An Alternative Approach to Differential-Difference Equations Using the Variational Iteration Method

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Although a variational iteration algorithm was proposed by Yildirim (Math. Prob. Eng. 2008 (2008), Article ID 869614) that successfully solves differential-difference equations, the method involves some repeated and unnecessary iterations in each step. An alternative iteration algorithm (variational iteration algorithm-II) is constructed in this paper that overcomes this shortcoming and promises to provide a universal mathematical tool for many differential-difference equations.

Key words: Variational Iteration Method; Discrete Korteweg-de Vries Equation; Discrete Volterra Equation; Lattice Equation.

1. Introduction

The differential-difference model has recently attracted much attention because of its ability to exactly describe many real-life problems in textile engineering [1], nanotechnology [2], and stratified hydrostatic flows [3]. Indeed, the differential-difference model is particularly suitable to describe discontinuous matter and is the preferred model describing flows through porous media over the classical approach which uses either fractal geometry [4 – 7] or fractional calculus [8 – 9]. Analytical methods for solving differential-difference equations have therefore been intensively investigated, for exmaple the exp-function method [10 – 12], the series solution method [13], the homotopy perturbation method [14, 15], and the variational iteration method [16 – 22] being the most widely used. An iteration algorithm, in particular, was proposed by Yildirim [17] in 2008 that solves differential-difference equations by using the variational iteration method in which some unnecessary and repeated calculations are involved in each iteration. This paper proposes an alternative iteration algorithm that overcomes this shortcoming.

2. A New Iteration Algorithm for Differential-Difference Equations

In [17], Yildirim considered the Volterra equation

$$\frac{\mathrm{d}u_n}{\mathrm{d}t} = u_n(u_{n+1} - u_{n-1}) \tag{1}$$

with initial conditions

$$u_n(0) = n \tag{2}$$

which admits the exact solution

$$u_n(t) = \frac{n}{1-2t}. \tag{3}$$

Yildirim constructed the iteration formula

$$u_{n,m+1}(t) = u_{n,m}(t) - \int_0^t \left[\frac{\mathrm{d}u_{n,m}(s)}{\mathrm{d}s} - (u_{n,m}(s))(u_{n+1,m}(s) - u_{n-1,m}(s))\right]\mathrm{d}s \tag{4}$$

for this problem by using the variational iteration method [23 – 26] and produced an iteration algorithm that is rather complex because of the unnecessary and repeated calculations that occur in each iteration. Various modifications of the variational iteration method

Table 1. Comparison between the exact and the approximate solutions for constant $k = 0.1$, and time $t = 0.5$.

n	Exact Solution	Approximate Solution
−25	−0.09804197167	−0.09804373331
−15	−0.08824837375	−0.08825728153
−5	−0.03789706091	−0.03788612415
0	0.009900946468	0.009933709149
5	0.05350310282	0.05351081023
15	0.091855587402	0.09184745591
25	0.09857365538	0.09857205219

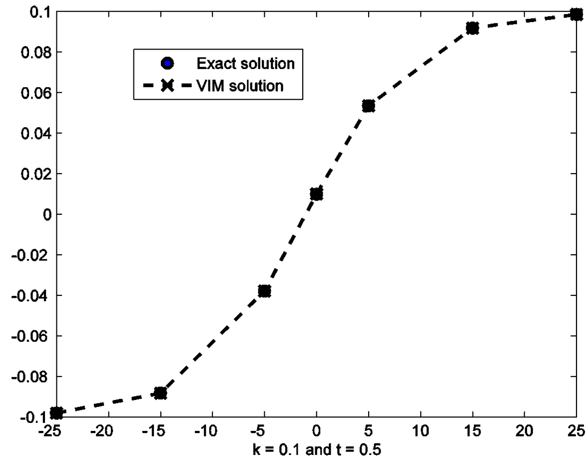

Fig. 1 (colour online). Graphical comparison between exact solution and VIM solution for constant $k = 0.1$ and time $t = 0.5$.

have since been suggested to overcome this shortcoming; including the incorporation of He's polynomials in the variational iteration method [27, 28]. An effective modification was also proposed by the inventor of the variational iteration method in [16, 29], where an alternative iteration formula

$$u_{n,m+1}(t) = u_{n,0}(t) + \int_0^t (u_{n,m}(s))(u_{n+1,m}(s) - u_{n-1,m}(s))\mathrm{d}s \tag{5}$$

was constructed instead of (4). This iteration formula is called the variational iteration algorithm-II according to a recent review article 'The Variational Iteration Method Which Should be Followed' [16]. Here, $u_{n,0}(t)$ is the initial guess that satisfies the initial condition, i. e. $u_{n,0}(t) = n$. The new iteration formula (5) produces the approximations

$$\begin{aligned} u_{n,0}(t) &= n, \\ u_{n,1}(t) &= n + 2nt, \\ u_{n,2}(t) &= n + 2nt + 4nt^2, \\ u_{n,3}(t) &= n + 2nt + 4nt^2 + 8nt^3, \\ u_{n,4}(t) &= n + 2nt + 4nt^2 + 8nt^3 + 16nt^4 \\ &= n(1 + 2t + 4t^2 + 8t^3 + 16t^4 \ldots), \\ &\vdots \\ u_{n,n}(t) &= \tfrac{n}{1-2t}. \end{aligned} \tag{6}$$

It is exact the solution appearing in (3). The results in (6) are clearly the same as those of Yildirim's, even though our proposed iteration algorithm is much simpler.

Yildirim also considered the discrete modified Korteweg-de Vries (mKdV) lattice equation

$$\frac{\mathrm{d}u_n}{\mathrm{d}t} = (1 - u_n^2)(u_{n+1} - u_{n-1}) \tag{7}$$

with the initial conditions

$$u_n(0) = \tanh(k)\tanh(kn) \tag{8}$$

and constructed the iteration formula

$$u_{n,m+1}(t) = u_{n,m}(t) - \int_0^t \left[\frac{\mathrm{d}u_{n,m}(s)}{\mathrm{d}s} - (1 - u_{n,m}^2(s))(u_{n+1,m}(s) - u_{n-1,m}(s))\right]\mathrm{d}s \tag{9}$$

for the problem. Again, we propose an alternative iteration formula

$$u_{n,m+1}(t) = \tanh(k)\tanh(kn) + \int_0^t (1 - u_{n,m}^2(s))(u_{n+1,m}(s) - u_{n-1,m}(s))\mathrm{d}s \tag{10}$$

which, if started with $u_{n,0}(t) = \tanh(k)\tanh(kn)$, pro-

n	Exact Solution	VIM Solution	ADM Solution	HPM Solution
−25	−0.09804197167	−0.09804373331	−0.09804384581	−0.09804373329
−15	−0.08824837375	−0.08825728153	−0.08826095868	−0.08825728153
−5	−0.03789706091	−0.03788612415	−0.03788809404	−0.03788612413
0	0.009900946468	0.009933709149	0.009933709152	0.00993370915
5	0.05350310282	0.05351081023	0.05350764842	0.05351081022
15	0.091855587402	0.09184745591	0.09184519212	0.09184745591
25	0.09857365538	0.09857205219	0.09857198864	0.09857205219

Table 2. Comparison between exact solution, VIM solution, ADM solution, and HPM solution for constant $k = 0.1$ and time $t = 0.5$.

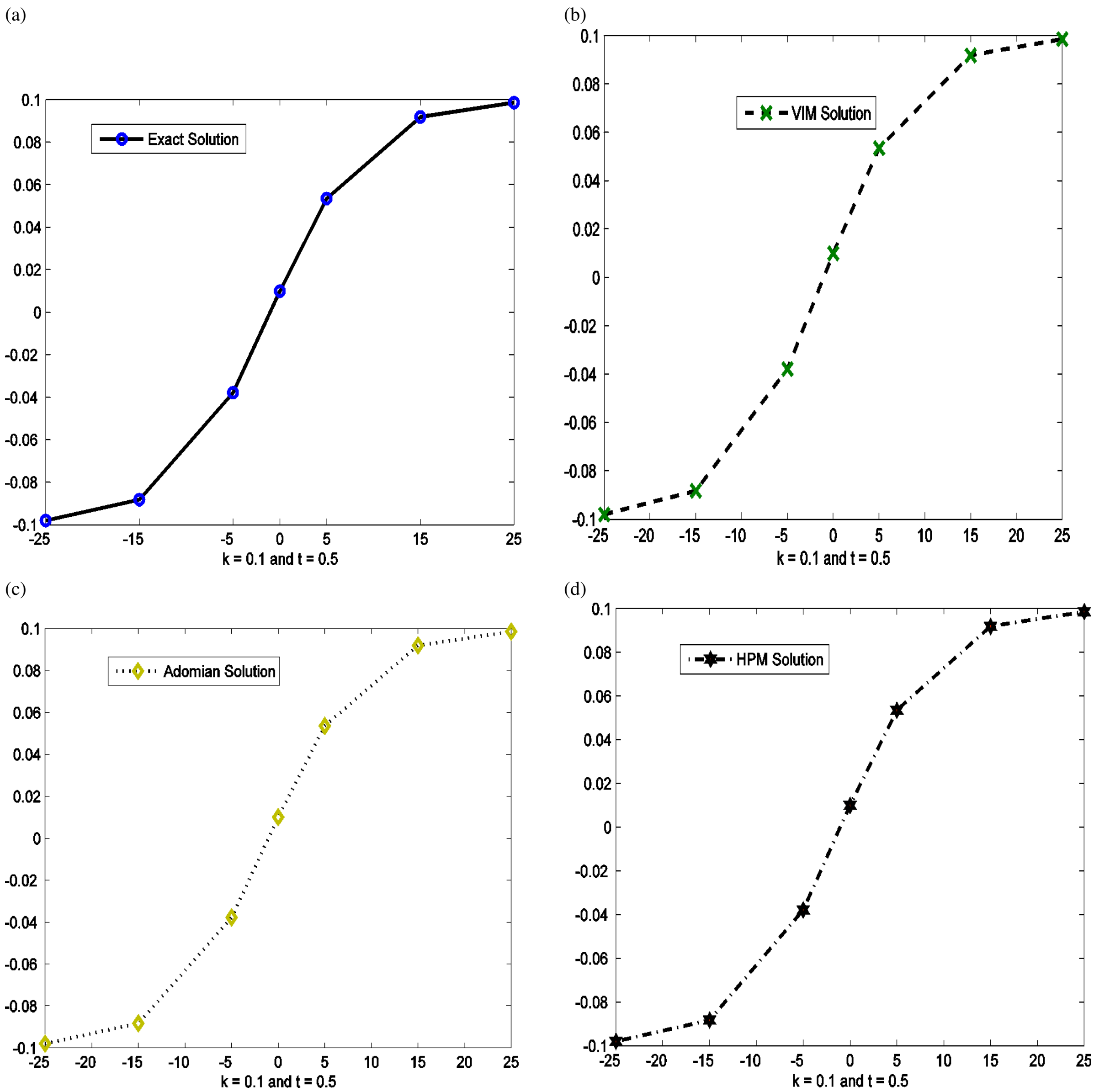

Fig. 2 (colour online). Graphical representation of (a) exact solution, (b) VIM solution, (c) ADM solution, and (d) HPM solution for constant $k = 0.1$ and time $t = 0.5$.

duces the following successive approximations:

$$
\begin{aligned}
u_{n,1}(t) = {} & \tanh(k)\tanh(kn) + [\tanh(k)(\tanh(k(n+1)) - \tanh(k(n-1)) \\
& - \tanh^2(k)\tanh^2(kn)(\tanh(k)(\tanh(k(n+1)) - \tanh(k(n-1))))]t,
\end{aligned} \tag{11}
$$

$$
\begin{aligned}
u_{n,2}(t) = {} & \tanh(k)\tanh(kn) + [\tanh(k)(\tanh(k(n+1)) - \tanh(k(n-1)) \\
& - \tanh^2(k)\tanh^2(kn)(\tanh(k)(\tanh(k(n+1)) - \tanh(k(n-1))))]t + [\tanh(k)\tanh(k(n+2))
\end{aligned}
$$

$$
\begin{aligned}
&-2\tanh(k)\tanh(kn)-\tanh^2(k)\tanh^2(k(n+1))(\tanh(k)\tanh(k(n+2)))-\tanh(k)\tanh(kn)\\
&+\tanh(k)\tanh(k(n-2))+\tanh^2(k)\tanh^2(k(n-1))(\tanh(k)\tanh(kn))-\tanh(k)\tanh(k(n-2))\\
&-2\tanh(k)\tanh(kn)\tanh(k)\tanh(k(n+1))-\tanh(k)\tanh(k(n-1))\tanh(k)\tanh(k(n+1))\\
&-\tanh(k)\tanh(k(n-1))-\tanh^2(k)\tanh^2(kn)\tanh(k)\tanh(k(n+1))-\tanh(k)\tanh(k(n-1))\\
&-\tanh^2(k)\tanh^2(kn)(\tanh(k)\tanh(k(n+2)))-2\tanh(k)\tanh(kn)-\tanh^2(k)\tanh^2(k(n+1)))\\
&\cdot\big(\tanh(k)\tanh(k(n+2))-\tanh(k)\tanh(kn)+\tanh(k)\tanh(k(n-2))\\
&+\tanh^2(k)\tanh^2(k(n-1))\tanh(k)\tanh(kn)-\tanh(k)\tanh(k(n-2))\big)0.5t^2.
\end{aligned}
\tag{12}
$$

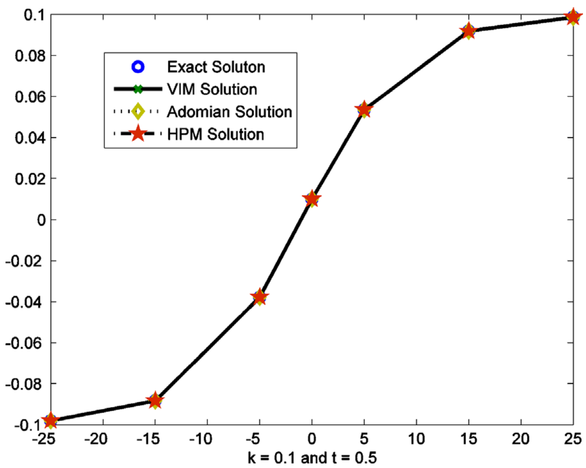

Fig. 3 (colour online). Graphical comparison between exact solution, VIM solution, ADM solution, and HPM solution for constant $k = 0.1$ and time $t = 0.5$.

The other components of $u_{n,m}(t)$ can be generated in an analogous way by using a mathematical software. The results obtained are identical to those produced by the iteration formula (9).

3. Conclusion

In this paper, we have proposed an effective variational iteration algorithm for solving differential-difference equations. The iteration procedure is much simpler than Yildirim's procedure and it produces the same results.

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