Generalized Couette Flow of a Third-Grade Fluid with Slip: The Exact Solutions

Rahmat Ellahi[a], Tasawar Hayat[b,c], and Fazal Mahmood Mahomed[d]

[a] Department of Mathematics, IIUI, H-10 Sector, Islamabad, Pakistan
[b] Department of Mathematics, Quaid-i-Azam University, Islamabad, Pakistan
[c] Department of Mathematics, College of Sciences, King Saud University, P. O. Box 2455, Riyadh 11451, Saudi Arabia
[d] Centre for Differential Equations, Continuum Mechanics and Applications, University of the Witwatersrand, Wits 2050, South Africa

Reprint requests to T. H.; E-mail: pensy_t@yahoo.com

Z. Naturforsch. **65a,** 1071 – 1076 (2010); received December 11, 2009 / revised February 6, 2010

The present note investigates the influence of slip on the generalized Couette flows of a third-grade fluid. Two flow problems are considered. The resulting equations and the boundary conditions are nonlinear. Analytical solutions of the governing nonlinear problems are found in closed form.

Key words: Third-Grade Fluid; Nonlinear Equations; Nonlinear Slip Conditions; Exact Solutions.

1. Introduction

Considerable interest has arisen recently in the flows of non-Newtonian fluids. This is because of their extensive applications in industry and technology. It is commonly known that the non-Newtonian fluids cannot be described as simply as the Newtonian fluids. Therefore, various non-Newtonian fluid models have been suggested due to their diverse effects. In general, the arising equations for such problems are of higher order, nonlinear, and complicated in comparison to the Navier-Stokes equations. Even the governing equations for the simplest subclass of the non-Newtonian fluids, namely the second-grade, poses various challenges to the workers in the field. Despite all these facts, several investigators are engaged in analyzing the flows of non-Newtonian fluids. Exact analytic solutions for these fluids are not many. Some recent representative studies in this direction have been mentioned through the works of Fetecau and Fetecau [1 – 4], Aksel [5], Aksel et al. [6], Tan and Masuoka [7, 8], Tan and Xu [9], Hayat and Kara [10], Hayat [11], and Hayat et al. [12, 13].

In the literature, not much attention has been given to examine the slip effects on the flows of non-Newtonian fluids. In fact there are two types of fluids which can exhibit slip effects. One class contains the rarefied gases [14] whereas the other fluids have more elastic properties. In such cases, the slippage appears under a large tangential traction. To be more specific, slippage can occur in the non-Newtonian fluids, concentrated polymer solution, and molten polymer. In flow of dilute suspension of particles, a layer next to the wall is clearly found. In experimental physiology Poiseuille noted such layer with a microscopic blood flow through capillary vessels [15]. The literature on the topic is scant. Some useful contributions in this direction have been made by Rao and Rajagopal [16], Hayat et al. [17] and several references therein. Furthermore, the steady one-dimensional second-grade flow for rigid boundaries is capable of describing the normal stress effects and is inadequate in describing shear thickening and shear thinning effects. The third-grade fluid model is able to predict such effects. By keeping all these facts in mind, the objective of the present note is to explore the slip effects on the flows of a third-grade fluid. Two problems, namely the generalized Couette flow and the thin film flow down an inclined plane, are considered. Note that the corresponding problems for the Newtonian and second-grade fluids are linear and identical. The paper is organized into four sections for which the introduction is the first. Section 2 includes the flow description for nonlinear problems. The exact analytic solutions of the problems consisting of the nonlinear equations and nonlinear boundary conditions are developed in Section 3. Section 4 contains numerical values and graphs. The main points of the flow analyses are included in Section 5.

2. Development of the Flow Problems

2.1. Flow between Two Rigid Plates

Let us consider the steady hydrodynamic flow of a fluid between two parallel rigid plates distant h apart. The lower plate at $y = 0$ is suddenly moved in the x-direction and the upper plate at $y = h$ is fixed. Moreover, a constant pressure gradient is applied in x-direction. The slip condition is taken into account in terms of the shear stress. The velocity field for the flow is $\boldsymbol{V} = (u(y), 0, 0)$. The continuity equation is automatically satisfied and the momentum equation after using the Cauchy stress tensor [18]

$$\mathbf{T} = -p_1\mathbf{I} + [\mu + \beta_3(\mathrm{tr}\mathbf{A}_1^2)]\mathbf{A}_1 + \alpha_1\mathbf{A}_2 + \alpha_2\mathbf{A}_1^2 \quad (1)$$

is

$$\mu\frac{\mathrm{d}u^2u}{\mathrm{d}y^2} + 2\beta_3\frac{\mathrm{d}}{\mathrm{d}y}\left[\frac{\mathrm{d}u}{\mathrm{d}y}\right]^3 = \frac{\mathrm{d}\hat{p}}{\mathrm{d}x}, \quad (2)$$

where the modified pressure

$$\hat{p} = p - (2\alpha_1 + \alpha_2)\left(\frac{\mathrm{d}u}{\mathrm{d}y}\right)^2 \quad (3)$$

and according to reference [18]

$$\mu \geq 0; \quad \beta_3 \geq 0,\ \alpha_1 \geq 0; \quad |\alpha_1 + \alpha_2| \leq \sqrt{24\mu\beta_3}.$$

The appropriate boundary conditions are

$$u(0) - \gamma\left[\frac{\mathrm{d}u}{\mathrm{d}y} + 2\frac{\beta_3}{\mu}\left(\frac{\mathrm{d}u}{\mathrm{d}y}\right)^3\right]_{y=0} = U_0, \quad (4)$$

$$u(h) + \gamma\left[\frac{\mathrm{d}u}{\mathrm{d}y} + 2\frac{\beta_3}{\mu}\left(\frac{\mathrm{d}u}{\mathrm{d}y}\right)^3\right]_{y=h} = 0, \quad (5)$$

where $\gamma = 0$ corresponds to a no-slip condition.

In order to carry out the non-dimensional analysis, we define

$$u^* = \frac{u}{U_0}, \quad y^* = \frac{y}{h}, \quad \gamma^* = \frac{\gamma}{h}, \qquad \beta = \frac{\beta_3 U_0^2}{\mu h^2}, \quad p^* = \frac{\hat{p}h}{\mu U_0}. \quad (6)$$

Invoking above equations and then omitting the asterisks for brevity one obtains

$$\frac{\mathrm{d}^2u}{\mathrm{d}y^2} + 2\beta\frac{\mathrm{d}}{\mathrm{d}y}\left[\frac{\mathrm{d}u}{\mathrm{d}y}\right]^3 = \frac{\mathrm{d}p}{\mathrm{d}x}, \quad (7)$$

$$u(0) - \gamma\left[\frac{\mathrm{d}u}{\mathrm{d}y} + 2\beta\left(\frac{\mathrm{d}u}{\mathrm{d}y}\right)^3\right]_{y=0} = 1, \quad (8)$$

$$u(1) + \gamma\left[\frac{\mathrm{d}u}{\mathrm{d}y} + 2\beta\left(\frac{\mathrm{d}u}{\mathrm{d}y}\right)^3\right]_{y=1} = 0. \quad (9)$$

2.2. Thin Film Flow

In this subsection, we consider a thin film flow of a third-grade fluid down an inclined plane. The pressure gradient and surface tension are not taken into account. The ambient air is at rest and flow is only due to gravity. Further, the thickness of the film δ is uniform.

With the definition of velocity in the previous subsection, the equation which governs the flow is

$$\mu\frac{\mathrm{d}^2u}{\mathrm{d}y^2} + 2\beta_3\frac{\mathrm{d}}{\mathrm{d}y}\left[\frac{\mathrm{d}u}{\mathrm{d}y}\right]^3 + \rho g\sin\alpha = 0, \quad (10)$$

where ρ is the density and g is the gravity. The boundary conditions that correspond to the present flow are

$$u(0) - \gamma\left[\frac{\mathrm{d}u}{\mathrm{d}y} + 2\frac{\beta_3}{\mu}\left(\frac{\mathrm{d}u}{\mathrm{d}y}\right)^3\right]_{y=0} = 0, \quad (11)$$

$$\left[\frac{\mathrm{d}u}{\mathrm{d}y} + 2\frac{\beta_3}{\mu}\left(\frac{\mathrm{d}u}{\mathrm{d}y}\right)^3\right]_{y=\delta} = 0. \quad (12)$$

Introducing

$$u^* = \frac{u\delta}{\nu}, \quad y^* = \frac{y}{\delta}, \quad \gamma^* = \frac{\gamma}{\delta}, \qquad \beta = \frac{\beta_3\nu}{\rho\delta^4}, \quad L = \frac{\delta^3 g\sin\alpha}{\nu^2} \quad (13)$$

and omitting the asterisks we can write

$$\frac{\mathrm{d}^2u}{\mathrm{d}y^2} + 2\beta\frac{\mathrm{d}}{\mathrm{d}y}\left[\frac{\mathrm{d}u}{\mathrm{d}y}\right]^3 + L = 0, \quad (14)$$

$$u(0) - \gamma\left[\frac{\mathrm{d}u}{\mathrm{d}y} + 2\beta\left(\frac{\mathrm{d}u}{\mathrm{d}y}\right)^3\right]_{y=0} = 0, \quad (15)$$

$$\left[\frac{\mathrm{d}u}{\mathrm{d}y} + 2\beta\left(\frac{\mathrm{d}u}{\mathrm{d}y}\right)^3\right]_{y=1} = 0. \quad (16)$$

3. Exact Solutions

3.1. Solution for Flow Between Two Rigid Boundaries

A first integral of (7) is

$$\frac{\mathrm{d}u}{\mathrm{d}y} + 2\beta\left(\frac{\mathrm{d}u}{\mathrm{d}y}\right)^3 = cy + A_1, \quad (17)$$

where $\mathrm{d}\hat{p}/dx = c$ and A_1 is an arbitrary constant of integration. We now invoke both of the boundary conditions (8) and (9) on the first integral (17). The applications of the boundary conditions (8) and (9) on the first integral (17) in turn result in

$$u(0) - \gamma A_1 = 1 \tag{18}$$

and

$$u(1) + \gamma(c + A_1) = 0. \tag{19}$$

We now require to solve (17) subject to the boundary conditions (18) and (19).

The linear transformation

$$\overline{y} = -(cy + A_1), \quad \overline{u} = -cu, \tag{20}$$

reduces (17) to

$$\frac{\mathrm{d}\overline{u}}{\mathrm{d}\overline{y}} + 2\beta\left(\frac{\mathrm{d}\overline{u}}{\mathrm{d}\overline{y}}\right)^3 + \overline{y} = 0 \tag{21}$$

and the boundary conditions (18) and (19) become

$$\overline{u}(-A_1) = -c - c\gamma A_1, \tag{22}$$

$$\overline{u}(-A_1 - c) = c\gamma(A_1 + c), \quad c \neq 0. \tag{23}$$

(21) has the exact solution

$$\begin{aligned}\overline{u} = {} & \frac{9\overline{y}}{8(6)^{\frac{2}{3}}}\left[\sqrt[3]{\frac{\sqrt{81\overline{y}^2 + \frac{6}{\beta}} - 9\overline{y}}{\beta}} - \sqrt[3]{\frac{\sqrt{81\overline{y}^2 + \frac{6}{\beta}} + 9\overline{y}}{\beta}}\right] \\ & + \frac{1}{24(6)^{\frac{2}{3}}}\sqrt{81\overline{y}^2 + \frac{6}{\beta}}\left[\sqrt[3]{\frac{\sqrt{81\overline{y}^2 + \frac{6}{\beta}} + 9\overline{y}}{\beta}} + \sqrt[3]{\frac{\sqrt{81\overline{y}^2 + \frac{6}{\beta}} - 9\overline{y}}{\beta}}\right] + A_2,\end{aligned} \tag{24}$$

where A_2 is a further constant of integration.

We choose $A_1 = 0$, (as it appears in the boundary conditions and we can select it) and then apply the boundary conditions (22) and (23) on our solution (24). This gives rise to

$$A_2 + \frac{1}{12\beta} + c = 0 \tag{25}$$

as well as the condition that c, β, and γ need to satisfy, viz.

$$\begin{aligned}-\gamma c = {} & \frac{9c}{8(6)^{\frac{2}{3}}}\left[\sqrt[3]{\frac{\sqrt{81c^2 + \frac{6}{\beta}} - 9c}{\beta}} - \sqrt[3]{\frac{\sqrt{81c^2 + \frac{6}{\beta}} + 9c}{\beta}}\right] \\ & - \frac{1}{24(6)^{\frac{2}{3}}c}\sqrt{81c^2 + \frac{6}{\beta}}\left[\sqrt[3]{\frac{\sqrt{81c^2 + \frac{6}{\beta}} + 9c}{\beta}} + \sqrt[3]{\frac{\sqrt{81c^2 + \frac{6}{\beta}} - 9c}{\beta}}\right] + 1 + \frac{1}{12c\beta}.\end{aligned} \tag{26}$$

Note that the choice $A_1 = 0$ is consistent and one can find c, β, and γ values that satisfy equation or condition (26). This is confirmed by Table 1. In fact, a one can select other values of A_1. One then gets corresponding relations for (25) and (26). However, the form of the exact solution (24) remains the same – simply A_2 changes. Reverting to the original variables, y and u, exact solutions of (7) subject to the boundary conditions (8) and (9)

Table 1. Values of c, β, and γ that satisfy condition (26).

γ	β	c	L	β	γ	c	L
0.5	0.5	-1.130	0.415	0.5	0.5	-1.130	0.415
	1	-1.200	0.293		1	-0.698	0.335
	2	-1.278	0.207		2	-0.405	0.258
	5	-1.392	0.131		5	-0.167	0.174
1	0.5	-0.698	0.335	1	0.5	-1.200	0.293
	1	-0.717	0.237		1	-0.717	0.237
	2	-0.740	0.167		2	-0.408	0.183
	5	-0.777	0.106		5	-0.182	0.123
2	0.5	-0.405	0.258	2	0.5	-1.278	0.207
	1	-0.408	0.183		1	-0.740	0.167
	2	-0.414	0.129		2	-0.414	0.129
	5	-0.423	0.081		5	-0.283	0.087
5	0.5	-0.167	0.174	5	0.5	-1.392	0.131
	1	-0.182	0.123		1	-0.777	0.106
	2	-0.183	0.087		2	-0.423	0.081
	5	-0.184	0.055		5	-0.184	0.055

are given by

$$u=\frac{9y}{8(6)^{\frac{2}{3}}}\left[\sqrt[3]{\frac{\sqrt{81c^2y^2+\frac{6}{\beta}}+9cy}{\beta}}-\sqrt[3]{\frac{\sqrt{81c^2y^2+\frac{6}{\beta}}-9cy}{\beta}}\right]$$
$$-\frac{1}{24(6)^{\frac{2}{3}}c}\sqrt{81c^2y^2+\frac{6}{\beta}}\left[\sqrt[3]{\frac{\sqrt{81c^2y^2+\frac{6}{\beta}}-9cy}{\beta}}+\sqrt[3]{\frac{\sqrt{81c^2y^2+\frac{6}{\beta}}+9cy}{\beta}}\right]+1+\frac{1}{12c\beta} \quad (27)$$

subject to conditions (26) being satisfied.

3.2. Solution for the Thin Film Problem

A first integral of equation (14) is

$$\frac{\mathrm{d}u}{\mathrm{d}y}+2\beta\left(\frac{\mathrm{d}u}{\mathrm{d}y}\right)^3=-Ly+B_1, \quad (28)$$

where B_1 is a constant. Imposition of the boundary conditions (15) and (16) on (28) yields

$$u(0)=\gamma L \quad (29)$$

in which $B_1=L$. The exact solution of (28) is

$$u=\frac{9(y-1)}{8(6)^{\frac{2}{3}}}\left[\sqrt[3]{\frac{\sqrt{81(Ly-L)^2+\frac{6}{\beta}}-9(Ly-L)}{\beta}}-\sqrt[3]{\frac{\sqrt{81(Ly-L)^2+\frac{6}{\beta}}+9(Ly-L)}{\beta}}\right]+\frac{1}{24(6)^{\frac{2}{3}}L}$$
$$\cdot\sqrt{81(Ly-L)^2+\frac{6}{\beta}}\left[\sqrt[3]{\frac{\sqrt{81(Ly-L)^2+\frac{6}{\beta}}+9(Ly-L)}{\beta}}+\sqrt[3]{\frac{\sqrt{81(Ly-L)^2+\frac{6}{\beta}}-9(Ly-L)}{\beta}}\right]+B_2, \quad (30)$$

where B_2 is a constant. The boundary condition (29) then results in B_2 being of the form

$$B_2 = \gamma L + \frac{9}{8(6)^{\frac{2}{3}}}\left[\sqrt[3]{\frac{\sqrt{81L^2+\frac{6}{\beta}}+9L}{\beta}} - \sqrt[3]{\frac{\sqrt{81L^2+\frac{6}{\beta}}-9L}{\beta}}\right] - \frac{1}{24(6)^{\frac{2}{3}}L}\sqrt{81L^2+\frac{6}{\beta}}\left[\sqrt[3]{\frac{\sqrt{81L^2+\frac{6}{\beta}}-9L}{\beta}} + \sqrt[3]{\frac{\sqrt{81L^2+\frac{6}{\beta}}+9L}{\beta}}\right]^{\frac{2}{3}}. \tag{31}$$

Remark: The flow in a semi-infinie space $y > 0$ with slip at the suddenly moved plate at $y = 0$ and no pressure gradient is

$$\mu\frac{d^2u}{dy^2} + 2\beta_3\frac{d}{dy}\left[\frac{du}{dy}\right]^3 = 0, \tag{32}$$

the appropriate boundary conditions are

$$u(0) - \gamma\left[\frac{du}{dy} + \frac{2\beta_3}{\gamma}\left(\frac{du}{dy}\right)^3\right]_{y=0} = U_0, \tag{33}$$

$$u \to 0 \text{ as } y \to \infty. \tag{34}$$

Defining

$$u^* = \frac{u}{U_0}, \quad y^* = \frac{U_0 y}{\nu}, \tag{35}$$

omitting the asterisk and then solving the resulting problem it is found that $u(y) = 0$ (trivial solution).

4. Discussion

We have obtained exact solutions for the generalized Couette flow problem in a third-grade fluid. From the present analysis we note the following features:

- Figure 1 represents the velocity profile for different values of the non-Newtonian parameter β and for fixed value of c. The velocity field u decreases for increasing third-grade parameter β.
- Figure 2 indicates that for a fixed value of β, the velocity increases for increasing c.
- Figure 3 shows the velocity profile for different values of β and fixed value of L. Here the velocity u decrease when the third-grade parameter β is increased.
- The velocity u increases by increasing L and fixed β (Fig. 4).

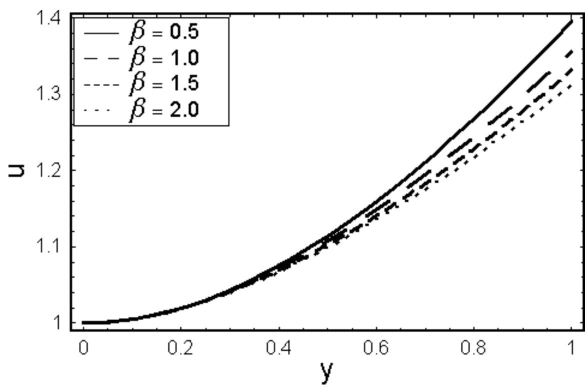

Fig. 1. Dimensionless velocity profiles with different values of the parameter β when γ and c are fixed for non-Newtonian fluid between two rigid boundaries.

- An increase in γ in Figure 5 results in a decrease in velocity u.
- Since our exact solutions are also valid for large β, we plot the velocity field for selected values of $\beta > 0$.
- The numerical relation of β and γ as a function of c and L is also given in Table 1.
- It is noted that c and L decrease by increasing β and fixed γ.
- For fixed β, and increasing γ, c increases and L decreases.

Acknowledgements

F. M. M. and R. E. gratefully acknowledges the warm hospitality of the Fluid Mechanics Group (FMG) of the Department of Mathematics, Quaid-i-Azam University 45320, Islamabad. They in particular thank Prof T. Hayat for his kind invitation and congratulates him for his selection as a Distinguished National Professor of the HEC of Pakistan. R. E. also thanks the higher education commission of Pakistan (HEC) to award him Best University Teacher Award.

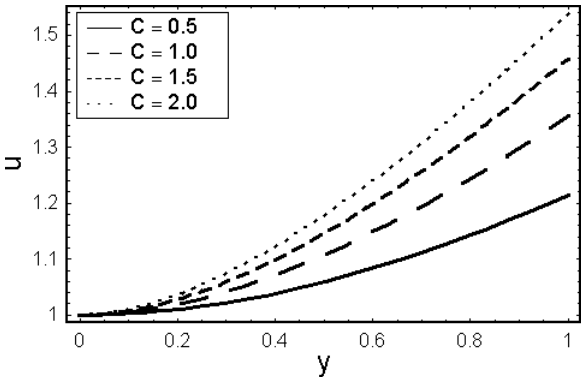

Fig. 2. Dimensionless velocity profiles with different values of the parameter c when γ and β are fixed for non-Newtonian fluid between two rigid boundaries.

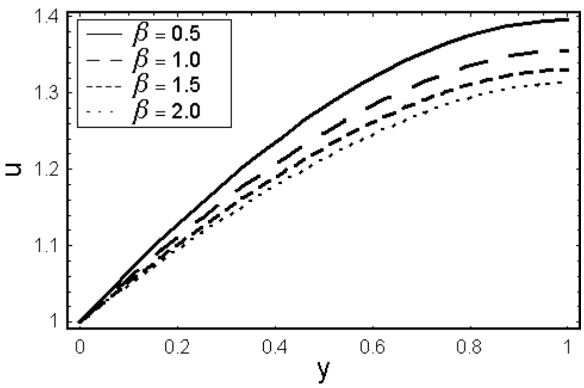

Fig. 3. Dimensionless velocity profiles with different values of the parameter β when L and γ are fixed for thin film problem.

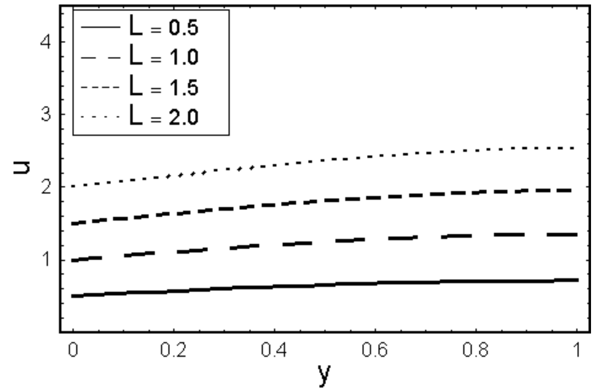

Fig. 4. Dimensionless velocity profiles with different values of the parameter L when β and γ are fixed for thin film problem.

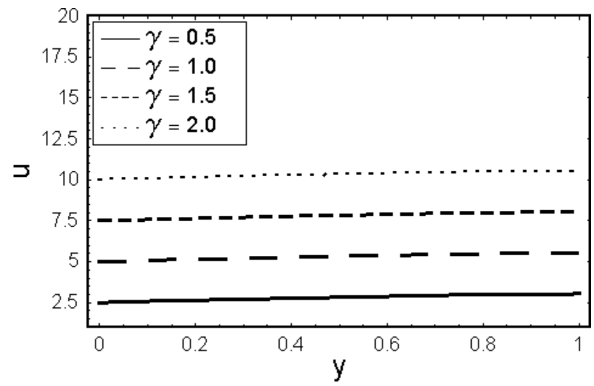

Fig. 5. Dimensionless velocity profiles with different values of the parameter γ when β and L are fixed for thin film problem.

[1] C. Fetecau and C. Fetecau, Z. Angew. Math. Phys. **56**, 1098 (2005).
[2] C. Fetecau and C. Fetecau, Int. J. Eng. Sci. **44**, 788 (2006).
[3] C. Fetecau and C. Fetecau, Int. J. Eng. Sci. **43**, 340 (2005).
[4] C. Fetecau and C. Fetecau, Int. J. Eng. Sci. **43**, 781 (2005).
[5] N. Aksel, Acta Mech. **157**, 235 (2006).
[6] N. Aksel, C. Fetecau, and M. Scholle, Z. Angew. Math. Phys. **57**, 815 (2006).
[7] W. C. Tan and T. Masuoka, Int. J. Nonlinear Mech. **40**, 512 (2005).
[8] W. C. Tan and T. Masuoka, Phys. Fluids **17**, 023101 (2005).
[9] W. C. Tan and M. Y. Xu, Mech. Res. Commun. **29**, 3 (2002).
[10] T. Hayat and A. H. Kara, Math. Comput. Model. **42**, 132 (2004).
[11] T. Hayat, Comput. Math. Appl. **52**, 1413 (2006).
[12] T. Hayat, R. Ellahi, and F. M. Mahomed, Acta Mech. **188**, 69 (2007).
[13] T. Hayat, R. Ellahi, and S. Asghar, Chem. Eng. Commun. **194**, 37 (2007).
[14] W. Kwang-Hua Chu and J. Fang, Europ. Phys. J. B **16**, 543 (2000).
[15] B. D. Coleman, H. Markowitz, and W. Noll, Springer-Verlag, Berlin, Heidelberg, New York 1966.
[16] I. J. Rao and K. R. Rajagopal, Acta Mech. **135**, 113 (1999).
[17] T. Hayat, M. Khan, and M. Ayub, J. Comput. Appl. Math. **202**, 402 (2007).
[18] R. L. Fosdick and K. R. Rajagopal, Proc. Roy. Soc. Lond. A **339**, 351 (1980).