Magnetohydrodynamic Flow and Mass Transfer of a Jeffery Fluid over a Nonlinear Stretching Surface

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This paper investigates the magnetohydrodynamic (MHD) boundary layer flow of a Jeffery fluid induced by a nonlinearly stretching sheet with mass transfer. The relevant system of partial differential equations has been reduced into ordinary differential equations by employing the similarity transformation. Series solutions of velocity and concentration fields are developed by using the homotopy analysis method (HAM). Effects of the various parameters such as Hartman number, Schmidt number, and chemical reaction parameter on velocity and concentration fields are discussed by presenting graphs. Numerical values of the mass transfer coefficient are also tabulated and analyzed.

Key words: Jeffery Fluid; Chemical Reaction; Series Solutions.

1. Introduction

In view of technological and industrial applications, the boundary layer flow of non-Newtonian fluids have been given considerable attention in recent years. Examples of non-Newtonian behaviour can be found in on processes for manufacturing coated sheets, foods, optical fibers, drilling muds, and plastic polymers. One of the non-Newtonian fluid models which has been accorded much attention by the investigators is the Jeffery fluid. This fluid model allows relaxation and retardation effects. The investigation of the boundary layer flows over a stretching surface has many important applications in extrusion processes, glass-fiber and paper production, electronic chips, crystal growing etc. Sakiadis [1] initiated the boundary layer flow over a continuous solid surface moving with constant speed. The work of Sakiadis was subsequently extended by many authors for boundary layer flows in viscous and non-Newtonian fluids boundary layer flows under varied conditions. Some of the recent contributions on the topic have been presented in the studies [2 – 10].

All the above mentioned studies take into account flows caused by a linear stretching sheet. Some investigations may be mentioned in this directions [11 – 15]. Recently, Raptis and Perdikis [16] have examined the MHD viscous flow over a nonlinear stretching sheet in the presence of a chemical reaction. The objective of present study is (i) to extend the flow analysis of [16] from viscous to Jeffery fluids and (ii) to provide an analytic solution for a highly nonlinear problem. A new developed technique, the homotopy analysis method (HAM) [17], has been employed for the analytic solution. This method has been successfully applied already for other problems [18 – 33]. Very recently, the optimal approach based on HAM is also presented. The readers may consult the studies [34, 35] in this direction. The structure of the paper is as follows. The formulation of the problem is described in Section 2. Series solutions of velocity and concentration are derived in Section 3. In Section 4 the convergence of the derived series solutions is explicitly discussed. Section 5 deals with the discussion of graphs and tables. Conclusions are reported in Section 6.

2. Mathematical Formulation

We consider the steady, incompressible, and MHD flow of a two-dimensional Jeffery fluid over a nonlinear stretching sheet. We choose x-axis parallel and y-axis normal to the stretching surface. A uniform magnetic field exerts in the y-direction. For a small magnetic Reynolds number, the induced magnetic field is neglected. We also considered the presence of a first-order chemical reaction. For the present problem, the equations governing the flow are given below:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0, \quad (1)$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+\lambda_1\left[u^2\frac{\partial^2 u}{\partial x^2}+v^2\frac{\partial^2 u}{\partial y^2}+2uv\frac{\partial^2 u}{\partial x\partial y}\right]= \nu\left[\frac{\partial^2 u}{\partial y^2}+\lambda_2\left\{u\frac{\partial^3 u}{\partial x\partial y^2}+v\frac{\partial^3 u}{\partial y^3}-\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial y^2}-\frac{\partial u}{\partial y}\frac{\partial^2 v}{\partial y^2}\right\}\right] -\frac{\sigma B_0^2}{\rho}u, \quad (2)$$

$$u\frac{\partial C}{\partial x}+v\frac{\partial C}{\partial y}=D\frac{\partial^2 C}{\partial y^2}-RC. \quad (3)$$

In the above equations, u and v are components of the velocity along x and y-directions, respectively, ν is the kinematic viscosity, ρ is the fluid density, σ is the electrical conductivity, λ_1 is the relaxation time, λ_2 is the retardation time. C is the species concentration in the fluid, D is the mass diffusion coefficient, and R is the first-order chemical reaction parameter.

The appropriate boundary conditions are

$$\begin{aligned}&u(x,y)=ax+bx^2,\\&v(x,y)=0, C(x,y)=C_w \text{ at } y=0,\\&u\to 0,\quad C\to 0 \text{ as } y\to\infty,\end{aligned} \quad (4)$$

where a and b are the dimensional constants.

In order to make the problem simpler, we introduce the following quantities:

$$\begin{aligned}&\eta=\sqrt{\frac{a}{\nu}}y,\ u=axf'(\eta)+bx^2g'(\eta),\\&v=-\sqrt{a\nu}f(\eta)-2bx\sqrt{\frac{\nu}{a}}g(\eta),\\&C=C_w\left\{C_0(\eta)+\frac{2bx}{a}C_1(\eta)\right\},\end{aligned} \quad (5)$$

where a prime denotes the derivative with respect to η. We note that (1) is satisfied identically and (2) – (4) are transformed as follows:

$$\begin{aligned}&f'''-M^2f'-f'^2+ff''\\&+\beta_1(2ff'f''-f^2f''')+\beta_2(f''^2-ff'''')=0,\end{aligned} \quad (6)$$

$$\begin{aligned}&g'''-M^2g'-3f'g'+2f''g+g''f+\beta_1\big(4ff'g''\\&+2fg'f''-f^2g'''-4fgf'''+4f'f''g-2f'^2g'\big)\\&+\beta_2\big(f'g'''-g'f'''-fg''''-2gf''''+3f''g''\big)=0,\end{aligned} \quad (7)$$

$$C_0''+ScfC_0'-Sc\gamma C_0=0, \quad (8)$$

$$C_1''+ScfC_1'-Scf'C_1-Sc\gamma C_1+ScgC_0'=0, \quad (9)$$

$$\begin{aligned}&f(0)=0,\quad f'(0)=1,\quad f'(\infty)=0,\\&g(0)=0,\quad g'(0)=1,\quad g'(\infty)=0,\\&C_0(0)=1,\ C_0(\infty)=0,\ C_1(0)=0,\ C_1(\infty)=0,\end{aligned} \quad (10)$$

in which the chemical reaction parameter γ, the Schmidt number Sc, the Hartman number M, and the Deborah numbers β_1,β_2 are

$$\begin{aligned}&\gamma=\frac{R}{a},\quad S=\frac{\nu}{D},\quad M^2=\frac{\sigma B_0^2}{a\rho},\\&\beta_1=\lambda_1 a,\quad \beta_2=\lambda_2 a.\end{aligned} \quad (11)$$

Here $\gamma>0$ indicates the destructive chemical reaction and $\gamma<0$ the generative chemical reaction. For $\gamma=0$ we have the case for a non-reactive species. The expressions of the mass transfer C_0' and C_1' at the wall are

$$\begin{aligned}&C_0'(0)=\left(\frac{\partial C_0}{\partial\eta}\right)_{\eta=0}\le 0,\\&C_1'(0)=\left(\frac{\partial C_1}{\partial\eta}\right)_{\eta=0}\le 0.\end{aligned} \quad (12)$$

3. Homotopy Analysis Solutions

For the series solutions, we express $f(\eta)$, $g(\eta)$, and concentration fields $C_0(\eta)$ and $C_1(\eta)$ by the set of base functions

$$\left\{\eta^k\exp(-n\eta)\,\middle|\,k\ge 0, n\ge 0\right\} \quad (13)$$

in the form

$$f(\eta)=a_{0,0}^0+\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}a_{m,n}^k\eta^k\exp(-n\eta), \quad (14)$$

$$g(\eta)=b_{0,0}^0+\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}b_{m,n}^k\eta^k\exp(-n\eta), \quad (15)$$

$$C_0(\eta)=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}d_{m,n}^k\eta^k\exp(-n\eta), \quad (16)$$

$$C_1(\eta)=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}e_{m,n}^k\eta^k\exp(-n\eta), \quad (17)$$

where $a^k_{m,n}$, $b^k_{m,n}$, $d^k_{m,n}$, and $e^k_{m,n}$ are the coefficients. The initial guesses $f_0(\eta)$, $g_0(\eta)$, $C_{0,0}(\eta)$, and $C_{1,0}(\eta)$ of $f(\eta)$, $g(\eta)$, $C_0(\eta)$, and $C_1(\eta)$ are

$$f_0(\eta) = 1 - \exp(-\eta), \tag{18}$$

$$g_0(\eta) = 1 - \exp(-\eta), \tag{19}$$

$$C_{0,0}(\eta) = \exp(-\eta), \tag{20}$$

$$C_{1,0}(\eta) = \eta \exp(-\eta) \tag{21}$$

with the following auxiliary linear operators:

$$\mathcal{L}_1(f) = \frac{\mathrm{d}^3 f}{\mathrm{d}\eta^3} - \frac{\mathrm{d}f}{\mathrm{d}\eta}, \tag{22}$$

$$\mathcal{L}_2(g) = \frac{\mathrm{d}^3 g}{\mathrm{d}\eta^3} - \frac{\mathrm{d}g}{\mathrm{d}\eta}, \tag{23}$$

$$\mathcal{L}_3(C_0) = \frac{\mathrm{d}^2 C_0}{\mathrm{d}\eta^2} - C_0, \tag{24}$$

$$\mathcal{L}_4(C_1) = \frac{\mathrm{d}^2 C_1}{\mathrm{d}\eta^2} - C_1. \tag{25}$$

The above operators have the following properties:

$$\mathcal{L}_1[c_1 + c_2\exp(\eta) + c_3\exp(-\eta)] = 0, \tag{26}$$

$$\mathcal{L}_2[c_4 + c_5\exp(\eta) + c_6\exp(-\eta)] = 0, \tag{27}$$

$$\mathcal{L}_3[c_7\exp(\eta) + c_8\exp(-\eta)] = 0, \tag{28}$$

$$\mathcal{L}_4[c_9\exp(\eta) + c_{10}\exp(-\eta)] = 0, \tag{29}$$

where c_i $(i = 1-10)$ are the arbitrary constants. If $p \in [0,1]$ is an embedding parameter and $\hbar_f$, $\hbar_g$, $\hbar_{C_0}$, and $\hbar_{C_1}$ indicate the non-zero auxiliary parameters, respectively, then the zeroth-order deformation problems are constructed as follows:

$$(1-p)\mathcal{L}_1[\hat{f}(\eta,p) - f_0(\eta)] = p\hbar_f \mathcal{N}_f\left[\hat{f}(\eta,p)\right], \tag{30}$$

$$\begin{aligned}&(1-p)\mathcal{L}_1[\hat{g}(\eta,p) - g_0(\eta)]\\ &= p\hbar_g \mathcal{N}_g\left[\hat{f}(\eta,p),\hat{g}(\eta,p)\right],\end{aligned} \tag{31}$$

$$\begin{aligned}&(1-p)\mathcal{L}_2\left[\widehat{C}_0(\eta,p) - C_{0,0}(\eta)\right]\\ &= p\hbar_{C_0}\mathcal{N}_{C_0}\left[\widehat{C}_0(\eta,p),\widehat{f}(\eta,p)\right],\end{aligned} \tag{32}$$

$$\begin{aligned}&(1-p)\mathcal{L}_2\left[\widehat{C}_1(\eta,p) - C_{1,0}(\eta)\right] =\\ &p\hbar_{C_1}\mathcal{N}_{C_1}\left[\widehat{C}_0(\eta,p),\widehat{C}_1(\eta,p),\widehat{f}(\eta,p),\widehat{g}(\eta,p)\right],\end{aligned} \tag{33}$$

$$\begin{aligned}&\hat{f}(\eta;p)\big|_{\eta=0} = 0,\\ &\left.\frac{\partial \hat{f}(\eta;p)}{\partial\eta}\right|_{\eta=0} = 1, \quad \left.\frac{\partial \hat{f}(\eta;p)}{\partial\eta}\right|_{\eta=\infty} = 0,\end{aligned} \tag{34}$$

$$\begin{aligned}&\hat{g}(\eta;p)\big|_{\eta=0} = 0,\\ &\left.\frac{\partial \hat{g}(\eta;p)}{\partial\eta}\right|_{\eta=0} = 1, \quad \left.\frac{\partial \hat{g}(\eta;p)}{\partial\eta}\right|_{\eta=\infty} = 0,\end{aligned} \tag{35}$$

$$\left.\frac{\partial \widehat{C}_0(\eta;p)}{\partial\eta}\right|_{\eta=0} = 1, \quad \left.\frac{\partial \widehat{C}_0(\eta;p)}{\partial\eta}\right|_{\eta=\infty} = 0, \tag{36}$$

$$\left.\frac{\partial \widehat{C}_1(\eta;p)}{\partial\eta}\right|_{\eta=0} = 0, \quad \left.\frac{\partial \widehat{C}_1(\eta;p)}{\partial\eta}\right|_{\eta=\infty} = 0, \tag{37}$$

in which the nonlinear differential operators $\mathcal{N}_f$, $\mathcal{N}_g$, $\mathcal{N}_{C_0}$, and $\mathcal{N}_{C_1}$ are

$$\begin{aligned}\mathcal{N}_f[\hat{f}(\eta;p)] &= \frac{\partial^3 \hat{f}(\eta;p)}{\partial\eta^3} - M^2\frac{\partial \hat{f}(\eta;p)}{\partial\eta}\\ &+\hat{f}(\eta;p)\frac{\partial^2 \hat{f}(\eta;p)}{\partial\eta^2} - \left(\frac{\partial \hat{f}(\eta;p)}{\partial\eta}\right)^2\\ &+\beta_1\Bigg(2\hat{f}(\eta;p)\frac{\partial \hat{f}(\eta;p)}{\partial\eta}\frac{\partial^2 \hat{f}(\eta;p)}{\partial\eta^2}\\ &\qquad -\hat{f}^2(\eta;p)\frac{\partial^3 \hat{f}(\eta;p)}{\partial\eta^3}\Bigg)\\ &+\beta_2\left[\left(\frac{\partial^2 \hat{f}(\eta;p)}{\partial\eta^2}\right)^2 - \hat{f}(\eta;p)\frac{\partial^4 \hat{f}(\eta;p)}{\partial\eta^4}\right],\end{aligned} \tag{38}$$

$$\begin{aligned}&\mathcal{N}_g[\hat{f}(\eta;p),\hat{g}(\eta,p)] = \frac{\partial^3 \hat{g}(\eta;p)}{\partial\eta^3} - M^2\frac{\partial \hat{g}(\eta;p)}{\partial\eta}\\ &-3\frac{\partial \hat{g}(\eta;p)}{\partial\eta}\frac{\partial \hat{f}(\eta;p)}{\partial\eta} + 2\hat{g}(\eta,p)\frac{\partial^2 \hat{f}(\eta;p)}{\partial\eta^2}\\ &+\hat{f}(\eta;p)\frac{\partial^2 \hat{g}(\eta;p)}{\partial\eta^2} + \beta_1\Bigg[4\hat{f}(\eta;p)\frac{\partial \hat{f}(\eta;p)}{\partial\eta}\frac{\partial^2 \hat{g}(\eta;p)}{\partial\eta^2}\\ &+2\hat{f}(\eta;p)\frac{\partial \hat{g}(\eta;p)}{\partial\eta}\frac{\partial^2 \hat{f}(\eta;p)}{\partial\eta^2} - \hat{f}^2(\eta;p)\frac{\partial^3 \hat{g}(\eta;p)}{\partial\eta^3}\\ &-4\hat{f}(\eta;p)\hat{g}(\eta,p)\frac{\partial^3 \hat{f}(\eta;p)}{\partial\eta^3} + 4\frac{\partial \hat{f}(\eta;p)}{\partial\eta}\\ &\quad\cdot\frac{\partial^2 \hat{f}(\eta;p)}{\partial\eta^2}\hat{g}(\eta,p) - 2\left(\frac{\partial \hat{f}(\eta;p)}{\partial\eta}\right)^2\frac{\partial \hat{g}(\eta;p)}{\partial\eta}\Bigg]\\ &+\beta_2\Bigg[\frac{\partial \hat{f}(\eta;p)}{\partial\eta}\frac{\partial^3 \hat{g}(\eta;p)}{\partial\eta^3} - \frac{\partial \hat{g}(\eta;p)}{\partial\eta}\frac{\partial^3 \hat{f}(\eta;p)}{\partial\eta^3}\\ &-\hat{f}(\eta;p)\frac{\partial^4 \hat{g}(\eta;p)}{\partial\eta^4} - 2\hat{g}(\eta,p)\frac{\partial^4 \hat{f}(\eta;p)}{\partial\eta^4}\\ &+3\frac{\partial^2 \hat{f}(\eta;p)}{\partial\eta^2}\frac{\partial^2 \hat{g}(\eta;p)}{\partial\eta^2}\Bigg],\end{aligned} \tag{39}$$

$$\mathcal{N}_{C_0}\left[\widehat{C}_0(\eta,p),\widehat{f}(\eta,p)\right]= \frac{\partial^2\widehat{C}_0(\eta,p)}{\partial\eta^2}+Sc\widehat{f}(\eta,p)\frac{\partial\widehat{C}_0(\eta,p)}{\partial\eta}-Sc\gamma C_0, \tag{40}$$

$$\mathcal{N}_{C_1}\left[\widehat{C}_0(\eta,p),\widehat{C}_1(\eta,p),\widehat{f}(\eta,p),\widehat{g}(\eta,p)\right]= \frac{\partial^2\widehat{C}_1(\eta,p)}{\partial\eta^2}-Sc\gamma C_1+Sc\bigg(-\frac{\partial\widehat{f}(\eta,p)}{\partial\eta}\widehat{C}_1(\eta,p) +\widehat{g}(\eta,p)\frac{\partial\widehat{C}_0(\eta,p)}{\partial\eta}+\widehat{f}(\eta,p)\frac{\partial\widehat{C}_1(\eta,p)}{\partial\eta}\bigg). \tag{41}$$

Obviously, for $p=0$ and $p=1$, we have

$$\hat{f}(\eta;0)=f_0(\eta),\ \hat{f}(\eta;1)=f(\eta), \tag{42}$$

$$\hat{g}(\eta,0)=g_0(\eta),\hat{g}(\eta,1)=g(\eta), \tag{43}$$

$$\widehat{C}_0(\eta,0)=C_{0,0}(\eta),\ \widehat{C}_0(\eta,1)=C_0(\eta), \tag{44}$$

$$\widehat{C}_1(\eta,0)=C_{1,0}(\eta),\ \widehat{C}_1(\eta,1)=C_1(\eta). \tag{45}$$

Expanding $\hat{f}(\eta;p)$, $\hat{g}(\eta,p)$, $\widehat{C}_0(\eta,p)$, and $\widehat{C}_1(\eta,p)$ in Taylor series with respect to the embedding parameter p, we obtain

$$\hat{f}(\eta;p)=f_0(\eta)+\sum_{m=1}^{\infty}f_m(\eta)p^m, \tag{46}$$

$$\hat{g}(\eta,p)=g_0(\eta)+\sum_{m=1}^{\infty}g_m(\eta)p^m, \tag{47}$$

$$\widehat{C}_0(\eta,p)=C_{0,0}(\eta)+\sum_{m=1}^{\infty}C_{0,m}(\eta)\,p^m, \tag{48}$$

$$\widehat{C}_1(\eta,p)=C_{1,0}(\eta)+\sum_{m=1}^{\infty}C_{1,m}(\eta)\,p^m, \tag{49}$$

$$f_m(\eta)=\frac{1}{m!}\frac{\partial^m f(\eta;p)}{\partial\eta^m}\bigg|_{p=0},\quad g_m(\eta)=\frac{1}{m!}\frac{\partial^m g(\eta;p)}{\partial\eta^m}\bigg|_{p=0}, \tag{50}$$

$$C_{0,m}(\eta)=\frac{1}{m!}\frac{\partial^m\widehat{C}_0(\eta,p)}{\partial p^m}\bigg|_{p=0},\quad C_{1,m}(\eta)=\frac{1}{m!}\frac{\partial^m\widehat{C}_1(\eta,p)}{\partial p^m}\bigg|_{p=0}, \tag{51}$$

in which the series convergence in (47) – (50) depends upon $\hbar_f$, $\hbar_g$, $\hbar_{C_0}$, and $\hbar_{C_1}$. The values of $\hbar_f$, $\hbar_g$, $\hbar_{C_0}$, and $\hbar_{C_1}$ are chosen in such a way that the series (47) – (50) are convergent at $p=1$ and hence

$$f(\eta)=f_0(\eta)+\sum_{m=0}^{\infty}f_m(\eta), \tag{52}$$

$$g(\eta)=g_0(\eta)+\sum_{m=0}^{\infty}g_m(\eta), \tag{53}$$

$$C_0(\eta)=C_{0,0}(\eta)+\sum_{m=1}^{\infty}C_{0,m}(\eta), \tag{54}$$

$$C_1(\eta)=C_{1,0}(\eta)+\sum_{m=1}^{\infty}C_{1,m}(\eta). \tag{55}$$

The problems corresponding to the mth-order deformation are

$$\mathcal{L}_1[f_m(\eta)-\chi_m f_{m-1}(\eta)]=\hbar_f\mathcal{R}_{f,m}(\eta), \tag{56}$$

$$\mathcal{L}_2[g_m(\eta)-\chi_m g_{m-1}(\eta)]=\hbar_g\mathcal{R}_{g,m}(\eta), \tag{57}$$

$$\mathcal{L}_3[C_{0,m}(\eta)-\chi_m C_{0,m-1}(\eta)]=\hbar_{C_0}\mathcal{R}_{C_{0,m}}(\eta), \tag{58}$$

$$\mathcal{L}_4[C_{1,m}(\eta)-\chi_m C_{1,m-1}(\eta)]=\hbar_{C_1}\mathcal{R}_{C_{1,m}}(\eta), \tag{59}$$

$$f_m(0)=0,f'_m(0)=0,f'_m(\infty)=0, \tag{60}$$

$$g_m(0)=0,g'_m(0)=0,g'_m(\infty)=0, \tag{61}$$

$$C_{0,m}(0)=C_{0,m}(\infty)=0, \tag{62}$$

$$C_{1,m}(0)=C_{1,m}(\infty)=0, \tag{63}$$

$$\mathcal{R}_{f,m}(\eta)=f'''_{m-1}-M^2f'_{m-1}+\sum_{k=0}^{m-1}[f_{m-1-k}f''_k -f'_{m-1-k}f'_k]+\beta_1\sum_{k=0}^{m-1}f_{m-1-k}\sum_{l=0}^{k}[2f'_{k-l}f''_l-f_{k-l}f'''_l] +\beta_2\sum_{k=0}^{m-1}f''_{m-1-k}f''_k-f_{m-1-k}f''''_k, \tag{64}$$

$$\mathcal{R}_{g,m}(\eta)=g'''_{m-1}-M^2g'_{m-1} +\sum_{k=0}^{m-1}[-3f'_{m-1-k}g'_k+2g_{m-1-k}f''_k+f_{m-1-k}g''_k] +\beta_1\left[\sum_{k=0}^{m-1}f_{m-1-k}\sum_{l=0}^{k}[4f'_{k-l}g''_l+2f''_{k-l}g'_l-f_{k-l}g'''_l -4f'''_{k-l}g_l]+\sum_{k=0}^{m-1}f'_{m-1-k}\sum_{l=0}^{k}[4f''_{k-l}g_l-2f'_{k-l}g'_l]\right] +\beta_2\sum_{k=0}^{m-1}[f'_{m-1-k}g'''_k-g'_{m-1-k}f'''_k-f_{m-1-k}g''''_k -2g_{m-1-k}f''''_k+3f''_{m-1-k}g''_k], \tag{65}$$

Table 1. Convergence of the HAM solution for different order of approximations when $M = 1.0$, $\beta = 0.2$, $S = 1.0$, and $\gamma = 0.2$.

Order of approximation	$-f''(0)$	$-g''(0)$	$-C_0'(0)$	$-C_1'(0)$
1	1.212500	1.787500	1.166667	0.250000
5	1.286329	1.729082	1.1602806	0.128032
10	1.286998	1.728359	1.1600012	0.128054
15	1.287000	1.728356	1.160005	0.128064
20	1.287000	1.728356	1.160005	0.128063
25	1.287000	1.728356	1.160005	0.128063

$$\mathcal{R}_{C_{0,m}}(\eta) = C''_{0,m-1} - Sc\gamma C_{0,m-1} + Sc\sum_{k=0}^{m-1} C'_{0,m-1-k} f_k, \quad (66)$$

$$\mathcal{R}_{C_{1,m}}(\eta) = C''_{1,m-1} - Sc\gamma C_{1,m-1} + Sc \cdot \sum_{k=0}^{m-1}\left(-C_{1,m-1-k} f'_k + C'_{1,m-1-k} f_k + C'_{0,m-1-k} g_k\right), \quad (67)$$

$$\chi_m = \begin{cases} 0, & m \le 1, \\ 1, & m > 1. \end{cases} \quad (68)$$

The general solutions of (57) – (64) can be written as

$$f_m(\eta) = f_m^*(\eta) + c_1 + c_2\exp(\eta) + c_3\exp(-\eta), \quad (69)$$

$$g_m(\eta) = g_m^*(\eta) + c_4 + c_5\exp(\eta) + c_6\exp(-\eta), \quad (70)$$

$$C_{0,m}(\eta) = C_{0,m}^*(\eta) + c_7\exp(\eta) + c_8\exp(-\eta), \quad (71)$$

$$C_{1,m}(\eta) = C_{1,m}^*(\eta) + c_9\exp(\eta) + c_{10}\exp(-\eta), \quad (72)$$

where $f_m^*(\eta)$, $g_m^*(\eta)$, $C_{0,m}^*(\eta)$, and $C_{1,m}^*(\eta)$ denote the special solutions of (57) – (64) and

$$\begin{aligned} &c_2 = c_5 = c_7 = c_9 = 0, \\ &c_1 = -c_3 - f_m^*(0), \quad c_3 = \left.\frac{\partial f^*(\eta)}{\partial \eta}\right|_{\eta=0}, \\ &c_4 = -c_6 - g_m^*(0), \quad c_6 = \left.\frac{\partial g^*(\eta)}{\partial \eta}\right|_{\eta=0}, \\ &c_8 = -C_{0,m}^*(0), \quad c_{10} = -C_{1,m}^*(0). \end{aligned} \quad (73)$$

Note that (57) – (60) can be solved by Mathematica one after the other in the order $m = 1, 2, 3, \ldots$.

4. Convergence of Homotopy Solutions

The convergence of series solutions are dependent upon the values of the auxiliary parameters $\hbar_f$, $\hbar_g$, $\hbar_{C_0}$, and $\hbar_{C_1}$. In order to determine the range of the admissible values of $\hbar_f$, $\hbar_g$, $\hbar_{C_0}$, and $\hbar_{C_1}$ for the functions $f''(0)$, $g''(0)$, $C_0'(0)$, and $C_1'(0)$, the $\hbar$-curves are plotted for the 15th order of approximation in Figures 1. It is clear that admissible values of $\hbar_f$, $\hbar_g$, $\hbar_{C_0}$, and $\hbar_{C_1}$ are $-1 \le \hbar_f \le -0.3$, $-0.7 \le \hbar_g \le -0.2$, $-1.5 \le \hbar_{C_0} \le -0.5$, and $-1.8 \le \hbar_{C_1} \le -0.5$. Furthermore, the series solutions (53) – (56) converge in the whole region of η when $\hbar_f = \hbar_g = -0.5$ and $\hbar_{C_0} = \hbar_{C_1} = -1$. Table 1 shows the convergence of the HAM solutions at different order of approximations.

Table 2. Values of the surface mass transfer $C_0'(0)$ and $C_1'(0)$ for some values of M, β_1, β_2, Sc, and γ.

Sc	γ	M	$\beta_1 = \beta_2$	$-C_0'(0)$	$-C_1'(0)$
1	0.4	1.2	0.2	0.83412	0.15646
	0.8			1.05823	0.13321
	1.0			1.15258	0.12582
	2.0			1.53473	0.10372
0.5	1.0			0.79841	0.07527
1.0				0.87943	0.08653
1.5				1.42758	0.16663
2.0				1.66074	0.20171
1.0	1.0	0		1.17773	0.12365
		0.3		1.17554	0.12416
		0.6		1.16966	0.12526
		0.9		1.16153	0.12605
		1.2		1.15258	0.12583
		1.2	0	1.14848	0.11841
			0.2	1.15258	0.12582
			0.4	1.55658	0.12992
			0.8	1.16029	0.13383
			1.0	1.16204	0.13469

5. Results and Discussion

In this section, the graphical results are presented for the effects of Deborah numbers β_1, β_2, Hartman number M, Schmidt number Sc, and the chemical reaction parameter γ on the velocity and concentration fields. Such effects are discussed by Figures 2 – 10. Figures 2 – 4 show the effects of β_1, β_2, and M on f' and g'. For different values of Deborah number β_1, the velocity profiles are plotted in Figure 2. It is obvious that the velocity distribution across the boundary layer decreases by increasing values of β. Figure 2 shows the variation of Deborah number β_1 on f' and g'. It is found that the velocity components f' and g' decrease as β_1 increases. However, such increase is small in f' when compared with g'. The boundary layer thickness decreases when Deborah number β_1 is increased. The effects of Deborah number β_2 on the velocity components f' and g' are opposite to β_1 (Fig. 3). It can be seen from Figure 4 that the effect of the Hartman number M is similar to that of β_1 on the velocity compo-

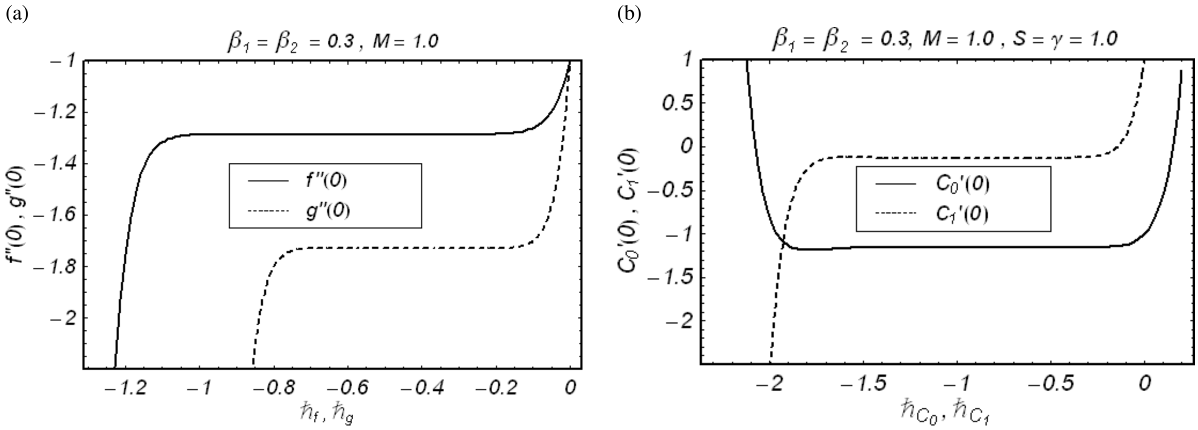

Fig. 1. $\hbar$-curves of the functions $f''(0)$, $g''(0)$, $C_0'(0)$, and $C_1'(0)$ at the 15th order of approximation.

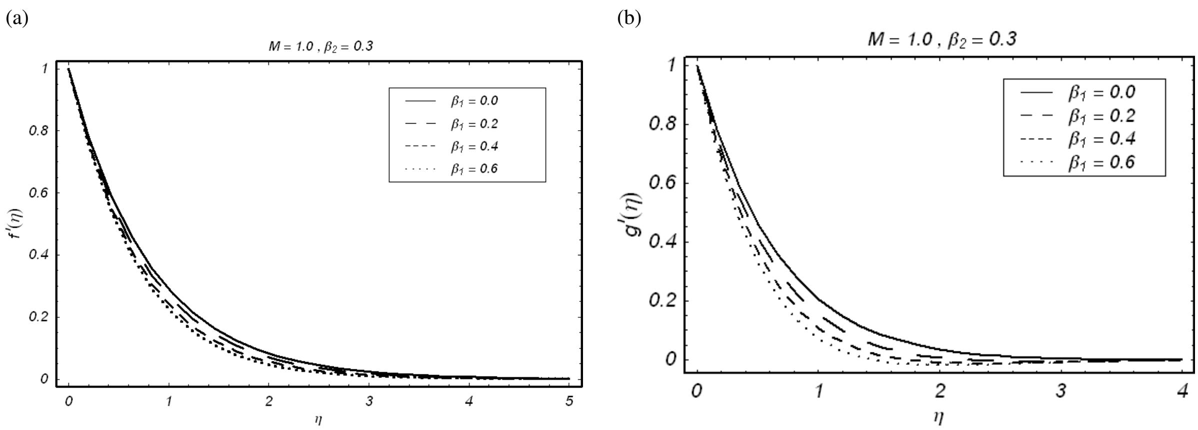

Fig. 2. Variation of Deborah number β_1 on f' and g'.

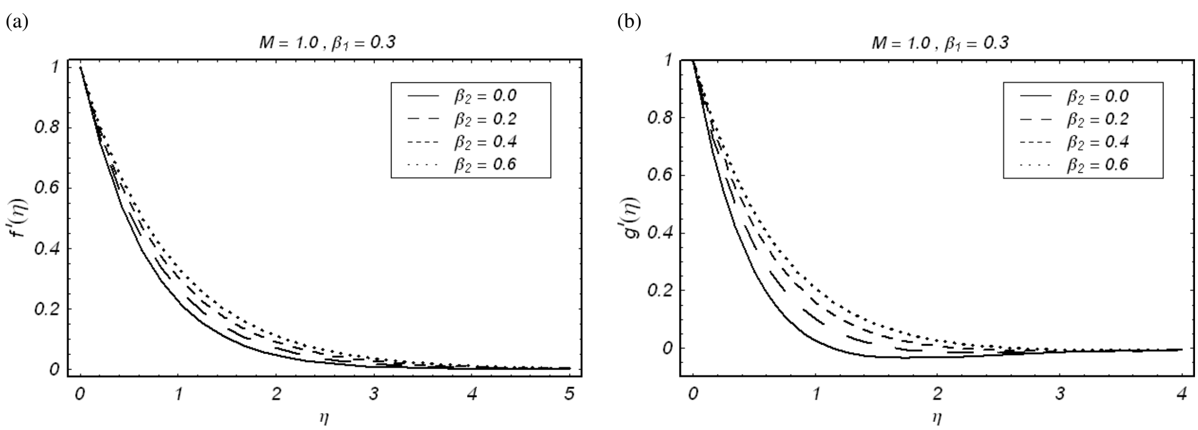

Fig. 3. Variation of Deborah number β_2 on f' and g'.

(a) (b)

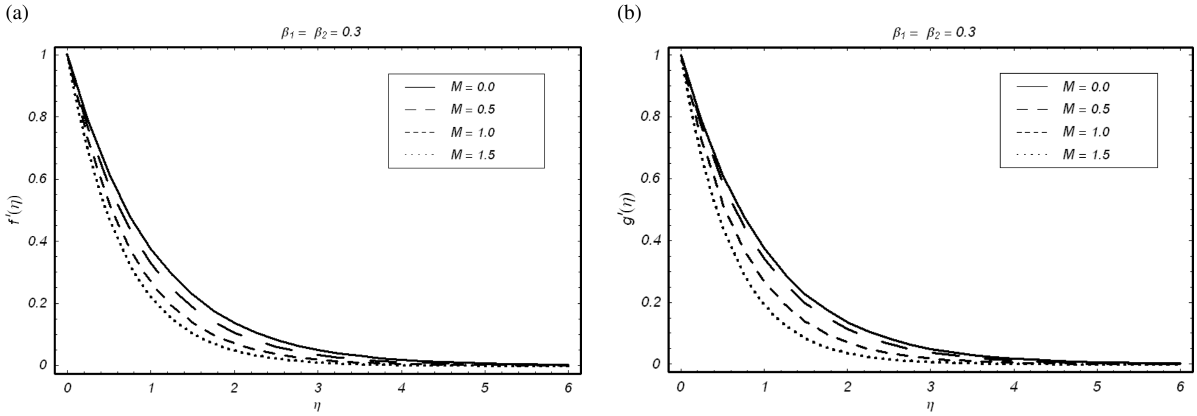

Fig. 4. Variation of Hartman number M on f' and g'.

(a) (b)

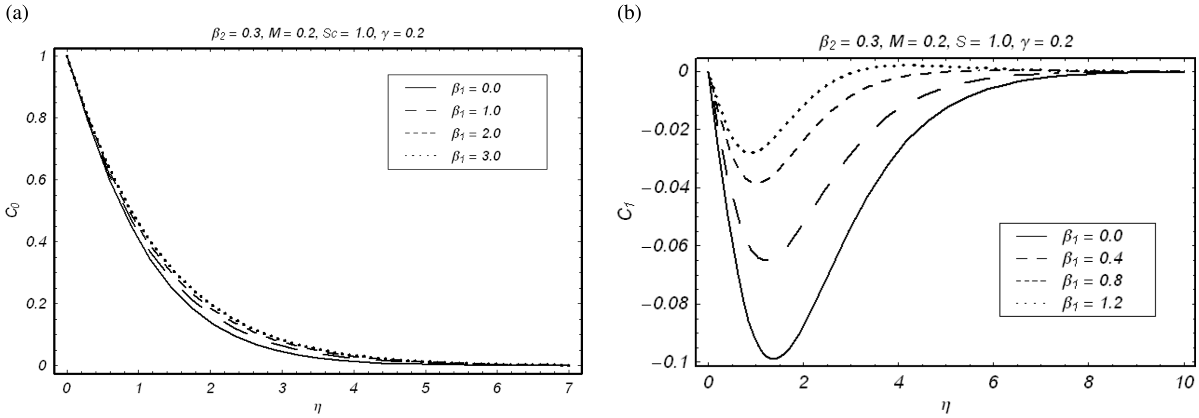

Fig. 5. Variation of Deborah number β_1 on C_0 and C_1.

(a) (b)

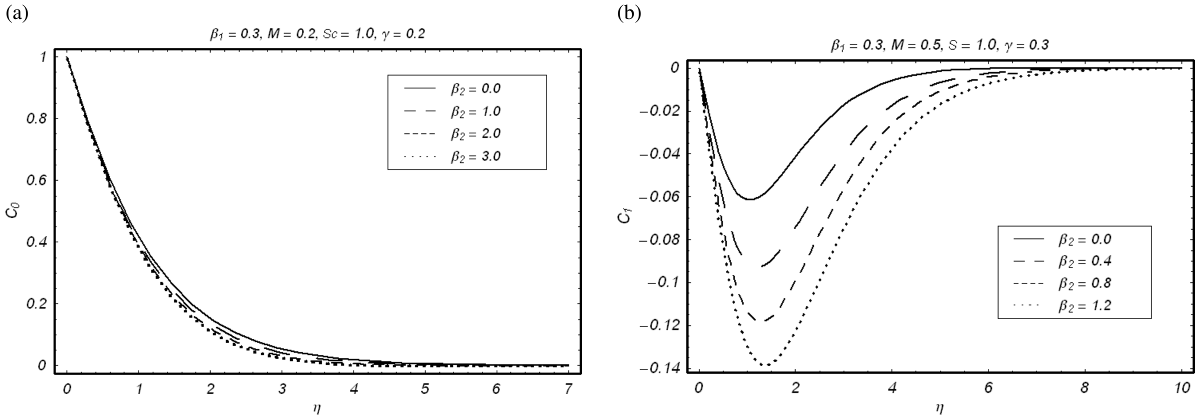

Fig. 6. Variation of Deborah number β_2 on C_0 and C_1.

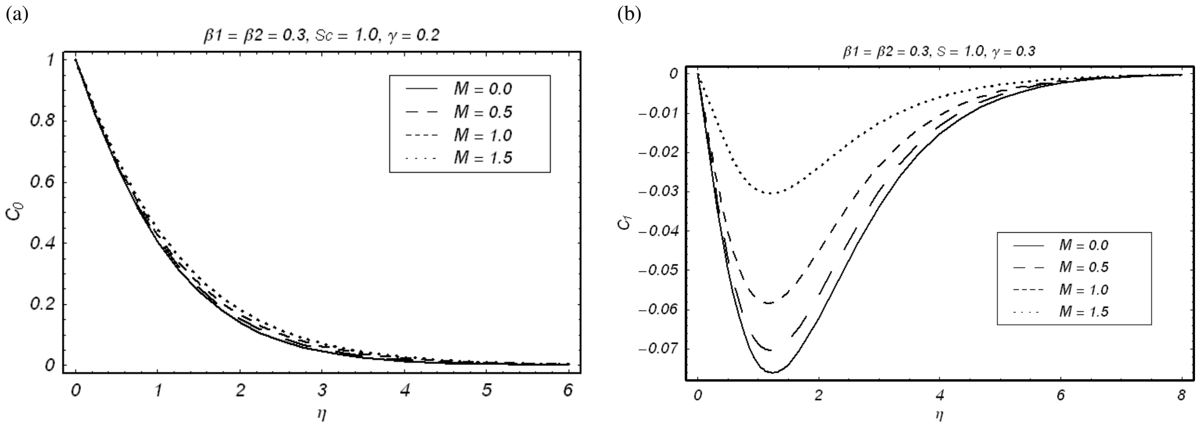

Fig. 7. Variation of Hartman number M on C_0 and C_1.

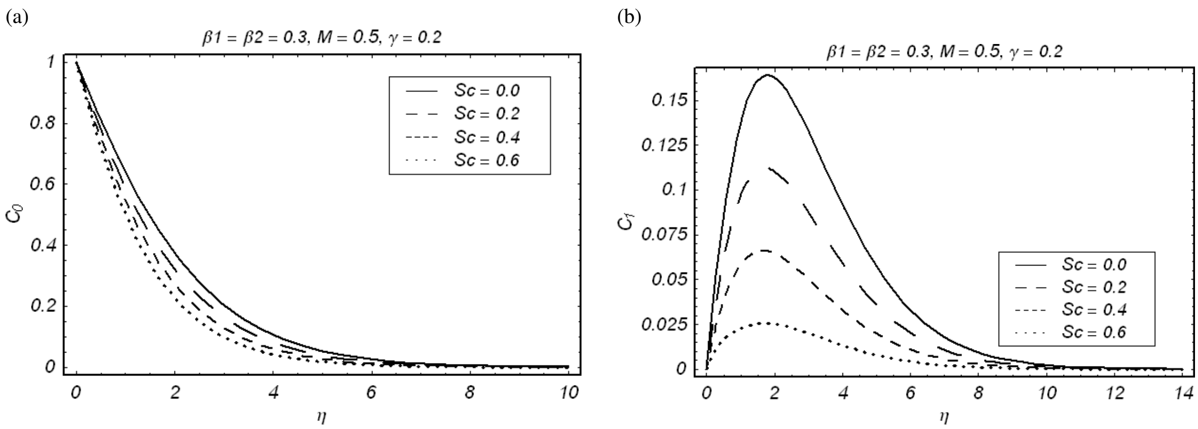

Fig. 8. Variation of Schmidt number Sc on C_0 and C_1.

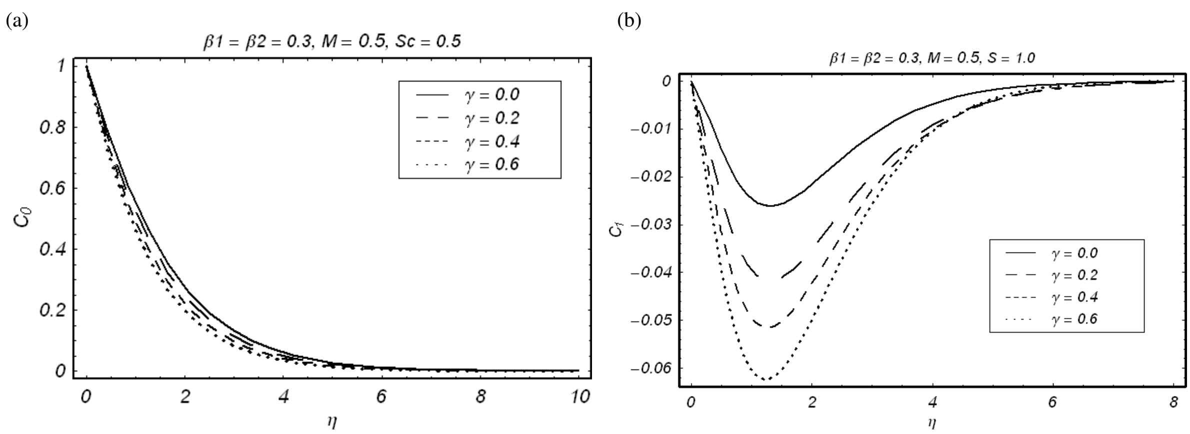

Fig. 9. Variation of chemical reaction parameter γ on C_0 and C_1 in the case of destructive chemical reaction.

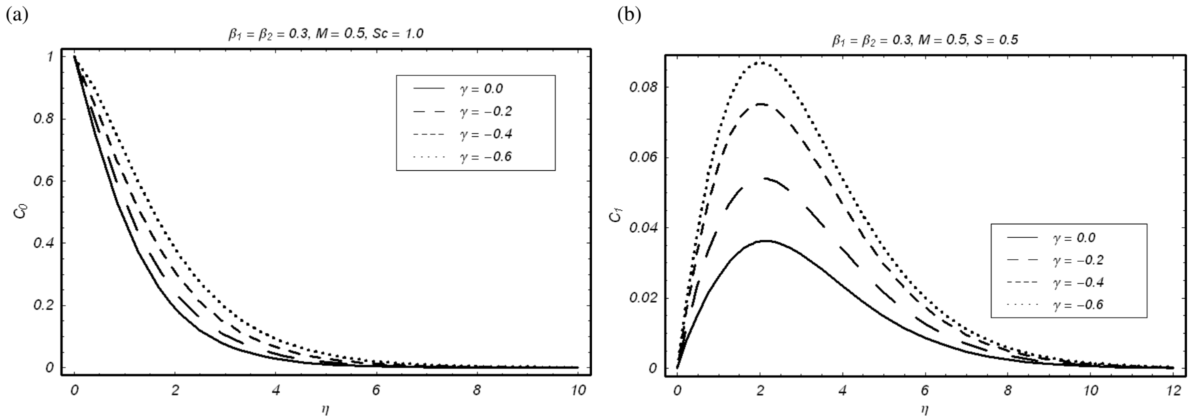

Fig. 10. Variation of chemical reaction parameter γ on C_0 and C_1 in case of generative chemical reaction.

nents f' and g'. The variations of the emerging parameters on the concentration fields C_0 and C_1 are plotted in Figures 5 – 10. Figure 5 is the graphical representation showing the effects of the Deborah number β_1 on C_0 and C_1 in the case of destructive chemical reaction ($\gamma > 0$). The concentration field C_0 is increased for large values of β_1 while the magnitude of C_1 decreases when β_1 increases. It should be pointed out that the variation in C_1 is larger in comparison to C_0 for large values of β_1. Figure 6 is displayed for the variations of β_2 on the concentration fields C_0 and C_1 in the case of destructive chemical reaction ($\gamma > 0$). It can be seen that Figure 6 shows the opposite qualitative effects when compared with Figure 5. Figure 7a shows the effects of M on C_0. It is observed that the concentration field C_0 is increased when M increases. The variation of M on C_1 is sketched in Figure 7b. Figure 8 gives the variations of the Schmidt number Sc on the concentration fields C_0 and C_1 for $\gamma = 0.2$. Both C_0 and C_1 decreases when Sc increases. The effects of destructive chemical reaction parameter ($\gamma > 0$) on the concentration fields C_0 and C_1 are displayed in Figure 9. It is found from Figure 9a that the concentration field C_0 is a decreasing function of γ. It is also clear from Figure 9b that the magnitude of C_1 decreases when γ increases. Figure 10 depicts the variation of a generative chemical reaction ($\gamma < 0$) on the concentration fields C_0 and C_1. It is found from Figure 10a that C_0 increases for large generative chemical reaction parameter. Figure 10b depicts that the magnitude of C_1 also increases as $\gamma\,(\gamma < 0)$ increases.

6. Closing Remarks

The present study investigates the mass transfer in the MHD flow of a Jeffery fluid bounded by a nonlinearly stretching surface. The velocity and the concentration fields are derived. The homotopy analysis method is utilized for the series solutions. The behaviours of various embedded parameters in the considered problem are analyzed. The gradient of mass transfer are also computed in the tabulated forms. The main observations are pointed out below.

- The behaviour of β_1 and M on $f'(\eta)$ and $g'(\eta)$ are the same.
- The effects of increasing the values of M is to decrease the boundary layer thickness.
- The concentrations fields C_0 and C_1 decreases as Sc increases.
- The influence of the destructive reaction ($\gamma > 0$) is to decrease the concentration fields.
- The concentration fields C_0 and C_1 has opposite results for destructive ($\gamma > 0$) and generative ($\gamma < 0$) chemical reactions.
- The surface mass transfer decreases by increasing M.

Acknowledgement

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