Travelling Wave Solutions for the Burgers Equation and the Korteweg-de Vries Equation with Variable Coefficients Using the Generalized (G'/G)-Expansion Method

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In this article, a generalized (G'/G)-expansion method is used to find exact travelling wave solutions of the Burgers equation and the Korteweg-de Vries (KdV) equation with variable coefficients. As a result, hyperbolic, trigonometric, and rational function solutions with parameters are obtained. When these parameters are taking special values, the solitary wave solutions are derived from the hyperbolic function solution. It is shown that the proposed method is direct, effective, and can be applied to many other nonlinear evolution equations in mathematical physics.

Key words: Nonlinear Evolution Equations; Generalized (G'/G)-Expansion Method; Hyperbolic Solution; Trigonometric Solution; Rational Solution; Burgers Equation; KdV Equation.

1. Introduction

Seeking exact solutions of nonlinear evolution equations (NLEEs) is of important significance in mathematical physics and becomes one of the most exciting and extremely active areas of research investigation. In the past several decades, many effective methods for obtaining exact solutions of NLEEs have been presented, such as the inverse scattering method [1], the Hirota bilinear method [2], the Bäcklund transformation [3], the Painlevé expansion [4], the sine-cosine method [5], the homogeneous balance method [6], the homotopy perturbation method [7 – 9], the variational iteration method [10 – 13], the Adomian decomposition method [14], the tanh function method [15 – 20], the algebraic method [21 – 25], the Jacobi elliptic function expansion method [26 – 28], the F-expansion method [29 – 35], the auxiliary equation method [36 – 39], the exp-function method [40 – 42], the (G'/G)-expansion method [43 – 53], and so on. Very recently, Wang et al. [49] introduced a new method, called the (G'/G)-expansion method, to look for travelling wave solutions of NLEEs with constant coefficients.

However, to our knowledge, most of the aforementioned methods are related to the constant-coefficient models. Recently, the study of variable-coefficients NLEEs has attracted much attention [12, 49, 54, 55] because most of real nonlinear physical equations possess variable coefficients. Zhang et al. [50] first proposed the generalized (G'/G)-expansion method to find exact solutions to the modified Korteweg-de Vries (mKdV) equation with variable coefficients. The (G'/G)-expansion method is based on the assumptions that the travelling wave solutions can be expressed by a polynomial in (G'/G), where $G = G(\xi)$ satisfies the following second-order linear ordinary differential equation:

$$G'' + \lambda G' + \mu G = 0, \tag{1}$$

where λ and μ are constants. The degree of this polynomial can be determined by considering the homogeneous balance between the highest-order derivatives and the nonlinear terms appearing in the given NLEE. The coefficients of this polynomial can be obtained by solving a set of algebraic equations resulted from the process of using the method. As two applications of the generalized (G'/G)-expansion method, we will consider the following nonlinear Burgers equation with variable coefficients:

$$u_t - d(t)\, u_{xx} + a(t) u\, u_x = 0, \tag{2}$$

as well as the following variable coefficients KdV equation:

$$u_t + f(t)\, u_{xxx} + g(t) u\, u_x = 0, \tag{3}$$

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where $d(t)$, $a(t)$, $f(t)$, and $g(t)$ are differentiable functions of t.

(2) is used to describe the spread of a sound wave in the medium with viscidity and heat exchange if we do not consider the medium's frequently dispersive character and the slack comfort process. At the same time, the Burgers equations with variable coefficients can be used to describe the cylinder and spherical waves in these equations such as overfull, traffic model, etc. (3) is the best typical representation of nonlinear dispersive wave equations as we all know. The KdV-equation (3) has been widely studied using different methods, see for example [56, 57].

The rest of this article is organized as follows. In Section 2, we describe the generalized (G'/G)-expansion method. In Section 3, we apply this method to (2) and (3). And in Section 4, some conclusions are given.

2. Description of the Generalized G'/G-Expansion Method

For a given NLEE with independent variables $X = (x,y,z,t)$ and dependent variable u, we consider the partial differential equation (PDE)

$$F(u,u_t,u_x,u_y,u_z,u_{xt},u_{yt},u_{zt},u_{tt},u_{xx},u_{yy},u_{zz},\ldots)=0. \tag{4}$$

The solution of (4) can be expressed by a polynomial in (G'/G) as follows:

$$u(X)=\alpha_0(X)+\sum_{i=1}^{m}\alpha_i(X)(G'/G)^i,\ \ \alpha_m(X)\neq 0, \tag{5}$$

where $G=G(\xi)$ satisfies (1) while $\xi=\xi(X)$ and $\alpha_i(X)$ are all functions of X to be determined later. To determine $u(X)$ explicitly, we take the following four steps:

Step 1. Determine the integer m by balancing the highest-order nonlinear term(s) and the highest-order partial derivatives of $u(X)$ in (4).

Step 2. Substitute (5) along with (1) into (4) and collect all terms with the same order of (G'/G) together. The left-hand side of (4) is converted into a polynomial in (G'/G). Then set each coefficient of this polynomial to zero to derive a set of over-determined differential equations for $\alpha_0(X)$, $\alpha_i(X)$, and ξ.

Step 3. Solve the system of over-determined differential equations obtained in Step 2 for $\alpha_i(X)$ and ξ by using the software Maple or Mathematica.

Step 4. Use the results obtained in above steps to derive a series of fundamental solutions of (4) depending on (G'/G), since the solutions of (1) have been well known for us, then we can obtain the exact solutions of (4).

Remark 1. It can be easily found that if $\alpha_i(X)$ is a constant and ξ is merely a linear function of x and t, then (5) becomes (2.4) constructed by Wang et al. in [49]. So, we may get more general exact solutions of (4).

Remark 2. The present method in this article has been applied by Zhang et al. [50] to the nonlinear mKdV equation with variable coefficients. They have obtained three travelling wave solutions for this equation.

3. Applications

In this section, we determine the exact travelling wave solutions of the nonlinear Burgers equation and the KdV equation with variable-coefficients which are attracted much attention.

3.1. Example 1. The Variable-Coefficients Burgers Equation

By balancing u_{xx} with uu_x in (2), we get $m=1$. In order to search for explicit solutions, we suppose that (2) has the following formal solution:

$$u(x,t)=\alpha_1(G'/G)(t)+\alpha_0(t),\ \alpha_1(G'/G)(t)\neq 0, \tag{6}$$

where $G=G(\xi)$ satisfies (1), $\xi=p(t)x+q(t)$ while $p(t)$ and $q(t)$ are functions to be determined.

From (1) and (6) we have:

$$\begin{aligned} u_t=&-\alpha_1(t)\left[x\frac{\mathrm{d}p(t)}{\mathrm{d}t}+\frac{\mathrm{d}q(t)}{\mathrm{d}t}\right](G'/G)^2\\ &-\left\{\lambda\alpha_1(t)\left[x\frac{\mathrm{d}p(t)}{\mathrm{d}t}+\frac{\mathrm{d}q(t)}{\mathrm{d}t}\right]-\frac{\mathrm{d}\alpha_1(t)}{\mathrm{d}t}\right\}(G'/G)\\ &-\mu\alpha_1(t)\left[x\frac{\mathrm{d}p(t)}{\mathrm{d}t}+\frac{\mathrm{d}q(t)}{\mathrm{d}t}\right]+\frac{\mathrm{d}\alpha_0(t)}{\mathrm{d}t}, \end{aligned} \tag{7}$$

$$\begin{aligned} u_x=&-\alpha_1(t)p(t)(G'/G)^2-\lambda\alpha_1(t)p(t)(G'/G)\\ &-\mu\alpha_1(t)p(t), \end{aligned} \tag{8}$$

$$\begin{aligned} u_{xx}=&\,2\alpha_1(t)p^2(t)(G'/G)^3+3\lambda\alpha_1(t)p^2(t)(G'/G)^2\\ &+[\lambda^2\alpha_1(t)p^2(t)+2\mu\alpha_1(t)p^2(t)](G'/G)\\ &+\mu\lambda\alpha_1(t)p^2(t), \end{aligned} \tag{9}$$

$$\begin{aligned} uu_x = &-\alpha_1^2(t)p(t)(G'/G)^3 \\ &+[-\lambda\alpha_1^2(t)p(t)-\alpha_0(t)\alpha_1(t)p(t)](G'/G)^2 \\ &+[-\lambda\alpha_0(t)\alpha_1(t)p(t)-\mu\alpha_1^2(t)p(t)](G'/G) \\ &-\mu\alpha_0(t)\alpha_1(t)p(t). \end{aligned} \tag{10}$$

Substituting (6)–(10) into (2) and collecting all terms with the same order of (G'/G) together, the left-hand side of (2) is converted into a polynomial in $x^j(G'/G)^i$, $(j=0,1,\ i=0,1,2,3)$. Setting each coefficient of this polynomial to zero, we get the following set of over-determined differential equations:

$$\begin{aligned} &x^0(G'/G)^3 : -2d(t)\alpha_1(t)p^2(t)-a(t)\alpha_1^2(t)p(t)=0, \\ &x^0(G'/G)^2 : -\alpha_1(t)\frac{\mathrm{d}q(t)}{\mathrm{d}t}-3\lambda d(t)\alpha_1(t)p^2(t) \\ &\qquad -\lambda a(t)\alpha_1^2(t)p(t) \\ &\qquad -a(t)\alpha_0(t)\alpha_1(t)p(t)=0, \\ &x^0(G'/G) : -\lambda\alpha_1(t)\frac{\mathrm{d}q(t)}{\mathrm{d}t}+\frac{\mathrm{d}\alpha_1(t)}{\mathrm{d}t}-\lambda^2 d(t)\alpha_1(t)p^2(t) \\ &\qquad -2\mu d(t)\alpha_1(t)p^2(t)-\mu a(t)\alpha_1^2(t)p(t) \\ &\qquad -\lambda a(t)\alpha_0(t)\alpha_1(t)p(t)=0, \\ &x^0(G'/G)^0 : -\mu\alpha_1(t)\frac{\mathrm{d}q(t)}{\mathrm{d}t}+\frac{\mathrm{d}\alpha_0(t)}{\mathrm{d}t} \\ &\qquad -\mu a(t)\alpha_0(t)\alpha_1(t)p(t) \\ &\qquad -\mu\lambda d(t)\alpha_1(t)p^2(t)=0, \\ &x(G'/G) : -\lambda\alpha_1(t)\frac{\mathrm{d}p(t)}{\mathrm{d}t}=0, \\ &x(G'/G)^0 : -\mu\alpha_1(t)\frac{\mathrm{d}p(t)}{\mathrm{d}t}=0. \end{aligned} \tag{11}$$

Solving the system (11) by Maple or Mathematica, we have

$$\begin{aligned} &\alpha_1(t)=R,\ \alpha_0(t)=Q,\ p(t)=P,\ a(t)=\frac{-2Pd(t)}{R}, \\ &q(t)=-P^2\Big(\lambda-\frac{2Q}{R}\Big)\int^t d(t)\mathrm{d}t, \end{aligned} \tag{12}$$

where P, Q, and R are arbitrary constants.

Substituting (12) into (6), we have

$$u(x,t)=R(G'/G)+Q, \tag{13}$$

where

$$\xi = Px-P^2\Big(\lambda-\frac{2Q}{R}\Big)\int^t d(t)\mathrm{d}t.$$

From the general solution of (1) we can find the ratio (G'/G). Consequently, we have the following three types of exact solutions of (2):

Case 1. When $\lambda^2-4\mu>0$, we obtain the hyperbolic function solution in the form

$$\begin{aligned} u(x,t)=R\Bigg[&\frac{-\lambda}{2}+\frac{\sqrt{\lambda^2-4\mu}}{2} \\ &\cdot\frac{C_1\cosh\Big(\frac{\sqrt{\lambda^2-4\mu}}{2}\Big)\xi+C_2\sinh\Big(\frac{\sqrt{\lambda^2-4\mu}}{2}\Big)\xi}{C_1\sinh\Big(\frac{\sqrt{\lambda^2-4\mu}}{2}\Big)\xi+C_2\cosh\Big(\frac{\sqrt{\lambda^2-4\mu}}{2}\Big)\xi}\Bigg] \\ &+Q. \end{aligned} \tag{14}$$

Case 2. When $\lambda^2-4\mu<0$, we have the trigonometric function solution in the form

$$\begin{aligned} u(x,t)=R\Bigg[&\frac{-\lambda}{2}+\frac{\sqrt{4\mu-\lambda^2}}{2} \\ &\cdot\frac{-C_1\sin\Big(\frac{\sqrt{4\mu-\lambda^2}}{2}\Big)\xi+C_2\cos\Big(\frac{\sqrt{4\mu-\lambda^2}}{2}\Big)\xi}{C_1\cos\Big(\frac{\sqrt{4\mu-\lambda^2}}{2}\Big)\xi+C_2\sin\Big(\frac{\sqrt{4\mu-\lambda^2}}{2}\Big)\xi}\Bigg] \\ &+Q. \end{aligned} \tag{15}$$

Case 3. When $\lambda^2-4\mu=0$, we get the rational solution in the form

$$u(x,t)=R\left[\frac{C_2}{C_1+C_2\xi}-\frac{\lambda}{2}\right]+Q. \tag{16}$$

Finally, we note that, if $\mu=0$, $\lambda>0$, $C_2=0$, and $C_1\neq 0$ then we deduce from (14) that

$$u(x,t)=\frac{-R\lambda}{2}\left[1-\coth\left(\frac{\lambda\xi}{2}\right)\right]+Q, \tag{17}$$

while if $\mu=0$, $\lambda>0$, $C_2\neq 0$, and $C_2^2>C_1^2$ we get

$$u(x,t)=\frac{-R\lambda}{2}\left[1-\tanh\left(\xi_0+\frac{\lambda\xi}{2}\right)\right]+Q, \tag{18}$$

where

$$\xi_0=\tanh^{-1}\left(\frac{C_1}{C_2}\right).$$

Note that (17) and (18) represent the solitary wave solutions of the Burgers equation (2).

3.2. Example 2. The Variable-Coefficients KdV Equation

By balancing u_{xxx} with uu_x in (3) we get $m = 2$. In order to search for explicit solutions, we suppose that (3) has the following formal solution:

$$u(x,t) = \alpha_2(t)(G'/G)^2 + \alpha_1(t)(G'/G) + \alpha_0(t), \quad \alpha_2(t) \neq 0, \tag{19}$$

where $G = G(\xi)$ satisfies (1), $\xi = p(t)x + q(t)$, while $p(t)$ and $q(t)$ are functions to be determined.

From (1) and (19) we have:

$$\begin{aligned} u_t = &-2\alpha_2(t)\left[x\frac{\mathrm{d}p(t)}{\mathrm{d}t} + \frac{\mathrm{d}q(t)}{\mathrm{d}t}\right](G'/G)^3 \\ &+ \left\{\frac{\mathrm{d}\alpha_2(t)}{\mathrm{d}t} - 2\lambda\alpha_2(t)\left[x\frac{\mathrm{d}p(t)}{\mathrm{d}t} + \frac{\mathrm{d}q(t)}{\mathrm{d}t}\right] - \alpha_1(t)\left[x\frac{\mathrm{d}p(t)}{\mathrm{d}t} + \frac{\mathrm{d}q(t)}{\mathrm{d}t}\right]\right\}(G'/G)^2 \\ &+ \left\{\frac{\mathrm{d}\alpha_1(t)}{\mathrm{d}t} - 2\mu\alpha_2(t)\left[x\frac{\mathrm{d}p(t)}{\mathrm{d}t} + \frac{\mathrm{d}q(t)}{\mathrm{d}t}\right] - \lambda\alpha_1(t)\left[x\frac{\mathrm{d}p(t)}{\mathrm{d}t} + \frac{\mathrm{d}q(t)}{\mathrm{d}t}\right]\right\}(G'/G) \\ &+ \frac{\mathrm{d}\alpha_0(t)}{\mathrm{d}t} - \mu\alpha_1(t)\left[x\frac{\mathrm{d}p(t)}{\mathrm{d}t} + \frac{\mathrm{d}q(t)}{\mathrm{d}t}\right], \end{aligned} \tag{20}$$

$$\begin{aligned} u_x = &-2\alpha_2(t)p(t)(G'/G)^3 + [-2\lambda\alpha_2(t)p(t) - \alpha_1(t)p(t)](G'/G)^2 + [-2\mu\alpha_2(t)p(t) \\ &- \lambda\alpha_1(t)p(t)](G'/G) - \mu\alpha_1(t)p(t), \end{aligned} \tag{21}$$

$$\begin{aligned} uu_x = &-2\alpha_2{}^2(t)p(t)(G'/G)^5 + [-2\lambda\alpha_2{}^2(t)p(t) -3\alpha_1(t)\alpha_2(t)p(t)](G'/G)^4 + [-2\mu\alpha_2{}^2(t)p(t) \\ &-3\lambda\alpha_1(t)\alpha_2(t)p(t) - \alpha_1^2(t)p(t) -2\alpha_0(t)\alpha_2(t)p(t)](G'/G)^3 + [-3\mu\alpha_1(t)\alpha_2(t)p(t) \\ &-\lambda\alpha_1{}^2(t)p(t) - 2\lambda\alpha_0(t)\alpha_2(t)p(t) -\alpha_0(t)\alpha_1(t)p(t)](G'/G)^2 + [-\mu\alpha_1^2(t)p(t) \\ &-2\mu\alpha_0(t)\alpha_2(t)p(t) - \lambda\alpha_0(t)\alpha_1(t)p(t)](G'/G) -\mu\alpha_0(t)\alpha_1(t)p(t), \end{aligned} \tag{22}$$

$$\begin{aligned} u_{xxx} = &-24\alpha_2(t)p^3(t)(G'/G)^5 + [-54\lambda\alpha_2(t)p^3(t) -6\alpha_1(t)p^3(t)](G'/G)^4 + [-40\mu\alpha_2(t)p^3(t) \\ &-38\lambda^2\alpha_2(t)p^3(t) - 12\lambda^2\alpha_1(t)p^3(t)](G'/G)^3 \\ &+[-52\mu\lambda\alpha_2(t)p^3(t) - 8\mu\alpha_1(t)p^3(t) - 8\lambda^3\alpha_2(t)p^3(t) -7\lambda^2\alpha_1(t)p^3(t)](G'/G)^2 + [-16\mu^2\alpha_2(t)p^3(t) \\ &-14\mu\lambda^2\alpha_2(t)p^3(t) - 8\mu\lambda\alpha_1(t)p^3(t) -\lambda^3\alpha_1(t)p^3(t)](G'/G) - 6\mu^2\lambda\alpha_2(t)p^3(t) \\ &-2\mu^2\alpha_1(t)p^3(t) - \mu\lambda^2\alpha_1(t)p^3(t). \end{aligned} \tag{23}$$

Substituting (19) – (23) into (3) and collecting all terms with the same order of (G'/G) together, the left-hand side of (3) is converted into a polynomial in $x^j(G'/G)^i$, $(j = 0, 1, i = 0, 1, \ldots, 5)$. Setting each coefficient of this polynomial to zero, we get the following set of over-determined differential equations:

$$\begin{aligned}
&x^0(G'/G)^5 : -24f(t)\alpha_2(t)p^3(t) - 2g(t)\alpha_2{}^2(t)p(t) = 0, \\
&x^0(G'/G)^4 : -54\lambda f(t)\alpha_2(t)p^3(t) - 6f(t)\alpha_1(t)p^3(t) -2\lambda g(t)\alpha_2{}^2(t)p(t) - 3g(t)\alpha_1(t)\alpha_2(t)p(t) = 0, \\
&x^0(G'/G)^3 : -2\alpha_2(t)\frac{\mathrm{d}q(t)}{\mathrm{d}t} - 40\mu f(t)\alpha_2(t)p^3(t) -38\lambda^2 f(t)\alpha_2(t)p^3(t) - 12\lambda f(t)\alpha_1(t)p^3(t) \\
&\quad -2\mu g(t)\alpha_2{}^2(t)p(t) - 3\lambda g(t)\alpha_1(t)\alpha_2(t)p(t) -g(t)\alpha_1{}^2(t)p(t) - 2g(t)\alpha_0(t)\alpha_2(t)p(t) = 0, \\
&x^0(G'/G)^2 : \frac{\mathrm{d}\alpha_2(t)}{\mathrm{d}t} - 2\lambda\alpha_2(t)\frac{\mathrm{d}q(t)}{\mathrm{d}t} - \alpha_1(t)\frac{\mathrm{d}q(t)}{\mathrm{d}t} -52\mu\lambda f(t)\alpha_2(t)p^3(t) - 8\mu f(t)\alpha_1(t)p^3(t) \\
&\quad -8\lambda^3 f(t)\alpha_2(t)p^3(t) - 7\lambda^2 f(t)\alpha_1(t)p^3(t) -3\mu g(t)\alpha_1(t)\alpha_2(t)p(t) - \lambda g(t)\alpha_1{}^2(t)p(t) \\
&\quad -2\lambda g(t)\alpha_0(t)\alpha_2(t)p(t) - g(t)\alpha_0(t)\alpha_1(t)p(t) = 0, \\
&x^0(G'/G) : \frac{\mathrm{d}\alpha_1(t)}{\mathrm{d}t} - 2\mu\alpha_2(t)\frac{\mathrm{d}q(t)}{\mathrm{d}t} - \lambda\alpha_1(t)\frac{\mathrm{d}q(t)}{\mathrm{d}t} -16\mu^2 f(t)\alpha_2(t)p^3(t) - 14\mu\lambda^2 f(t)\alpha_2(t)p^3(t) \\
&\quad -8\mu\lambda f(t)\alpha_1(t)p^3(t) - \lambda^3 f(t)\alpha_1(t)p^3(t) -\mu g(t)\alpha_1{}^2(t)p(t) - 2\mu g(t)\alpha_0(t)\alpha_2(t)p(t) -\lambda g(t)\alpha_0(t)\alpha_1(t)p(t) = 0, \\
&x^0(G'/G)^0 : \frac{\mathrm{d}\alpha_0(t)}{\mathrm{d}t} - \mu\alpha_1(t)\frac{\mathrm{d}q(t)}{\mathrm{d}t} -6\mu^2\lambda f(t)\alpha_2(t)p^3(t) - 2\mu^2 f(t)\alpha_1(t)p^3(t) \\
&\quad -\mu\lambda^2 f(t)\alpha_1(t)p^3(t) - \mu g(t)\alpha_0(t)\alpha_1(t)p(t) = 0, \\
&x(G'/G)^3 : -2\alpha_2(t)\frac{\mathrm{d}p(t)}{\mathrm{d}t} = 0, \\
&x(G'/G)^2 : -2\lambda\alpha_2(t)\frac{\mathrm{d}p(t)}{\mathrm{d}t} - \alpha_1(t)\frac{\mathrm{d}p(t)}{\mathrm{d}t} = 0, \\
&x(G'/G) : -2\mu\alpha_2(t)\frac{\mathrm{d}p(t)}{\mathrm{d}t} - \lambda\alpha_1(t)\frac{\mathrm{d}p(t)}{\mathrm{d}t} = 0, \\
&x(G'/G)^0 : -\mu\alpha_1(t)\frac{\mathrm{d}p(t)}{\mathrm{d}t} = 0.
\end{aligned} \tag{24}$$

Solving the system (24) by Maple or Mathematica, we have

$$\begin{aligned} &P(t)=P,\ \ \alpha_0(t)=Q,\ \ \alpha_1(t)=R,\ \ \alpha_2(t)=\frac{R}{\lambda},\\ &g(t)=\frac{-12\lambda P^2 f(t)}{R},\\ &q(t)=-P^3\left(\lambda^2+8\mu-\frac{12\lambda Q}{R}\right)\int^t f(t)\mathrm{d}t, \end{aligned} \tag{25}$$

where P, Q, and R are arbitrary constants.

Substituting (25) into (19), we have

$$u(x,t)=\frac{R}{\lambda}(G'/G)^2+R(G'/G)+Q. \tag{26}$$

where

$$\xi=Px-P^3\left(\lambda^2+8\mu-\frac{12\lambda Q}{R}\right)\int^t f(t)\mathrm{d}t.$$

Consequently, we have the following three types of exact solutions of (3):

Case 1. When $\lambda^2-4\mu>0$, we obtain the hyperbolic function solution in the form

$$\begin{aligned} u(x,t)=&\frac{R}{\lambda}\left[\frac{-\lambda}{2}+\frac{\sqrt{\lambda^2-4\mu}}{2}\left(\frac{C_1\cosh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\right)\xi+C_2\sinh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\right)\xi}{C_1\sinh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\right)\xi+C_2\cosh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\right)\xi}\right)\right]^2\\ &+R\left[\frac{-\lambda}{2}+\frac{\sqrt{\lambda^2-4\mu}}{2}\left(\frac{C_1\cosh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\right)\xi+C_2\sinh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\right)\xi}{C_1\sinh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\right)\xi+C_2\cosh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\right)\xi}\right)\right]+Q. \end{aligned} \tag{27}$$

Case 2. When $\lambda^2-4\mu<0$, we have the trigonometric function solution in the form

$$\begin{aligned} u(x,t)=&\frac{R}{\lambda}\left[\frac{-\lambda}{2}+\frac{\sqrt{4\mu-\lambda^2}}{2}\left(\frac{-C_1\sin\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\right)\xi+C_2\cos\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\right)\xi}{C_1\cos\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\right)\xi+C_2\sin\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\right)\xi}\right)\right]^2\\ &+R\left[\frac{-\lambda}{2}+\frac{\sqrt{4\mu-\lambda^2}}{2}\left(\frac{-C_1\sin\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\right)\xi+C_2\cos\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\right)\xi}{C_1\cos\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\right)\xi+C_2\sin\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\right)\xi}\right)\right]+Q. \end{aligned} \tag{28}$$

Case 3. When $\lambda^2-4\mu=0$, we get the rational function solution in the form

$$\begin{aligned} u(x,t)=&\frac{R}{\lambda}\left[\frac{C_2}{C_1+C_2\xi}-\frac{\lambda}{2}\right]^2\\ &+R\left[\frac{C_2}{C_1+C_2\xi}-\frac{\lambda}{2}\right]+Q. \end{aligned} \tag{29}$$

Finally, we note that, if $\mu=0$, $\lambda>0$, $C_2=0$, and $C_1\neq 0$ then we deduce from (27) that

$$u(x,t)=\frac{R\lambda}{4}\mathrm{csch}^2\left(\frac{\lambda\xi}{2}\right)+Q, \tag{30}$$

while if $\mu=0$, $\lambda>0$, $C_2\neq 0$, and $C_2^2>C_1^2$ we get

$$u(x,t)=\frac{-R\lambda}{4}\mathrm{sech}^2\left(\xi_0+\frac{\lambda\xi}{2}\right)+Q, \tag{31}$$

where $\xi_0=\tanh^{-1}\left(\frac{C_1}{C_2}\right)$.

Note that (30) and (31) represent the solitary wave solutions of the KdV equation (3).

4. Conclusion

In this article, the generalized (G'/G)-expansion method is used to obtain more general exact solutions for NLEEs. By using the proposed method we have successfully obtained exact solutions with parameters of the Burgers equation and the KdV equation with variable coefficients. When these parameters are taking special values, the solitary wave solutions are derived from the hyperbolic solutions.

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