Exact Solutions of Flows of an Oldroyd 8-Constant Fluid with Nonlinear Slip Conditions

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This communication is concerned with the nonlinear flows of an Oldroyd 8-constant fluid when the no-slip condition is not valid. Due to slip effects in terms of shear stress, the arising slip conditions are nonlinear. The resulting mathematical problems involves nonlinear differential equations and nonlinear boundary conditions. To the best of my knowledge, no such analysis for the flows of an Oldroyd 8-constant fluid is available in the literature. Graphs are plotted for the velocity profiles and examined with respect to the sundry emerging parameters.

Key words: Nonlinear Conditions; Analytic Solutions; Oldroyd 8-Constant Fluid.

1. Introduction

Many fluids, for instance, yoghurt, ketchup, shampoo, blood, and several fluids in industry cannot be described by the well-known Navier-Stokes equations. Mathematical problems for the flow of such fluids involve equations which are very complicated and of higher order in comparison with the Navier-Stokes equations [1, 2]. Such problems are further complicated when slip conditions are considered. Consequently, such nonlinear problems offer interesting challenges to the applied mathematicians, physicists, modelers, and computer scientists alike. In spite of all these difficulties, a good list of references on the topic can be found in the recent studies [3 – 10].

Much attention to non-Newtonian fluid dynamics is given to the flows with no-slip conditions. However, there are fluids, for example, polymeric materials that slip or stick-slip on solid surfaces. In view of such considerations, some studies on flows of non-Newtonian fluids are available when slip condition is chosen [11 – 14]. It is well accepted now that slip effects may appear for two types of fluids (i. e., rare field gases [15] and fluids having much more elastic character). In these fluids, slippage appear subject to large tangential traction. It is noticed through experimental observations [16 – 22] that the occurrence of slippage is possible in the non-Newtonian fluids, polymer solution, and molten polymer. In addition, a clear layer is sometimes found next to the wall when flow of a dilute suspension of particles is examined. In experimental physiology such a layer is observed when blood flow through capillary vessels is studied [23]. The fluids that exhibit slip effect have many applications, for instance, the polishing of artificial heart valves and internal cavities [24]. Moreover, the slip phenomenon is supported by the molecular theories [25 – 28].

The objective of the present paper is to investigate the two nonlinear flow cases of an Oldroyd 8-constant fluid with slip conditions, without and with pressure gradient. Flow is considered between two concentric cylinders. In the first problem, the inner cylinder moves and the outer cylinder remains stationary. Second problem deals with the flow situation when the inner cylinder is at rest and the outer cylinder is in motion. Both differential systems are subjected to nonlinear differential equations and nonlinear boundary conditions. Exact solutions are developed and computations have been made for the salient features of the involved pertinent parameters.

2. Problem Formulation

I consider the steady flow between two concentric cylinders. The inner and outer cylinders have radii R_0 and R_1, respectively, and $k = R_1/R_0 < 1$. In a (r, θ, z) system, we select the z-axis along the common axis of the cylinders and r normal to it. I also assume that no flow occurs in the θ-direction. The velocity field $\mathbf{V}$ for the flow under investigation satisfies the following expression:

$$\mathbf{V} = (0, 0, u(r)). \tag{1}$$

The Cauchy stress $\mathbf{T}$ and extra stress $\mathbf{S}$ tensors in an Oldroyd 8-constant fluid can be expressed as

$$\mathbf{T} = -p\mathbf{I} + \mathbf{S}, \tag{2}$$

in which p is the pressure, $\mathbf{I}$ the identity tensor, and $\mathbf{S}$ satisfies

$$\begin{aligned} &\mathbf{S} + \lambda_1 \frac{\mathrm{D}\mathbf{S}}{\mathrm{D}t} + \frac{\lambda_3}{2}(\mathbf{S}\mathbf{A}_1 + \mathbf{A}_1\mathbf{S}) + \frac{\lambda_5}{2}(\mathrm{tr}\,\mathbf{S})\mathbf{A}_1 \\ &+ \frac{\lambda_6}{2}[\mathrm{tr}(\mathbf{S}\mathbf{A}_1)]\mathbf{I} = \mu\left[\mathbf{A}_1 + \lambda_2 \frac{\mathrm{D}\mathbf{A}_1}{\mathrm{D}t} + \lambda_4 \mathbf{A}_1^2 \right. \\ &\left. \qquad + \frac{\lambda_7}{2}[\mathrm{tr}(\mathbf{A}_1^2)]\mathbf{I}\right], \end{aligned} \tag{3}$$

where

$$\frac{\mathrm{D}\mathbf{S}}{\mathrm{D}t} = \frac{\partial \mathbf{S}}{\partial t} - \mathbf{L}\mathbf{S} - \mathbf{S}\mathbf{L}^T, \quad \mathbf{L} = \mathrm{grad}\,\mathbf{V},$$
$$\mathbf{A}_1 = \mathbf{L} + \mathbf{L}^T,$$

and $\mathrm{D}/\mathrm{D}t$ is the material derivative, μ and λ_i $(i = 1-8)$ are the dynamic viscosity and material constants, respectively.

One can easily note that the continuity equation is satisfied by (1) and the z-component of the balance of linear momentum after using (1) – (3) yields

$$\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}(r(S_{rz})) = \frac{\mathrm{d}p}{\mathrm{d}z}, \tag{4}$$

where

$$S_{rz} = \mu\left[\frac{1 + \alpha_1\left(\frac{\mathrm{d}u}{\mathrm{d}r}\right)^2}{1 + \alpha_2\left(\frac{\mathrm{d}u}{\mathrm{d}r}\right)^2}\right]\frac{\mathrm{d}u}{\mathrm{d}r},$$
$$\alpha_1 = \lambda_1(\lambda_4 + \lambda_7) - (\lambda_3 + \lambda_5)(\lambda_4 + \lambda_7 - \lambda_2) - \frac{\lambda_5\lambda_7}{2},$$
$$\alpha_2 = \lambda_1(\lambda_3 + \lambda_6) - (\lambda_3 + \lambda_5)(\lambda_3 + \lambda_6 - \lambda_1) - \frac{\lambda_5\lambda_6}{2}.$$

In the above equation the subjected boundary conditions are

$$u(r) - \frac{\xi}{\mu}S_{rz} = U_0 \text{ at } r = R_0, \tag{5}$$

$$u(r) + \frac{\xi}{\mu}S_{rz} = 0 \text{ at } r = R_1. \tag{6}$$

By the rescaling

$$u^* = \frac{u}{U_0}, \quad r^* = \frac{r}{R_0}, \quad z^* = \frac{z}{R_0}, \quad p^* = \frac{pR_0}{\rho U_0},$$
$$\alpha_1^* = \frac{\alpha_1 U_0^2}{R_0^2}, \quad \alpha_2^* = \frac{\alpha_2 U_0^2}{R_0^2}, \quad \gamma = \frac{\xi}{R_0},$$

(4) – (6) can be put in dimensionless form as

$$\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left\{r\left[\frac{1 + \alpha_1\left(\frac{\mathrm{d}u}{\mathrm{d}r}\right)^2}{1 + \alpha_2\left(\frac{\mathrm{d}u}{\mathrm{d}r}\right)^2}\right]\frac{\mathrm{d}u}{\mathrm{d}r}\right\} = c, \tag{7}$$

$$u(1) - \gamma\left\{\left[\frac{1 + \alpha_1\left(\frac{\mathrm{d}u}{\mathrm{d}r}\right)^2}{1 + \alpha_2\left(\frac{\mathrm{d}u}{\mathrm{d}r}\right)^2}\right]\frac{\mathrm{d}u}{\mathrm{d}r}\right\}\Bigg|_{r=1} = 1, \tag{8}$$

$$u(k) + \gamma\left\{\left[\frac{1 + \alpha_1\left(\frac{\mathrm{d}u}{\mathrm{d}r}\right)^2}{1 + \alpha_2\left(\frac{\mathrm{d}u}{\mathrm{d}r}\right)^2}\right]\frac{\mathrm{d}u}{\mathrm{d}r}\right\}\Bigg|_{r=k} = 0, \tag{9}$$

where the asterisks have been omitted for simplicity.

In the next section, the problem consisting of (7) – (9) is solved for the two cases, i. e. for $c = 0$ and for $c \neq 0$.

3. Exact Solutions of the Problems

3.1. Zero Pressure Gradient

First, I obtain exact solutions for the case when the pressure gradient c is zero. In this case (7) has the first integral

$$\frac{1 + \alpha_1\left(\frac{\mathrm{d}u}{\mathrm{d}r}\right)^2}{1 + \alpha_2\left(\frac{\mathrm{d}u}{\mathrm{d}r}\right)^2}\frac{\mathrm{d}u}{\mathrm{d}r} = \frac{A}{r}, \tag{10}$$

where A is an arbitrary constant. The boundary conditions (8) and (9) become

$$u(1) = \gamma A + 1, \quad u(k) = \frac{-\gamma A}{k}. \tag{11}$$

The general solution of (10) is

$$u = \Lambda(\alpha_1, \alpha_2, A)r + B, \tag{12}$$

where B is a constant and $\Lambda(\alpha_1,\alpha_2,A)$ is given by

$$\Lambda(\alpha_1,\alpha_2,A)=\left\{\frac{1}{2}\alpha_1 A\left(\alpha_1-\frac{1}{3}\alpha_2\right)+\frac{1}{3^3}A^3\alpha_2^3+\frac{1}{2}\left[\frac{4}{3^3}\alpha_1^2\alpha_2^3A^4+\left(\alpha_1^4-\frac{2}{3}\alpha_1^2\alpha_2-\frac{1}{3^3}\alpha_1^2\alpha_2^2\right)A^2+\frac{4}{3^3}\alpha_1^3\right]^{1/2}\right\}^{1/3}+\left\{\frac{1}{2}\alpha_1 A\left(\alpha_1-\frac{1}{3}\alpha_2\right)+\frac{1}{3^3}A^3\alpha_2^3-\frac{1}{2}\left[\frac{4}{3^3}\alpha_1^2\alpha_2^3A^4+\left(\alpha_1^4-\frac{2}{3}\alpha_1^2\alpha_2-\frac{1}{3^3}\alpha_1^2\alpha_2^2\right)A^2+\frac{4}{3^3}\alpha_1^3\right]^{1/2}\right\}^{1/3}. \tag{13}$$

The imposition of the boundary condition (11a) on (12) results in

$$u=(r-1)\Lambda(\alpha_1,\alpha_2,A)+\gamma A+1. \tag{14}$$

The second boundary condition (11b) in the solution (14) yields the relation

$$(k-1)\Lambda(\alpha_1,\alpha_2,A)+\gamma A\left(1+\frac{1}{k}\right)+1=0 \tag{15}$$

among the constants and parameters k, α_1, α_2, A, and γ with $\Lambda(\alpha_1,\alpha_2,A)$ are given by (13).

Note that if $A=0$, then we get the solution

$$u=\pm(-\alpha_1)^{-\frac{1}{2}}(r-1),\quad \alpha_1<0 \tag{16}$$

with the relation between α_1 and k as

$$-\alpha_1=(1-k)^2. \tag{17}$$

The solution u = constant is not possible for $\Lambda=0$ as it violates the boundary condition (5) for $A=0$. This trivial solution can be obtained from (13) and (14) by setting $A=0$.

3.2. Non-Zero Pressure Gradient

Now, I solve (7)–(9) for non-zero pressure gradient c.

A first integral of (7) for $c\neq 0$ is

$$r\left[\frac{1+\alpha_1\left(\frac{du}{dr}\right)^2}{1+\alpha_2\left(\frac{du}{dr}\right)^2}\right]\frac{du}{dr}=\frac{cr^2}{2}+c_1, \tag{18}$$

where c_1 is an arbitrary constant. As a consequence the boundary conditions (8) and (9) become

$$u(1)=\frac{\gamma c}{2}+\gamma c_1+1,\quad u(k)=-\frac{\gamma ck}{2}-\frac{\gamma c_1}{k}. \tag{19}$$

The constant c_1 appears in the boundary conditions (19). Previously the choice $A=0$ in (11) led to a restricted solution. However, here we can select c_1 to be any particular value. The simple choice $c_1=0$ results in (18) becoming

$$\frac{1+\alpha_1\left(\frac{du}{dr}\right)^2}{1+\alpha_2\left(\frac{du}{dr}\right)^2}\frac{du}{dr}=\frac{cr}{2}. \tag{20}$$

By means of the transformation

$$\bar{r}=\frac{c}{2}r,\quad c\neq 0,\quad \bar{u}=\frac{c}{2}u, \tag{21}$$

(20) transforms to

$$\left[1+\alpha_1\left(\frac{d\bar{u}}{d\bar{r}}\right)^2\right]\frac{d\bar{u}}{d\bar{r}}=\bar{r}\left[1+\alpha_2\left(\frac{d\bar{u}}{d\bar{r}}\right)^2\right]. \tag{22}$$

One can then obtain

$$\frac{d\bar{u}}{d\bar{r}}=\left[\frac{1}{2}\alpha_1\bar{r}\left(\alpha_1-\frac{1}{3}\alpha_2\right)+\frac{1}{3^3}\bar{r}^3\alpha_2^3+3^{-\frac{3}{2}}\left(\alpha_1\alpha_2^{\frac{3}{2}}\bar{r}^2+\alpha_1^{\frac{3}{2}}\right)\right]^{1/3}+\left[\frac{1}{2}\alpha_1\bar{r}\left(\alpha_1-\frac{1}{3}\alpha_2\right)+\frac{1}{3^3}\bar{r}^3\alpha_2^3-3^{-\frac{3}{2}}\left(\alpha_1\alpha_2^{\frac{3}{2}}\bar{r}^2+\alpha_1^{\frac{3}{2}}\right)\right]^{1/3}\approxeq\Omega(\alpha_1,\alpha_2,\bar{r}) \tag{23}$$

provided the relation

$$\frac{64}{3^6}\alpha_1^5\alpha_2^3=\left(\alpha_1^4-\frac{3}{2}\alpha_1^3\alpha_2-\frac{1}{3^3}\alpha_1^2\alpha_2^2\right)^2 \tag{24}$$

holds. One can deduce a more general equation than (23) without the need for the relation (24). However, this unduly complicates the resulting solution.

The boundary conditions (19) with $c_1=0$ in the new coordinates $(\bar{r},\bar{u})$ are

$$\bar{u}\left(\frac{c}{2}\right)=\frac{\gamma c^2}{4}+\frac{c}{2},\quad \bar{u}\left(\frac{ck}{2}\right)=-\frac{\gamma c^2k}{4},\quad k\neq 1. \tag{25}$$

The solution of (23) subject to the boundary conditions (25) is

$$\bar{u}=\int_{\frac{c}{2}}^{\bar{r}}\Omega(\alpha_1,\alpha_2,z)dz+\frac{\gamma c^2}{4}+\frac{c}{2} \tag{26}$$

with the condition

$$\int_{\frac{c}{2}}^{\frac{c}{2}k}\Omega(\alpha_1,\alpha_2,\bar{r})d\bar{r}=-\frac{\gamma c^2k}{4}-\frac{\gamma c^2}{4}-\frac{c}{2} \tag{27}$$

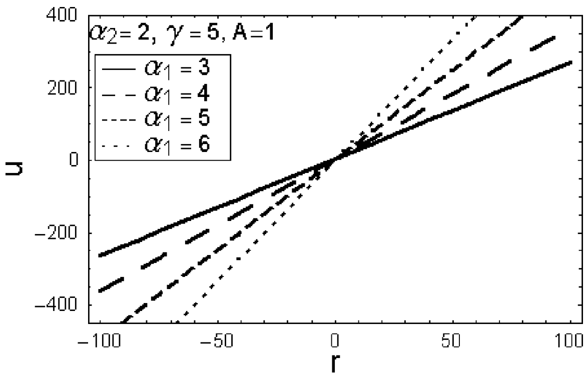

Fig. 1. Profiles of dimensionless velocity $u(y)$ for zero pressure gradient with various values of α_1, when α_2, γ, and A are fixed.

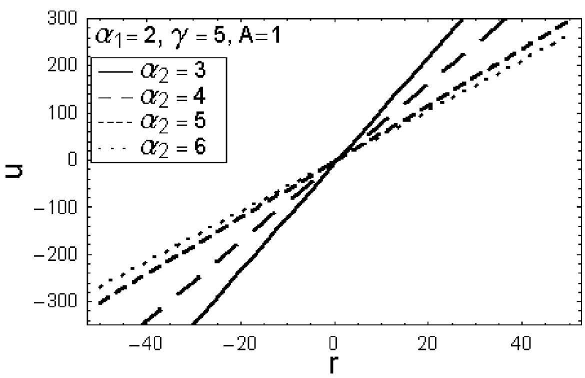

Fig. 2. Profiles of dimensionless velocity $u(y)$ for zero pressure gradient with various values of α_2, when α_1, γ, and A are fixed.

being satisfied. The solutions are discussed and interpreted in the next section.

4. Discussion

In order to illustrate the influences of α_1, α_2, slip parameter γ, and pressure gradient c on the velocity u, we have plotted seven figures. Figures 1 – 3 are for zero pressure gradient and Figures 4 – 7 are for the non-zero pressure gradient flow.

5. Concluding Remarks

The present study contributes an exact solution for the nonlinear flows of an Oldroyd 8-constant fluid when the no-slip condition is not valid. In addition, the analysis for the non-zero pressure gradient is also made. As a result, the following observations are stated.

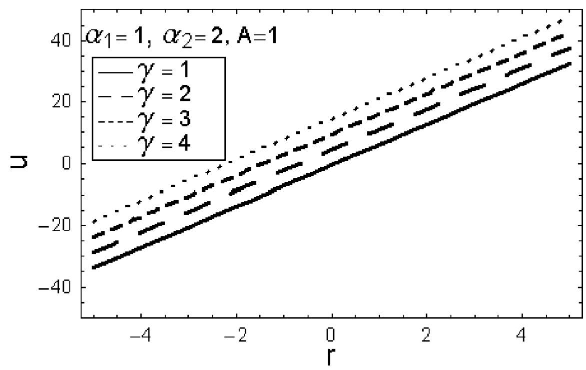

Fig. 3. Profiles of dimensionless velocity $u(y)$ for zero pressure gradient with various values of γ, when α_2, α_1, and A are fixed.

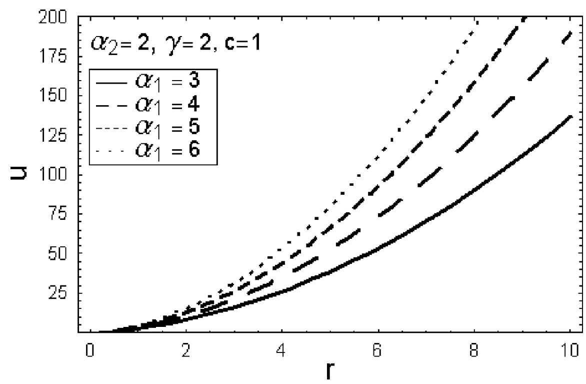

Fig. 4. Profiles of dimensionless velocity $u(y)$ for non-zero pressure gradient with various values of α_1, when α_2, γ, and c are fixed.

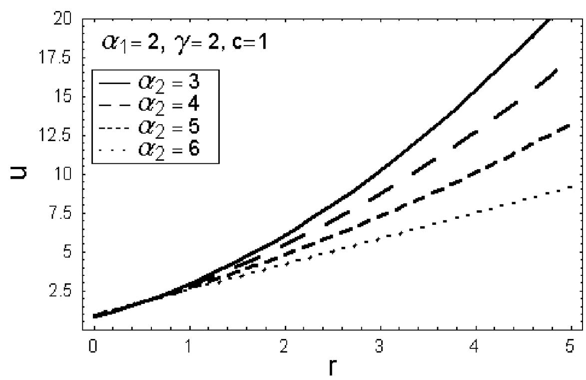

Fig. 5. Profiles of dimensionless velocity $u(y)$ for zero pressure gradient with various values of α_2, when α_1, γ, and c are fixed.

- For zero pressure gradient, an increase in α_1 leads to a decrease in the velocity profile u on the left and increase on the right of the origin whereas the behaviour is reversed in the case of α_2.

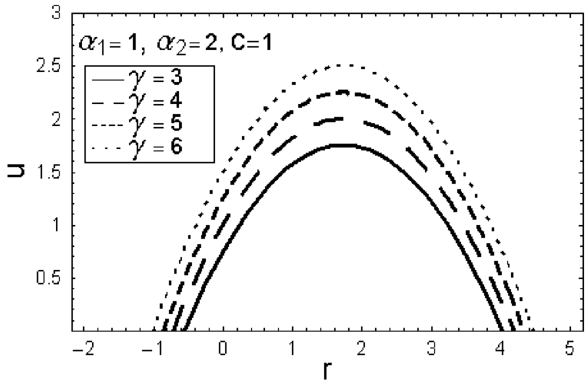

Fig. 6. Profiles of dimensionless velocity $u(y)$ for non-zero pressure gradient with various values of γ, when α_2, α_1, and c are fixed.

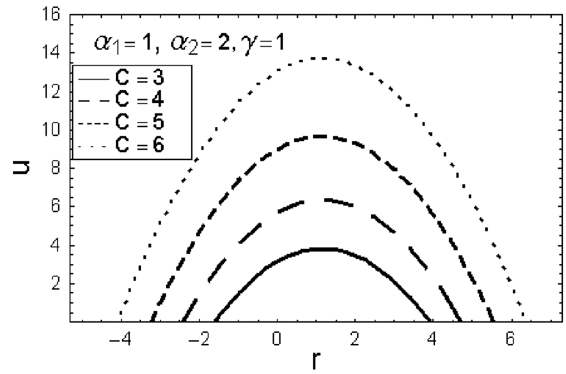

Fig. 7. Profiles of dimensionless velocity $u(y)$ for non-zero pressure gradient with various values of c, when α_2, α_1, and γ are fixed.

• For non-zero pressure gradient, by increasing the values of α_1 gives rise to decreasing the velocity, whereas the behaviour of u is reversed in the case of α_2.

• The increase in slip parameter γ and the pressure gradient c results in the velocity increasing.

• It is noted that the solution of the first problem is linear whereas in the second case the solution is nonlinear.

• The velocity profile u for zero pressure gradient is greater than to non-zero pressure gradient. The curvature of the velocity profile depends on the amplitude of the parameters.

• It is also worth mentioning that our exact solutions are more general with variable boundary conditions. Therefore, one can easily obtain other exact solutions for different parameters with different boundaries. Moreover, our exact analytical solutions are not only valid for small but also for large values of all emerging parameters.

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