the sphere be $U_{0}$. The point $x=0$, $y=0$ is a stagnation point. Watson distinguished four regions of flow.

In Region 1, $x=O(b)$. The fluid outside the boundary layer is approximately inviscid and its velocity rises rapidly from zero at the stagnation point to $U_{0}$. In Region $2, x>O(b)$. The fluid velocity outside the boundary layer is now approximately $U_{0}$. Since the boundary layer is matched to the free stream $U_{0}$ it is a Blasius boundary layer with thickness

$$
\begin{equation*}
\delta(x)=O\left[\left(\frac{v x}{U_{0}}\right)^{1 / 2}\right] \tag{61}
\end{equation*}
$$

Let $h(x)$ be the total depth of the flow. Since $2 \pi F$ is the volume flow rate

$$
\begin{equation*}
2 \pi F=\pi b^{2} U_{0}=2 \pi a \sin \left(\frac{x}{a}\right) h(x) U_{0} \tag{62}
\end{equation*}
$$

and therefore, neglecting terms $O\left[\left(\frac{x}{a}\right)^{3}\right]$,

$$
\begin{equation*}
h(x)=O\left(\frac{b^{2}}{x}\right) \tag{63}
\end{equation*}
$$

In Region 2, $\delta(x)<h(x)$. In Region 3, the viscous stresses are important up to the free surface and
$\delta(x)=O(h(x))$. The boundary layer changes from Blasius to a liquid film on a sphere. In Region 4, the way the liquid film was formed is no longer important and the solution for the liquid film applies with a suitable choice for $c_{2} / c_{1}$.

From (61) and (63), the boundary layer thickness $\delta(x)$ becomes of order $h(x)$ when

$$
\begin{equation*}
x^{3}=O\left(\frac{b^{4} U_{0}}{v}\right) . \tag{64}
\end{equation*}
$$

The Reynolds number of the incident jet is

$$
\begin{equation*}
\operatorname{Re}=\frac{U_{0} b}{v} \tag{65}
\end{equation*}
$$

and expressing $U_{0}$ in terms of Re , (64) becomes

$$
\begin{equation*}
x=O\left(b \operatorname{Re}^{1 / 3}\right) . \tag{66}
\end{equation*}
$$

Thus in Region 3, (66) is satisfied and the speed of the film on the free surface is $O\left(U_{0}\right)$. But the velocity on the free surface of the liquid film is given by (57) with $\eta=1$. Since $f^{\prime}(1)=1$, (57) yields

$$
\begin{equation*}
O\left(U_{0}\right)=O\left[\frac{F^{2}}{v a^{3}\left(\left(\frac{x}{a}\right)^{3}+\frac{3}{a} \frac{c_{2}}{c_{1}}\right)}\right], \tag{67}
\end{equation*}
$$

where terms of $O\left(\left(\frac{x}{a}\right)^{5}\right)$ were neglected in the expansion of $\sin \left(\frac{x}{a}\right)$. Using (62) for $F$ and (65) for $U_{0}$, (67) becomes

$$
\begin{equation*}
\left(\frac{x}{a}\right)^{3}+\frac{3}{a} \frac{c_{2}}{c_{1}}=O\left(\left(\frac{b}{a}\right)^{3} \operatorname{Re}\right) . \tag{68}
\end{equation*}
$$

But in Region 3, (66) is satisfied and therefore

$$
\begin{equation*}
\frac{3}{a} \frac{c_{2}}{c_{1}}=O\left(\left(\frac{b}{a}\right)^{3} \operatorname{Re}\right) \tag{69}
\end{equation*}
$$

Introduce the length $l$ defined by

$$
\begin{equation*}
\frac{3}{a} \frac{c_{2}}{c_{1}}=\left(\frac{l}{a}\right)^{3} . \tag{70}
\end{equation*}
$$

Then from (69),

$$
\begin{equation*}
\frac{l}{a}=O\left(\frac{b}{a} \operatorname{Re}^{1 / 3}\right) . \tag{71}
\end{equation*}
$$

Equation (71) gives an order of magnitude for $\frac{l}{a}$ and was derived by considering the development of the flow from the impact point of the film. It is the same estimate


Fig. 4. Graph of the free surface $y=\phi(x)$ of the liquid film when $\left(\frac{l}{a}\right)^{3}=5 \times 10^{-2}$.
for $l$ as obtained by Watson [5] for an axisymmetric jet falling vertically on a horizontal plane. Expressed in terms of $\frac{l}{a}$, (56) and (57) become
$\phi(x)=\frac{\pi v a^{2}}{4 \sqrt{3} F}\left[\frac{\frac{2 x}{a}-\sin \left(\frac{2 x}{a}\right)+\frac{4}{3}\left(\frac{l}{a}\right)^{3}}{\sin \left(\frac{x}{a}\right)}\right]$,
$u(x, y)=\frac{36 F^{2}}{v a^{3} \pi^{2}\left(\frac{2 x}{a}-\sin \left(\frac{2 x}{a}\right)+\frac{4}{3}\left(\frac{l}{a}\right)^{3}\right)} f^{\prime}(\eta)$,
where

$$
\begin{equation*}
\eta=\frac{4 \sqrt{3} F}{\pi v a^{2}}\left[\frac{\sin \left(\frac{x}{a}\right)}{\left(\frac{2 x}{a}-\sin \left(\frac{2 x}{a}\right)+\frac{4}{3}\left(\frac{l}{a}\right)^{3}\right)} y\right] . \tag{74}
\end{equation*}
$$

The value of the film Reynolds number is assumed large because (66) holds only for $x \gg b$. A graph of $\phi(x)$ against $x$ for $\left(\frac{l}{a}\right)^{3}=5 \times 10^{-2}$ is plotted in Figure 4. In an actual flow the group invariant solution will only hold sufficiently far from the impact point for the conditions in the inner regions to no longer have an effect as illustrated in Figure 1.
As the radius $a$ of the sphere tends to infinity the results for an axisymmetric jet falling vertically on a horizontal plane are rederived. For expanding (72) - (74) for $\frac{x}{a}$ small gives to lowest order,

$$
\begin{align*}
& \phi(x)=\frac{\pi v}{3 \sqrt{3} F} \frac{\left(x^{3}+l^{3}\right)}{x}  \tag{75}\\
& u(x, y)=\frac{27 F^{2}}{v \pi^{2}} \frac{f^{\prime}(\eta)}{\left(x^{3}+l^{3}\right)} \tag{76}
\end{align*}
$$

where

$$
\begin{equation*}
\eta=\frac{3 \sqrt{3} F}{\pi v} \frac{x y}{\left(x^{3}+l^{3}\right)} . \tag{77}
\end{equation*}
$$

The results agree with the results of Watson [5] if $2 \pi F=Q$ and $f^{\prime}(\eta)$ is replaced by $f(\eta), y$ by $z$ and $x$ by $r$. The expansion of (58) and (59) for $\frac{x}{a}$ small is obtained by setting $l=0$ in (75) and (76).

## 5. Conclusions

The problem of flow in a film of liquid on a sphere was formulated. The conserved quantity for the liquid film was constructed with the help of a conserved vector. A symmetry was associated with the conserved vector that gave the conserved quantity for the liquid film. That symmetry was then used to construct the group invariant solution for the liquid film on the sphere. The velocity profile $f^{\prime}(\eta)$ plotted against $\eta$ for a liquid film on a sphere is the same as the velocity profile for twodimensional and radial liquid jets.
The group invariant solution contained one remaining constant $\frac{c_{2}}{c_{1}}$ after the boundary conditions and con-
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served quantity had been imposed. This constant depended on the conditions in the incident jet and its order of magnitude was obtained by considering the growth of the boundary layer from the impact point of the jet on the sphere as described by Watson [5] for the spreading of a film of liquid on a horizontal plane. We can expect the group invariant solution to only apply when $x$ is sufficiently large for the conditions in the inner regions to no longer affect the flow. We estimated this distance to be $O\left(b \operatorname{Re}^{1 / 3}\right)$ where $b$ is the radius and Re is the Reynolds number of the incident jet.

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