Homotopy Perturbation Method for One-Dimensional Hyperbolic Equation with Integral Conditions

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In this study, we use the homotopy perturbation method (HPM) to solve an initial-boundary value problem with a non-classic condition for the one-dimensional wave equation. We will deal with a new type of non-local boundary value problems which are the solution of hyperbolic partial differential equations with a non-standard boundary specification. The method is very reliable and effective and provides the solution in terms of rapid convergent series. Several examples are tested to support our study.

Key words: Homotopy Perturbation Method; Wave Equation; Non-Standard Boundary Conditions; Closed Form Solution; Hyperbolic Partial Differential Equation.

1. Introduction

In this paper, an initial-boundary value problem with a non-classic condition for the one-dimensional wave equation will be approached analytically to obtain the exact solutions for these problems. Our approach stems mainly from the homotopy perturbation method. Mathematical modelling of many frontier physical problems leads to hyperbolic partial differential equations with a given initial condition, a standard boundary condition, and an integral condition replacing the classic boundary condition [1, 2]. An effective method is required to analyze the mathematical model which provides solutions confirming to physical reality, i. e. the real of physics. Generally, the numerical techniques such as finite difference formulae, finite element procedures, and spectral schemes are based on the discretization methods, and they only permit us to calculate the approximate solutions for some values of time and space variables, which cause to overlook some important phenomena such as chaos and bifurcation, because generally non-classical partial differential problems exhibit some delicate structures in very small time and space intervals. In addition, the numerical techniques require computer-intensive calculations. Recently, Dehghan [3] used the Adomian decomposition method for solving the governing problem.

In recent years a lot of attention has been devoted to the study of the homotopy perturbation method (HPM) because it provides a promising tool in the series solution field. The results from these investigations indicate that the homotopy perturbation method and the related phenomena can offer practical advantages over the methods currently available. It is well known in the literature that the homotopy perturbation method provides the solution in a rapidly convergent series where the series may yield the solution in a closed form. The homotopy perturbation method was first proposed by the Chinese mathematician Ji-Huan He [4, 5]. This technique has been employed to solve a large variety of linear and nonlinear problems [6 – 11]. The interested reader can see [12 – 28] for latest developments of HPM. It is worth mentioning that Dehghan and co-workers [16 – 26] used a variety of reliable techniques for the numerical solution of one-dimensional hyperbolic equations with non-local boundary specifications of various types. It has to be highlighted that Geng [27, 28] was the first scientist who pointed out the effective approach to such problems using the homotopy perturbation, variational iteration, and numerical methods.

In this paper, we considered the following hyperbolic problem with a non-local constraint in place of a standard boundary condition:

$$\frac{\partial^2 u}{\partial t^2} - p(x,t)\frac{\partial^2 u}{\partial x^2} = q(x,t), \quad 0 < x < l, \quad 0 < t \le T, \tag{1}$$

with the initial conditions

$$u(x,0) = f(x), \quad 0 \le x \le l, \tag{2}$$

and

$$\frac{\partial u}{\partial t}(x,0) = g(x), \quad 0 \le x \le l, \tag{3}$$

the Dirichlet boundary condition

$$u(x,t) = h(t), \tag{4}$$

and the non-local condition

$$\int_0^l u(x,t)\mathrm{d}x = m(t), \quad 0 < t \le T, \tag{5}$$

where q, f, and g are known functions.

The purpose of the present article is to demonstrate that the homotopy perturbation method can be used as a powerful tool for solving hyperbolic partial differential equations with non-local boundary conditions. The method is useful for obtaining both closed form of the explicit solution and numerical approximations of linear or nonlinear differential equations and it is also quite straightforward to write computer codes.

2. Test Examples [3]

Example 1. Consider (1) – (5) with $l = 1, T = 1$, and

$$f(x) = 1, \quad 0 < x < 1, \tag{6}$$

$$g(x) = 0, \quad 0 < x < 1, \tag{7}$$

$$h(t) = 1, \tag{8}$$

$$p(x,t) = 1, \quad 0 < t < 1, \tag{9}$$

$$q(x,t) = (t^2 - x^2)\mathrm{e}^{tx}, \quad 0 < t < 1, \tag{10}$$

$$m(t) = \frac{\mathrm{e}^t - 1}{t}. \tag{11}$$

The exact solution of this equation is:

$$u(x,t) = \exp(tx). \tag{12}$$

To solve the problem by the homotopy perturbation method, we construct the following homotopy:

$$\frac{\partial^2 u}{\partial t^2} = p\left(\mathrm{e}^{tx}(t^2 - x^2) + \frac{\partial^2 u}{\partial x^2}\right), \quad 0 < x < l, \quad 0 < t \le T. \tag{13}$$

Assume the solution of (13) to be in the form

$$u = u_0 + pu_1 + p^2u_2 + p^3u_3 + \dots. \tag{14}$$

Substituting (14) into (13) and collecting terms of the same power of p gives

$$p^0 : \frac{\partial^2 u_0}{\partial t^2} = 0,$$

$$p^1 : \frac{\partial^2 u_1}{\partial t^2} - (t^2 - x^2)\mathrm{e}^{tx} - \frac{\partial^2 u_0}{\partial x^2} = 0,$$

$$p^2 : \frac{\partial^2 u_2}{\partial t^2} - \frac{\partial^2 u_1}{\partial x^2} = 0,$$

$$p^3 : \frac{\partial^2 u_3}{\partial t^2} - \frac{\partial^2 u_2}{\partial x^2} = 0,$$

$$\vdots$$

The solution reads

$$u_0(x,t) = 1 + tx + \frac{t^2x^2}{2} - \frac{x^4 + t^4}{24},$$

$$u_1(x,t) = \frac{x^4 + t^4}{24} - \frac{x^2t^2}{2},$$

$$u_2(x,t) = \frac{x^2t^2}{4} - \frac{x^4 + t^4}{48},$$

$$u_3(x,t) = \frac{x^4 + t^4}{48} - \frac{x^2t^2}{8},$$

$$\vdots$$

The solution of the problem can be obtained by setting $p = 1$ in (14):

$$u(x,t) = 1 + tx + \frac{t^2x^2}{2} + \dots. \tag{15}$$

Therefore, the solution in a closed form is (12). We found that the homotopy perturbation technique is quite efficient to determine solutions in closed form when they exist.

Example 2. We consider (1) – (5) with $T = 1, l = 1$, and

$$f(x) = x^2, \quad 0 < x < 1, \tag{16}$$

$$g(x) = 0, \quad 0 < x < 1, \tag{17}$$

$$h(t) = t^2, \tag{18}$$

$$p(x,t) = 1, \tag{19}$$

$$q(x,t) = 0, \tag{20}$$

$$m(t) = t^2 + \frac{1}{3}. \tag{21}$$

The exact solution is

$$u(x,t) = x^2 + t^2. \tag{22}$$

To solve the problem by the homotopy perturbation method, we construct the following homotopy:

$$\frac{\partial^2 u}{\partial t^2} = p\frac{\partial^2 u}{\partial x^2}, \quad 0 < x < l, \quad 0 < t \le T. \qquad (23)$$

Substituting (14) into (23) and collecting terms of the same power of p gives

$$p^0 : \frac{\partial^2 u_0}{\partial t^2} = 0,$$
$$p^1 : \frac{\partial^2 u_1}{\partial t^2} - \frac{\partial^2 u_0}{\partial x^2} = 0,$$
$$p^2 : \frac{\partial^2 u_2}{\partial t^2} - \frac{\partial^2 u_1}{\partial x^2} = 0,$$
$$p^3 : \frac{\partial^2 u_3}{\partial t^2} - \frac{\partial^2 u_2}{\partial x^2} = 0,$$
$$\vdots$$

The solution reads

$$u_0(x,t) = \frac{1}{2}(x^2+t^2),$$
$$u_1(x,t) = \frac{1}{4}(x^2+t^2),$$
$$u_2(x,t) = \frac{1}{8}(x^2+t^2),$$
$$\vdots$$

So we obtain the following iterative formula:

$$u_i(x,t) = \frac{1}{2^{i+1}}(x^2+t^2), \qquad (24)$$

and then, we obtain

$$u(x,t) = (x^2+t^2)\sum_{n=0}^{\infty}\frac{1}{2^{n+1}}. \qquad (25)$$

The method avoids the difficulties and massive computational work by determining the analytic solution.

3. Conclusion

In this study, the homotopy perturbation method was employed successfully for solving the one-dimensional second-order wave equation with given initial conditions and subject to a boundary integral condition replacing the classical boundary condition. This method worked very well for one-dimensional wave equation with an integral condition. It has been demonstrated that the homotopy perturbation method performs well in the context of two simple examples for which the exact solutions are known and can be used for comparison. Indeed, these examples indicate that the homotopy perturbation method provides an excellent approximation for the exact solution with the use of only a small number of components. Accuracy can be improved by simply adding additional terms in the homotopy perturbation method, which involves routine calculations. Naturally, these calculations can be facilitated with some form of computer algebra system.

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