Non-Perturbative Solution of the Magnetohydrodynamic Flow over a Nonlinear Stretching Sheet by Homotopy Perturbation Method-Padé Technique

Ahmet Yıldırım and Sefa Anıl Sezer

Ege University, Science Faculty, Department of Mathematics, 35100 Bornova-İzmir, Turkey

Reprint requests to A. Y.; E-mail: ahmet.yildirim@ege.edu.tr

Z. Naturforsch. **65a,** 1106 – 1110 (2010); received December 15, 2009 / revised March 26, 2010

In this study, we investigate the magnetohydrodynamic (MHD) boundary layer flow by employing the homotopy perturbation method (HPM) and Padé approximation. The series solution of the governing nonlinear problem is developed. Generally, the truncated series solution is adequate only in a small region when the exact solution is not reached. We overcame this limitation by using the Padé techniques, which have the advantage in turning the polynomials approximation into a rational function, and applied it to the series solution to improve the accuracy and enlarge the convergence domain. A comparison of the present solution with the existing solution is made and excellent agreement is noted.

Key words: Nonlinear Stretching; Padé Approximation; Homotopy Perturbation Method; Magnetohydrodynamic Flow; Non-Perturbative Solution.

1. Introduction

The aim of the present investigation is to analyze the magnetohydrodynamic (MHD) flow caused by a sheet with nonlinear stretching. The boundary layer flow of an incompressible viscous fluid over a continuously stretching sheet is often encountered in many engineering and industrial processes. Such processes include the aerodynamic extrusion of plastic sheets, hot rolling, glass-fiber production, and so on [1 – 3]. The pioneering work in this area was done by Sakiadis [4, 5]. After that various aspects of the stretching flow problem were discussed by various investigators. Amongst these, Chiam [6] analyzed the MHD flow of a viscous fluid bounded by a stretching surface with power law velocity. Recently, Hayat et al. [7] used the modified Adomian decomposition method for solving the MHD flow caused by a sheet with nonlinear stretching.

In this study, the solution of the nonlinear problem is obtained by combining the homotopy perturbation method (HPM) with Padé approximation. The HPM-Padé technique has been successfully implemented to solve these problems by converting the series solutions into the diagonal Padé approximants. Numerical and figurative illustrations show that it is a promising tool for solving nonlinear problems.

The homotopy perturbation method (HPM) was first proposed by the Chinese mathematician Ji-Huan He [8, 9]. The essential idea of this method is to introduce a homotopy parameter, say p, which takes values from 0 to 1. When $p = 0$, the system of equations usually reduces to a sufficiently simplified form, which normally admits a rather simple solution. As p is gradually increased to 1, the system goes through a sequence of ‘deformations’, the solution for each of which is ‘close’ to that at the previous stage of ‘deformation’. Eventually at $p = 1$, the system takes the original form of the equation and the final stage of ‘deformation’ gives the desired solution. One of the most remarkable features of the HPM is that usually just a few perturbation terms are sufficient for obtaining a reasonably accurate solution. This technique has been employed to solve a large variety of linear and nonlinear problems [10 – 25]. The interested reader can see [26 – 29] for last development of HPM. The main idea in the using of Padé approximation for extension of the convergence of the series solutions has been proposed for the first time in [30].

2. Mathematical Formulation

Let us consider the MHD flow of an incompressible viscous fluid over a stretching sheet at $y = 0$. The fluid is electrically conducting under the influence of an applied magnetic field $B(x)$ normal to the scretching sheet. The induced magnetic field is neglected. The resulting boundary layer equations are

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0, \tag{1}$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\nu\frac{\partial^2 u}{\partial y^2}-\frac{\sigma B^2(x)}{\rho}u. \tag{2}$$

Here u and v are the velocity components in the x and y-directions, respectively, ν is the kinematic viscosity, ρ is the fluid density, and σ is the electrical conductivity of the fluid. In (2), the external electric field and the polarization effects are negligible and [6]

$$B(x)=B_0x^{(n-1)/2}.$$

The boundary conditions corresponding to the nonlinear stretching of a sheet are

$$u(x,0)=cx^n,\quad v(x,0)=0, \tag{3}$$

$$u(x,y)\to 0 \text{ as } y\to\infty. \tag{4}$$

Upon making use of the substitutions

$$\eta=\sqrt{\frac{c(n+1)}{2\nu}}x^{\frac{(n-1)}{2}}y, u=cx^nf'(\eta), \tag{5}$$

$$v=-\sqrt{\frac{c\nu(n+1)}{2}}x^{\frac{n-1}{2}}\left[f(\eta)+\frac{n-1}{n+1}\eta f'(\eta)\right], \tag{6}$$

the resulting nonlinear differential system is of the following form:

$$f'''+ff''-\beta f'^2-Mf'=0, \tag{7}$$

$$f(0)=0,\quad f'(0)=1,\quad f'(+\infty)=0, \tag{8}$$

where the primes denote differentiation with respect to η and

$$\beta=\frac{2n}{1+n},\quad M=\frac{2\sigma B_0^2}{\rho c(1+n)}.$$

3. The Padé Approximants to the Series Solution

We denote L and M as Padé approximants to $f(z)$ by

$$[L/M]=\frac{P_L(z)}{Q_M(z)}, \tag{9}$$

where $P_L(z)$ is a polynomial of degree at most L and $Q_M(z)$, $Q_M(z)\neq 0$ is a polynomial of degree at most M. The former power series is

$$f(z)=\sum_{k=0}^{\infty}c_kz^k. \tag{10}$$

We write the $P_L(z)$ and $Q_M(z)$ as

$$\begin{aligned}P_L(z)&=p_0+p_1z+p_2z^2+p_3z^3+\ldots+p_Lz^L,\\ Q_M(z)&=q_0+q_1z+q_2z^2+q_3z^3+\ldots+q_Mz^M,\end{aligned} \tag{11}$$

so

$$f(z)-\frac{P_L(z)}{Q_M(z)}=O(z^{L+M+1}) \text{ as } z\to 0, \tag{12}$$

and the coefficients of $P_L(z)$ and $Q_M(z)$ are determined by the equation. From (12), we have

$$f(z)Q_M(z)-P_L(z)=O(z^{L+M+1}), \tag{13}$$

which is a system of $L+M+1$ homogeneous equations with $L+M+2$ unknown quantities. We impose the normalization condition

$$Q_M(0)=1. \tag{14}$$

Then we can write out (13) as

$$\begin{aligned}&c_{L+1}+c_Lq_1+\ldots+c_{L-M+1}q_M=0,\\ &c_{L+2}+c_{L+1}q_1+\ldots+c_{L-M+2}q_M=0,\\ &\vdots\\ &c_{L+M}+c_{L+M-1}q_1+\ldots+c_Lq_M=0,\end{aligned} \tag{15}$$

$$\begin{aligned}&c_0=p_0,\\ &c_1+c_0q_1=p_1,\\ &c_2+c_1q_1+c_0q_2=p_2,\\ &\vdots\\ &c_L+c_{L-1}q_1+\ldots+c_0q_L=p_L.\end{aligned} \tag{16}$$

From (15) we can obtain the q_i $(1\leq i\leq M)$. Once the values of $q_1, q_2, \ldots, q_M$ are all known, (16) gives an explict formula for the unknown quantities $p_1, p_2, \ldots, p_L$. Since the diagonal approximants like [2/2], [3/3], [4/4], [5/5] or [6/6] are the most acccurate approximants by the built-in utilities of Maple software.

4. Solution by the Homotopy Perturbation Method

In order to solve (7) and (8) by the homotopy perturbation method, we construct the following homotopy:

$$f'''(\eta)=p(-f(\eta)f''(\eta)+\beta(f'(\eta))^2+Mf'(\eta)). \quad (17)$$

Assume the solution of (17) in the form

$$f=f_0+pf_1+p^2f_2+p^3f_3+\dots, \quad (18)$$

then substituting (18) into (17) and collecting terms of the same power of p, we get the following set of differential equations:

$$\begin{aligned}
p^0 &: f_0'''=0,\\
p^1 &: f_1'''=-(f_0f_0'')+\beta(f_0'^2)+M(f_0'),\\
p^2 &: f_2'''=-(f_0f_1''+f_1f_0'')+\beta(2f_0'f_1')+M(f_1'),\\
p^3 &: f_3'''=-(f_0f_2''+f_1f_1''+f_2f_0'')\\
&\qquad +\beta(f_1'^2+2f_0'f_2')+M(f_2'),\\
p^4 &: f_4'''=-(f_0f_3''+f_1f_2''+f_2f_1''+f_3f_0'')\\
&\qquad +\beta(2f_1'f_2'+2f_0'f_3')+M(f_3'),\\
p^5 &: f_5'''=-(f_0f_4''+f_1f_3''+f_2f_2''+f_3f_1''+f_4f_0'')\\
&\qquad +\beta(2f_0'f_4'+2f_1'f_3'+f_2'^2)+M(f_4'),\\
&\vdots
\end{aligned} \quad (19)$$

From (19), we obtain the following recursive iterative formula

$$f_0'''=0, \quad (20)$$

$$f_{k+1}'''=-A_k+\beta B_k+M(f_k'), \quad k\geq 0, \quad (21)$$

where $A_k=\sum_{i=0}^{k} f_i f_{k-i}''$ and $B_k=\sum_{i=0}^{k} f_i' f_{k-i}'$.

We now can solve the above ordinary differential equations and make use of the initial conditions $f(0)=0$ and $f'(0)=1$. Also $f''(0)=\alpha$ is to be determined from the conditions at infinity.

Thus, we successively obtain

$$f_0(\eta)=\eta,$$

$$f_1(\eta)=\frac{\alpha}{2}\eta^2+\left(\frac{\beta}{6}+\frac{M}{6}\right)\eta^3,$$

$$f_2(\eta)=\left(-\frac{\alpha}{24}+\frac{\beta\alpha}{12}+\frac{M\alpha}{24}\right)\eta^4 +\left(\frac{\beta^2}{60}-\frac{M}{60}+\frac{\beta M}{40}+\frac{M^2}{120}-\frac{\beta}{60}\right)\eta^5,$$

$$\begin{aligned}
f_3(\eta)=&\left(\frac{\beta\alpha^2}{60}-\frac{\alpha^2}{120}\right)\eta^5+\left(\frac{M^2\alpha}{720}+\frac{\alpha}{240}+\frac{\beta^2\alpha}{72}\right.\\
&\left.-\frac{\beta\alpha}{60}-\frac{M\alpha}{90}+\frac{\beta M\alpha}{72}\right)\eta^6+\left(\frac{\beta^3}{504}-\frac{\beta^2}{315}\right.\\
&-\frac{M^2}{504}+\frac{11\beta M^2}{5040}+\frac{\beta^2 M}{252}+\frac{M}{630}-\frac{13\beta M}{2520}\\
&\left.+\frac{\beta}{630}+\frac{M^3}{5040}\right)\eta^7,
\end{aligned}$$

$$\begin{aligned}
f_4(\eta)=&\left(\frac{11\alpha^2}{5040}-\frac{M\alpha^2}{630}-\frac{2\beta\alpha^2}{315}+\frac{\beta M\alpha^2}{504}\right.\\
&\left.+\frac{\beta^2\alpha^2}{252}\right)\eta^7+\left(\frac{103\beta\alpha}{40320}+\frac{\beta^2 M\alpha}{336}+\frac{M^3\alpha}{40320}\right.\\
&-\frac{83\beta^2\alpha}{20160}+\frac{\beta M^2\alpha}{960}-\frac{7\beta M\alpha}{1440}-\frac{\alpha}{2688}+\frac{M\alpha}{504}\\
&\left.+\frac{\beta^3\alpha}{504}-\frac{13M^2\alpha}{13440}\right)\eta^8+\left(-\frac{M^3}{8640}+\frac{43\beta M^3}{362880}\right.\\
&+\frac{71\beta M}{90720}-\frac{17\beta^3}{30240}+\frac{M^2}{3024}-\frac{73\beta^2 M}{60480}\\
&+\frac{5\beta^3 M}{9072}+\frac{\beta^2 M^2}{2240}+\frac{\beta^4}{4536}-\frac{\beta}{7560}+\frac{41\beta^2}{90720}\\
&\left.-\frac{23\beta M^2}{30240}-\frac{M}{7560}+\frac{M^4}{362880}\right)\eta^9,
\end{aligned}$$

$$\begin{aligned}
f_5(\eta)=&\left(\frac{\beta^2\alpha^3}{2016}-\frac{\beta\alpha^3}{1260}+\frac{11\alpha^3}{40320}\right)\eta^8+\left(\frac{5\alpha^2\beta^3}{6048}\right.\\
&-\frac{43\alpha^2}{120960}+\frac{\beta M^2\alpha^2}{8640}+\frac{5\beta^2 M\alpha^2}{6048}+\frac{3M\alpha^2}{4480}\\
&-\frac{13M^2\alpha^2}{120960}-\frac{13\beta M\alpha^2}{8640}-\frac{71\beta^2\alpha^2}{36288}\\
&\left.+\frac{137\beta\alpha^2}{90720}\right)\eta^9+\left(\frac{-31\beta^2 M\alpha}{22680}+\frac{\alpha}{34560}\right.\\
&-\frac{83M^3\alpha}{1814400}-\frac{233\beta\alpha}{725760}+\frac{5\beta^3 M\alpha}{9072}+\frac{359M^2\alpha}{1209600}\\
&+\frac{3929\beta M\alpha}{3628800}+\frac{13\beta^2M^2\alpha}{40320}+\frac{1469\beta^2\alpha}{1814400}\\
&-\frac{373\beta M^2\alpha}{604800}+\frac{5\beta^4\alpha}{18144}-\frac{121\beta^3\alpha}{151200}-\frac{989M\alpha}{3628800}\\
&\left.+\frac{17\beta M^3\alpha}{362880}+\frac{M^4\alpha}{3628800}\right)\eta^{10}+\left(\frac{-47M\beta^3}{207900}\right.\\
&-\frac{17M^4}{3991680}-\frac{109M^2}{2494800}+\frac{95\beta^2 M}{399168}+\frac{37\beta M^2}{221760}\\
&+\frac{5M\beta^4}{66528}-\frac{421\beta^4}{4989600}+\frac{13\beta^3}{124740}-\frac{331\beta M^3}{4989600}\\
&+\frac{M^3}{30240}-\frac{581\beta^2M^2}{2851200}+\frac{M^5}{39916800}+\frac{5\beta^5}{199584}
\end{aligned}$$

Table 1. Comparision of the values of $f''(0) = \alpha$ obtained by the HPM and the exact solution.

β	M	[11/11]	[12/12]	[13/13]	[14/14]	[15/15]	Exact [30] Solution
1	0	0.99999	−1.00005	−1.00000	−1.00000	1.00000	1.00000
	1	−1.41426	−1.07323	−1.41421	−1.41421	−1.41421	−1.41421
	5	−2.44949	−2.44947	−2.44948	−2.44948	−2.44948	−2.44948
	10	−3.31662	−3.31662	−3.31662	−3.31662	−3.31662	−3.31662
	50	−7.14142	−7.14142	−7.14142	−7.14142	−7.14142	−7.14142
	100	−10.04987	−10.04985	−10.04987	−10.04987	−10.04987	−10.04987
	500	−22.38302	−22.38302	−22.38302	−22.38302	−22.38302	−22.38302
	1000	−31.63858	−31.63858	−31.63895	−31.63858	−31.63858	−31.63858

Table 2. Numerical values of $f''(0) = \alpha$ for different values of β and M.

β	M	[6]	[15/15]	β	M	[6]	[15/15]
1.5	0	−1.1486	−1.1547	5	0	−1.9025	−1.9098
	1	−1.5252	−1.5252		1	−2.1529	−2.1528
	5	−2.5161	−2.5161		5	−2.9414	−2.9414
	10	−3.3663	−3.3663		10	−3.6956	−3.6956
	50	−7.1647	−7.1647		50	−7.3256	−7.3256
	100	−10.0664	−10.0776		100		−10.1816
	500		−22.3904		500		−22.4425
	1000		−31.6438		1000		−31.6806
−1	0	0.0	0.0	−1.5	0	0.72725	
	1	−0.8511	−0.8511		1	−0.6529	−0.6532
	5	−2.1628	−2.1628		5	−2.0852	−2.0852
	10	−3.1100	−3.1100		10	−3.0562	−3.0562
	50		−7.0475		50		−7.0238
	100	−9.9833	−9.9833		100	−9.9666	−9.9666
	500		−22.3532		500		−22.3457
	1000		−31.6175		1000		−31.6122

$$+\frac{19\beta M^4}{4435200}+\frac{\beta}{103950}-\frac{67\beta^2}{1247400}+\frac{317\beta^3M^2}{3991680} +\frac{M}{103950}+\frac{67\beta^2M^3}{1995840}-\frac{3\beta M}{30800}\Bigg)\eta^{11}$$

...

and so on; in this manner, the rest of the components of the homotopy perturbation series can be obtained. By setting $p \to 1$ in (18), we obtain

$$f(\eta) = f_0(\eta)+f_1(\eta)+f_2(\eta)+f_3(\eta) + f_4(\eta)+f_5(\eta)+\ldots.$$

5. Results and Discussions

The algorithm (20)–(21) is coded in the computer algebra package Maple and we employ Maple's built-in Padé approximants procedure. The Maple environment variable 'Digits' controlling the number of significant digits is set to 16 in all the calculations done in this paper. The undetermined value of α is calculated from the boundary condition at infinity. The difficulty at infinity is overcome by employing the diagonal Padé approximants that approximate $f'(\eta)$.

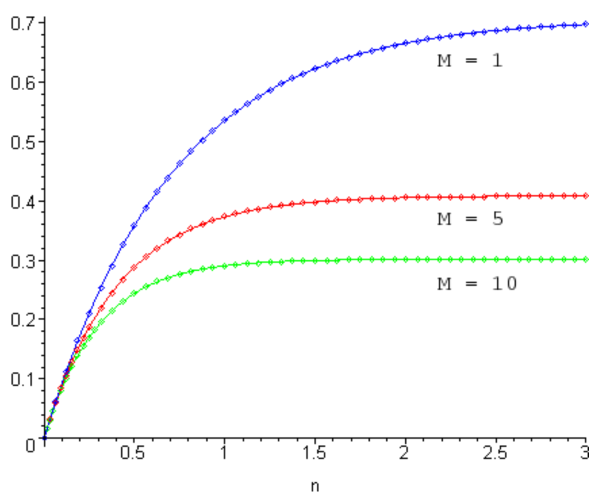

Fig. 1 (colour online). Comparison of the solution obtained by the HPM-Padé technique (circle) and the exact solution. M = 1 (top), M = 5 (middle), M = 10 (bottom).

In Table 1 and Table 2 we present the value of $\alpha = f''(0)$ at different orders of Padé approximations.

It is noted that we have no convergent value of $f''(0) = \alpha$ when $\beta = -1.5$ and M = 0.

For the special case $\beta = 1$, the exact solution of (7) – (8) is [31].

$$f(\eta) = \frac{1 - \mathrm{e}^{\left(-\sqrt{1+M}\eta\right)}}{\sqrt{1+M}}, \quad f''(0) = -\sqrt{1+M}.$$

6. Conclusions

In this paper, we used the HPM-Padé technique to obtain a non-perturbative solution of the MHD flow over a nonlinear stretching sheet. We used the Padé technique to enlarge the convergence domain. A comparison of the present solution with the existing exact solution for $\beta = 1$ is made [31] (Fig. 1). A good agreement between the present and exact solutions is achieved. In this work, we used the well-known software Maple to calculate the series and the rational functions obtained from the proposed technique.

Acknowledgement

The authors sincerely thank the unknown reviewers for their constructive comments and suggestions.

[1] T. Altan, S. Oh, and H. Gegrl, American Society of Metals, Metals Park 1979.
[2] E. G. Fisher, Extrusion of Plastics, Wiley, New York 1976.
[3] Z. Tadmor and I. Klein, Engineering Principles of Plasticating Extrusion, Polymer Science and Engineering Series, Reinhold, New York, Van Nostrand 1970.
[4] B. C. Sakiadis, AIChE J. **7**, 26 (1961).
[5] B. C. Sakiadis, AICHE. J. **7**, 221 (1961).
[6] T. C. Chiam, Int. J. Eng. Sci. **33**, 429 (1995).
[7] T. Hayat, Q. Hussain, and T. Javed, Nonlinear Anal.: Real World Appl. **10**, 966 (2009).
[8] J. H. He, Comput. Methods Appl. Mech. Eng. **178**, 257 (1999).
[9] J. H. He, Int. J. Nonlinear Mech. **35**, 37 (2000).
[10] H. Koçak and A. Yıldırım, Phys. Lett. A **373**, 3145 (2009).
[11] H. Koçak and A. Yıldırım, Commun. Numer. Methods Eng. **25**, 1084 (2009).
[12] M. E. Berberler and A. Yıldırım, Appl. Anal. (2009).
[13] A. Yıldırım and T. Öziş, Phys. Lett. A **369**, 70 (2007).
[14] A. Yıldırım, Comput. Math. Appl. **56**, 3175 (2008).
[15] A. Yıldırım, Int. J. Comput. Math. **87**, 992 (2008).
[16] A. Yıldırım, Z. Naturforsch. **63a**, 621 (2008).
[17] A. Yıldırım, Commun. Numer. Methods Eng. **25**, 1127 (2008).
[18] L. Cveticanin, Chaos, Solitons, and Fractals **30**, 1221 (2006).
[19] A. Yıldırım, Chaos, Solitons, and Fractals **42**, 2760 (2009).
[20] J. Biazar and H. Ghazvini, Appl. Math. Comput. **195**, 681 (2008).
[21] J. Biazar and H. Ghazvini, Phys. Lett. A **366**, 79 (2007).
[22] M. Dehghan and F. Shakeri, PIER **78**, 361 (2008).
[23] F. Shakeri and M. Dehghan, Physica Scripta **75**, 551 (2007).
[24] M. Dehghan and F. Shakeri, Phys. Scripta **75**, 778 (2007).
[25] F. Shakeri and M. Dehghan, Math. Comput. Model. **48**, 486 (2008).
[26] J. H. He, Int. J. Modern Phys. B **22**, 3487 (2008).
[27] J. H. He, Topol. Methods Nonlinear Anal. **31**, 205 (2008).
[28] J. H. He, Int. J. Mod. Phys. B **20**, 1141 (2006).
[29] J. H. He, Int. J. Mod. Phys. B **20**, 2561 (2006).
[30] Y. C. Jiao, Y. Yamamoto, C. Dang, and Y. Hao, Comput. Math. Appl. **43**, 783 (2002).
[31] K. B. Pavlov, Magn. Gidrod. **4**, 146 (1975).