Variational Iteration Method for Nonlinear Age-Structured Population Models Using Auxiliary Parameter

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In this paper, we apply He's variational iteration method (VIM) coupled with an auxiliary parameter which proves very effective to control the convergence region of an approximate solution. Moreover, a convenient way is considered for choosing a suitable auxiliary parameter via a residual function. The proposed algorithm is tested on some nonlinear age-structured population models, and numerical results explicitly reveal the complete reliability, efficiency, and accuracy of the suggested technique. It is observed that the approach may be implemented on other nonlinear models of physical nature.

Key words: Variational Iteration Method; Auxiliary Parameter; Nonlinear Age-Structured Population Models.

1. Introduction

The rapid development of nonlinear sciences [1 – 28] witnesses of a number of new analytical and numerical methods. Most of these introduced techniques are coupled with the inbuilt deficiencies including calculation of the so-called Adomian's polynomials, divergent results, limited convergence, lengthy calculations, small parameter assumption, and noncompatibility with the physical nature of the problems. In order to cope with such deficiencies, He [5 – 14] developed the variational iteration method (VIM) which was completely independent of the above mentioned inadequacies and has been applied to a wide class of nonlinear problems, see [1 – 17, 28] and the references therein. The exponential success rate of He's VIM is a true index of its credibility. With the passage of time, several modifications have been introduced in He's method which has further enhanced the accuracy and efficacy of this algorithm to a tangible level.

In this paper, we will discuss how to solve the following nonlinear age-structured population model [21]:

$$\begin{aligned}
&\frac{\partial p(t,x)}{\partial t}+\frac{\partial p(t,x)}{\partial x}=-\left[\mathrm{d}_1(x)+\mathrm{d}_2(x)P(t)\right]p(t,x),\\
&t\geq 0,\quad 0\leq x<A,\\
&p(0,x)=P_0(x),\quad 0\leq x<A,\\
&p(t,0)=\int_a^A\left[b_1(\xi)-b_2(\xi)P(t)\right]p(t,\xi)\mathrm{d}\xi,\ \ t\geq 0,\\
&P(t)=\int_0^A p(t,x)\mathrm{d}x,\quad t\geq 0,
\end{aligned}\tag{1}$$

where t, x denote time and age, respectively, $P(t)$ denotes the total population number at time t, $p(t,x)$ is the age-specific density of individuals of age x at time t, which means that $\int_a^{a+\Delta a}p(t,x)\mathrm{d}x$ gives the number of individuals that have an age between a and $a+\Delta a$ at time t, $d_1(x)$ is the natural death rate (without considering competition), $d_2(x)P(t)$ is the increase of death rate considering competition, $b_1(x)$ is the natural fertility rate (without considering competition), $b_2(x)P(t)$ is the decrease of fertility rate considering competition, a denotes the lowest age when an individual can bear, and A is the maximum age that an individual of the population may reach.

In general, model (1) can be written as follows:

$$\begin{aligned}
&\frac{\partial p(t,x)}{\partial t}+\frac{\partial p(t,x)}{\partial x}=-\mu(x,I_\mu(t),t)p(t,x),\\
&t\geq 0,\quad 0\leq x<A,\\
&p(0,x)=p_0(x),\quad 0\leq x<A,\\
&p(t,0)=\int_a^A\beta(\xi,I_\beta(t),t)p(t,\xi)\mathrm{d}\xi,\quad t\geq 0,
\end{aligned}\tag{2}$$

where β and μ denote fertility and death rate, respectively, $I_\mu(t) = \int_a^A \gamma_\mu(x)p(t,x)\mathrm{d}x$, $I_\beta(t) = \int_a^A \gamma_\beta(x)$ $p(t,x)\mathrm{d}x$, and $\gamma_\mu(x)$, $\gamma_\beta(x)$ are weight functions.

If $\gamma_\mu(x) = \gamma_\beta(x) \equiv 1$, $\mu(x,z,t) = d_1(x) + d_2(x)z$, $\beta(x,z,t) = b_1(x) - b_2(x)z$, then (2) will become (1).

The presence of an integral term in a boundary condition can greatly complicate the application of standard numerical techniques such as finite difference, finite elements, spectral methods, and so on. Therefore, to apply them widely to problems of practical interests, in general, it is important to convert non-local boundary value problems into more desirable forms. However, this is hard work in many cases [28].

Thus, for nonlinear age-structured population models, there are few valid methods for solving them. Cui and Chen presented a reproducing kernel method [21], Abia and López-Marcos brought forward difference schemes based on the Runge-Kutta method [22 – 24], Iannelli and Kim introduced a spline algorithm [25], Kim and Park developed an upwind scheme [26], Krzyzanowski et al. gave a discontinuous Galerkin method for nonlinear age-structured population models [27], and Li applied VIM for solving nonlinear age-structured population model (1) [28].

Inspired and motivated by the ongoing research in this area, we propose an insertion of an unknown auxiliary parameter h into the correction functional of VIM for some nonlinear age-structured population models (1). Furthermore, a convenient way is considered for choosing a suitable auxiliary parameter. It is observed that the suggested coupling provides a simple way to adjust and control the convergence region of the approximate solution for any values of t and x. Numerical results explicitly reveal the complete reliability, efficiency, and accuracy of the suggested technique. It is observed that the approach may be implemented on other nonlinear models of physical nature. It is worth mentioning that Yilmaz and Inc [29] proposed a modified version of VIM using a very similar auxiliary parameter for the numerical simulation of squeezing flow between two infinite plates. Moreover, the details of use of such an auxiliary parameter are also summarized in [30]. It needs to be highlighted that the proposed variational iteration algorithm involving an auxiliary parameter [31, 32] is particularly suitable for inverse problems and differential-difference equations.

2. Analysis of the Proposed Algorithm

Consider the following functional equation [1 – 21, 24 – 26]:

$$Lu + Ru + Nu = g(x), \tag{3}$$

where L is the highest-order derivative that is assumed to be easily invertible, R is a linear differential operator of order less than L, Nu represents the nonlinear terms, and g is the source term. The basic characteristic of He's method is to construct a correction functional for (3), which reads

$$\begin{aligned} &u_{n+1}(t) = \\ &u_n(t) + \int_0^t \lambda(s)\{Lu_n(s) + R\tilde{u}_n + N\tilde{u}_n - g(s)\}\,\mathrm{d}s, \end{aligned} \tag{4}$$

where λ is a Lagrange multiplier which can be identified optimally via variational theory [5 – 14], u_n is the nth approximate solution, and $\tilde{u}_n$ denotes a restricted variation, i.e., $\delta\tilde{u}_n = 0$. To solve (3) by He's VIM, we first determine the Lagrange multiplier λ that can be identified optimally via variational theory. Then, the successive approximations $u_n(x)$, $n \geq 0$, of the solution $u(x)$ can be readily obtained upon using the obtained Lagrange multiplier and by using any selective function u_0. The zeroth approximation u_0 may be selected by any function that just satisfies at least the initial and boundary conditions. With λ determined, several approximations $u_n(x), u_n \geq 0$, follow immediately. Consequently, the exact solution may be obtained by using

$$u(x) = \lim_{n\to+\infty} u_n(x). \tag{5}$$

In summary, we have the following variational iteration formula for (3):

$$\begin{aligned} &u_0(x) \ \text{ is an arbitrary function,} \\ &u_{n+1}(x) = u_n(x) \\ &+ \int_0^x \lambda(s)\{Lu_n(s) + Ru_n(s) + Nu_n(s) - g(s)\}\mathrm{d}s,\ n \geq 0. \end{aligned} \tag{6}$$

2.1. Coupling of Auxiliary Parameter and Correction Functional

An unknown auxiliary parameter h can be inserted into the correction functional (6) of He's VIM. According to this assumption, we construct the following variational iteration formula:

$$u_0(x) \text{ is an arbitrary function,}$$
$$u_1(x,h) = u_0(x) + h\int_0^x \lambda(s)\{Lu_0(s)+Ru_0(s) + Nu_0(s) - g(s)\}\mathrm{d}s,\ n \geq 0, \quad (7)$$
$$u_{n+1}(x,h) = u_n(x,h) + h\int_0^x \lambda(s)\{Lu_n(s,h)+Ru_n(s,h) + Nu_n(s,h) - g(s)\}\mathrm{d}s,\ n \geq 1.$$

Of course, assuming the Lagrange multiplier λ has been already identified. It should be emphasized that $u_n(x,h)$, $n \geq 1$, can be computed by symbolic computation software such as Maple or Mathematica. The approximate solutions $u_n(x,h), n \geq 1$, contain the auxiliary parameter h. The validity of the method is based on such an assumption that the approximations $u_n(x,h)$, $u_n \geq 0$, converge to the exact solution $u(x)$. It is the auxiliary parameter h which ensures that the assumption can be satisfied. In general, by means of the so-called h-curve, it is straightforward to choose a proper value of h which ensures that the approximate solutions are convergent. In fact, the proposed combination is very simple, easier to implement, and is cable to approximate the solution more accurately in a bigger interval.

3. Numerical Examples

In this section, we apply He's VIM coupled with the unknown auxiliary parameter to solve two nonlinear age-structured population models. Numerical results are compared with those achieved by the original variational iteration method (VIM).

Example 3.1 Consider the following nonlinear age-structured population model [21, 28]:

$$\frac{\partial p(t,x)}{\partial t} + \frac{\partial p(t,x)}{\partial x} = -P(t)p(t,x),$$
$$t \geq 0, \quad 0 \leq x < A,$$
$$p(0,x) = \frac{\mathrm{e}^{-x}}{2}, \quad 0 \leq x < A, \quad (8)$$
$$p(t,0) = P(t), \quad t \geq 0,$$
$$P(t) = \int_0^A p(t,x)\mathrm{d}x \quad t \geq 0,$$

where $A = +\infty$. And it is easy to verify that $p(t,x) = \frac{\mathrm{e}^{-x}}{1+\mathrm{e}^{-t}}$, $t \geq 0$, $x \geq 0$, is the exact solution of (8).

Solution: Take $\Omega = [0,10]\times[0,100]$, which denotes $(10 \text{ unit time}) \times (100 \text{ unit age})$. According to the original VIM, we have the following variational iteration formula:

$$p_{n+1}(t,x) = p_n(t,x) - \int_0^t \left\{\frac{\partial p_n(s,x)}{\partial s} + \frac{\partial p_n(s,x)}{\partial x} + p_n(s,x)\int_0^A p_n(s,x)\mathrm{d}x\right\}\mathrm{d}s, \quad n \geq 0. \quad (9)$$

Figure 1 shows the absolute error of $p_4(t,x)$ by the original VIM, when $p_0(t,x) = p(0,x) = \frac{\mathrm{e}^{-x}}{2}$, which confirms that the obtained results by original VIM are not valid for large values of t and x in Example 3.1. Now, using the recursive scheme (7) and by selecting

$$p_0(t,x) = p(0,x) = \frac{\mathrm{e}^{-x}}{2},$$

we successively obtain

$$p_1(t,x,h) = \frac{\mathrm{e}^{-x}}{2} + h\frac{t\mathrm{e}^{-x}}{4},$$

and, in general,

$$p_{n+1}(t,x,h) = p_n(t,x,h) - h\int_0^t \left\{\frac{\partial p_n(s,x,h)}{\partial s} + \frac{\partial p_n(s,x,h)}{\partial x} + p_n(s,x,h)\int_0^A p_n(s,x,h)\mathrm{d}x\right\}\mathrm{d}s, \quad (10)$$
$$n \geq 1.$$

First, to find the proper value of h for the approximate solutions (10), we plot the so-called h-curve of $\frac{\partial}{\partial t}p_4(t,x,h)$ for the case $t = 0$ and $x = 0$ as shown in Figure 2. According to this h-curve, it is easy to discover the valid region of h, which corresponds to the line segment nearly parallel to the horizontal axis. Furthermore, according to (8) we define the residual func-

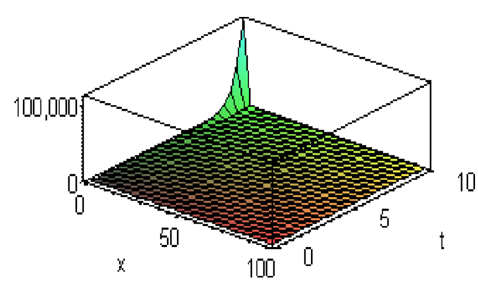

Fig. 1 (colour online). Absolute error for the 4th-order approximation by original VIM for $p(t,x)$.

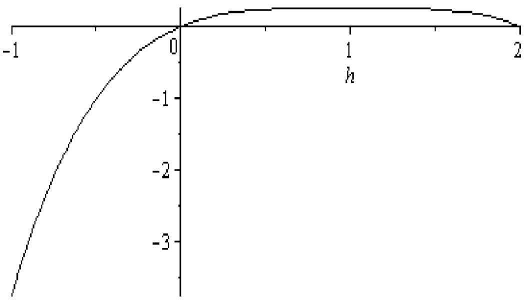

Fig. 2. h-curve of $\frac{\partial}{\partial t} p_4(t,x,h)$ given by (10) when $t = 0$ and $x = 0$.

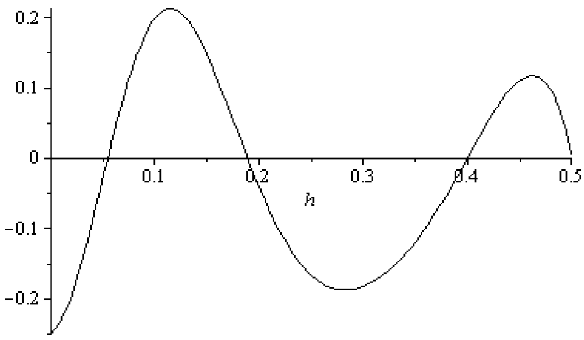

Fig. 3. h-curve of $r_4(t,x,h)$ given by (11) when $t = 10$ and $x = 0$.

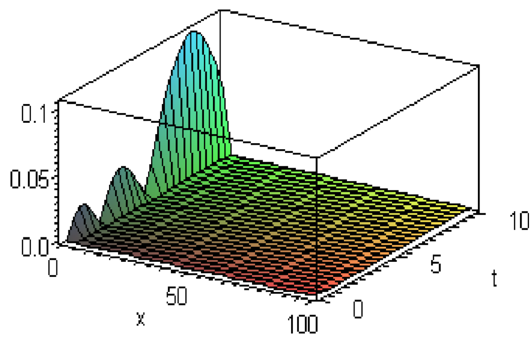

Fig. 4 (colour online). Absolute error for the 4th-order approximation by the coupled form of VIM for $p(t,x)$ and $h = 0.4$.

tion,

$$r_4(t,x,h) = \frac{\partial p_4(t,x,h)}{\partial t} + \frac{\partial p_4(t,x,h)}{\partial x} + p_4(t,x,h) \int_0^A p_4(s,x,h)\mathrm{d}x \tag{11}$$

for obtaining a suitable value of h. Figure 3 shows $r_4(10,0,h)$.

According to Figures 2 and 3, we select $h = 0.4$. The absolute error of the coupling of VIM and auxiliary parameter results for $p_4(t,x)$, $(t,x) \in [0,10] \times [0,100]$ is plotted in Figure 4.

It must be noted that the same situation will be occurre if we select $h = 0.05$, $h = 0.2$ or $h = 0.5$.

Example 3.2 Consider the following nonlinear age-structured population model [25, 28]:

$$\begin{aligned}
&\frac{\partial p(t,x)}{\partial t} + \frac{\partial p(t,x)}{\partial x} = -(P(t)+1)p(t,x),\\
&t \geq 0, \quad 0 \leq x < A,\\
&p(0,x) = \frac{\mathrm{e}^{-x}}{2}, \quad 0 \leq x < A,\\
&p(t,0) = P(t), \quad t \geq 0,\\
&P(t) = \int_0^A p(t,x)\mathrm{d}x, \quad t \geq 0,
\end{aligned} \tag{12}$$

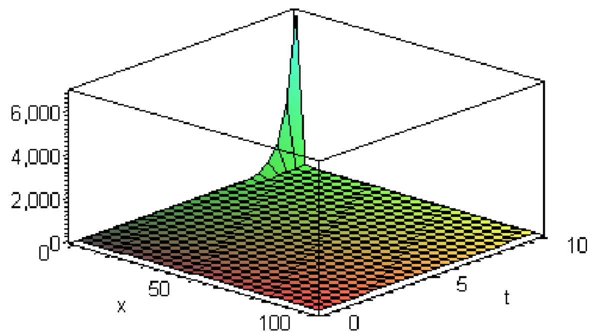

Fig. 5 (colour online). Absolute error for the 4th-order approximation by original VIM for $p(t,x)$.

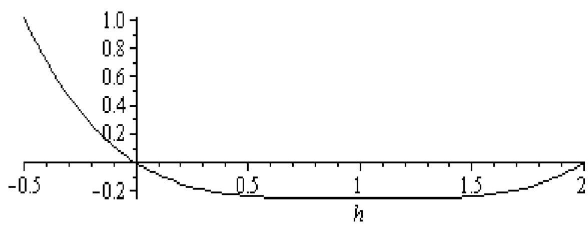

Fig. 6. h-curve of $\frac{\partial}{\partial t} p_4(t,x,h)$ when $t = 0$ and $x = 0$.

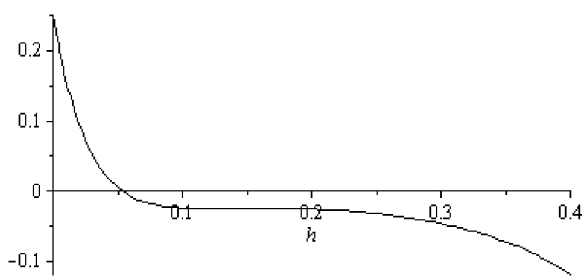

Fig. 7. h-curve of $r_4(t,x,h)$ when $t = 10$ and $x = 0$.

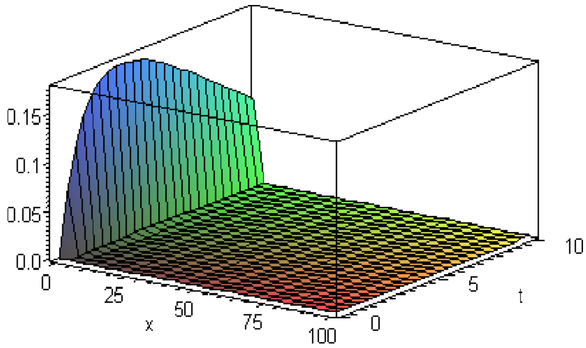

Fig. 8 (colour online). Absolute error for the 4th-order approximation by the coupled form of VIM for $p(t,x)$ and $h = 0.05$.

where $A = +\infty$. And it is easy to verify that $p(t,x) = \frac{e^{-x}}{2+t}$, $t \geq 0$, $x \geq 0$, is the exact solution of (12).

Solution: Take $\Omega = [0,10] \times [0,100]$, which denotes $(10 \text{ unit time}) \times (100 \text{ unit age})$. Proceeding as before, the absolute error of $p_4(t,x)$, $(t,x) \in [0,10] \times [0,100]$, by the original VIM is plotted in Figure 5 which confirms that the obtained results by original VIM is not valid for large values of t and x in Example 3.2. Now, we plot the h-curve of $\frac{\partial}{\partial t} p_4(t,x,h)$ for the case $t = 0$ and $x = 0$, as shown in Figure 6, and according to (12), we define the residual function,

$$r_4(t,x,h) = \frac{\partial p_4(t,x,h)}{\partial t} + \frac{\partial p_4(t,x,h)}{\partial x} + p_4(t,x,h)\left(1 + \int_0^A p_4(s,x,h)\mathrm{d}x\right). \quad (13)$$

Figure 7 shows the h-curve for $r_3(10,0,h)$.

By considering Figures 6 and 7, we select $h = 0.05$. The absolute error of the coupling of VIM and auxiliary parameter results for $p_4(t,x)$, $(t,x) \in [0,10] \times [0,100]$ is plotted in Figure 8.

4. Conclusion

In this paper, we coupled an unknown auxiliary parameter in the correction functional of He's VIM for some nonlinear age-structured population models. The suitable auxiliary parameter can be obtained by the related h-curve and the residual function. Numerical results and graphical representations explicitly reveal the complete reliability of the proposed method. It is observed that the used coupling can be very effective in solving complicated nonlinear problems of physical nature.

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