New Soliton and Periodic Solutions for Two Nonlinear Physical Models

Changbum Chun and Rathinasamy Sakthivel

Department of Mathematics, Sungkyunkwan University, Suwon 440-746, Republic of Korea

Reprint requests to C. C.; Fax: 82-31-290-7033; E-mail: cbchun@skku.edu

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In this paper, the exp-function method is applied by using symbolic computation to construct a variety of new generalized solitonary and periodic solutions for the Burgers-Kadomtsev-Petviashvili and Vakhnenko equations with distinct physical structures. The results reveal that the exp-function method is very effective and powerful for solving nonlinear evolution equations in mathematical physics.

Key words: Burgers-Kadomtsev-Petviashvili Equation; Solitonary Solutions; Vakhnenko Equation; Travelling Wave Solutions; Exp-Function Method.

1. Introduction

In the theoretical investigation, directly seeking exact solutions for nonlinear partial differential equations has become one of the central themes of perpetual interest in mathematical physics. Nonlinear wave phenomena appear in many fields, such as fluid mechanics, biomathematics, plasma physics, optical fibers, chemical physics, and other areas of engineering. These nonlinear phenomena are often related to nonlinear wave equations. In order to understand better these phenomena as well as to apply them in the practical life, it is important to seek their approximate solutions or more exact solutions. A variety of powerful methods have been developed for obtaining approximate and exact solutions for various nonlinear equations like the variational iteration method [1 – 3], F-expansion method [4, 5], tanh-function method [6 – 8], Adomian decomposition method [9, 10], homotopy perturbation method [11, 12], and so on.

In this paper, we consider the Burgers-Kadomtsev-Petviashvili (Burgers-KP) equation in the following form [13]:

$$(u_t + uu_x + \gamma u_{xx})_x + \lambda u_{yy} = 0, \tag{1}$$

where γ and λ are arbitrary constants. The Kadomtsev and Petviashvili equation is used to model shallow-water waves with weakly nonlinear restoring forces. If the surface tension is weak compared to the gravitational forces, then $\lambda = 1$ is used. However, if the surface tension is strong, then $\lambda = -1$ is used. The Burgers equation is a second-order partial differential equation and it is used to describe the structure of shock waves, traffic flow, and acoustic transmission [14]. The multiple-front solutions for the Burgers equation and the coupled Burgers equations are obtained in [15]. Very recently, Wazwaz [13] has obtained single and multiple-front solutions for the Burgers-Kadomtsev-Petviashvili equation using Hirota's bilinear method and the tanh-coth method.

Next, we consider the propagation of high-frequency waves in a relaxing medium, governed by the so-called Vakhnenko equation [16]

$$uu_{xxt} - u_x u_{xt} + u^2 u_t = 0. \tag{2}$$

In [16], the soliton solutions for the Vakhnenko equation have been obtained by the inverse scattering method. The one-loop soliton solution of the Vakhnenko equation by means of the homotopy analysis method is obtained in [17], but the obtained solution is just an approximate solution. However, the Vakhnenko equation has loop-like soliton solutions, and thus it is not easy to solve and obtain more exact solutions [16].

Another important method used to obtain exact solutions of nonlinear partial differential equation is the exp-function method [18]. The exp-function method is one of the most effective methods for solving nonlinear equations and it has been successfully applied to many kinds of nonlinear evolution equations [19 – 32]. The main purpose of this paper is to obtain generalized exact solutions, which include new periodic and soliton solutions of (1) and (2) by using the exp-function method. By using this method, it is possible to obtain a variety of generalized solitonary and periodic solutions for nonlinear evolution equations with distinct physical structures.

2. Solution of the Burgers-Kadomtsev-Petviashvili Equation

In order to obtain the solution of (1), we consider the transformation $u = v(\eta)$, $\eta = kx + ly + \omega t$, where k, l, and ω are constants to be determined later. Then we can rewrite Burgers-KP Equation (1) as the following nonlinear ordinary differential equation of the form

$$k\omega v'' + k^2(v')^2 + k^2 vv'' + k^3\gamma v''' + \lambda l^2 v'' = 0, \quad (3)$$

where the prime denotes the differential with respect to η. According to the exp-function method [18], we assume that the solution of (3) can be expressed as

$$v(\eta) = \frac{\sum_{n=-c}^{d} a_n \exp(n\eta)}{\sum_{m=-p}^{q} b_m \exp(m\eta)}, \quad (4)$$

where c, d, p, and q are positive integers which are unknown and have to be determined further, a_n and b_m are unknown constants.

(4) can be re-written in an alternative form [18] as follows:

$$v(\eta) = \frac{a_c \exp(c\eta) + \cdots + a_{-d} \exp(-d\eta)}{b_p \exp(p\eta) + \cdots + b_{-q} \exp(-q\eta)}. \quad (5)$$

In order to determine the values of c and p, we balance the linear term of highest order in (3) with the highest-order nonlinear term. By simple calculation, we have

$$v''' = \frac{c_1 \exp[(7p + c)\eta] + \cdots}{c_2 \exp(8p\eta) + \cdots} \quad (6)$$

and

$$vv' = \frac{c_3 \exp[(p + 2c)\eta] + \cdots}{c_4 \exp(3p\eta) + \cdots} = \frac{c_3 \exp[(6p + 2c)\eta] + \cdots}{c_4 \exp(8p\eta) + \cdots}, \quad (7)$$

where c_i are determined coefficients only for simplicity. Balancing highest order of the exp-function in (6) and (7), we obtain

$$7p + c = 6p + 2c \quad (8)$$

which gives

$$p = c. \quad (9)$$

Similarly, to determine the values of d and q, we balance the linear term of lowest order in (3) with the lowest-order nonlinear term

$$v''' = \frac{\cdots + d_1 \exp[-(7q + d)\eta]}{\cdots + d_2 \exp[(-8q)\eta]} \quad (10)$$

and

$$vv' = \frac{\cdots + d_3 \exp[-(q + 2d)\eta]}{\cdots + d_4 \exp[(-3q)\eta]} = \frac{\cdots + d_3 \exp[-(6q + 2d)\eta]}{\cdots + d_4 \exp[(-8q)\eta]}, \quad (11)$$

where d_i are determined coefficients only for simplicity. Balancing lowest-order of the exp-function in (10) and (11), we obtain

$$-(7q + d) = -(6q + 2d) \quad (12)$$

which gives

$$q = d. \quad (13)$$

Case 1. *$p = c = 1$, $d = q = 1$*:

We can freely choose the values of c and d. For simplicity, we set $p = c = 1$, $b_1 = 1$, and $d = q = 1$, then (5) reduces to

$$v(\eta) = \frac{a_1 \exp(\eta) + a_0 + a_{-1} \exp(-\eta)}{\exp(\eta) + b_0 + b_{-1} \exp(-\eta)}. \quad (14)$$

Substituting (14) in (3) and using Maple software and equating to zero the coefficients of all powers of $\exp(n\eta)$ gives a set of algebraic equations for $a_1, a_0, a_{-1}, b_0, b_{-1}, k, l$, and ω. Solving this systems of algebraic equations using Maple gives the following sets of nontrivial solutions:

$$a_1 = \frac{4\gamma k b_{-1} + a_{-1}}{b_{-1}}, \quad a_0 = 0, \quad a_{-1} = a_{-1}, \quad b_0 = 0, \quad b_{-1} = b_{-1}, \quad k = k, \quad l = l, \quad \omega = -\frac{\lambda l^2 b_{-1} + k^2 a_{-1} + 2\gamma k^3 b_{-1}}{k b_{-1}}, \quad (15)$$

$$a_1 = a_1, \quad a_0 = a_0, \quad a_{-1} = -\frac{A}{4\gamma^2 k^2}, \quad b_0 = b_0, \quad b_{-1} = -\frac{\tilde{A}}{4\gamma^2 k^2}, \quad k = k, \quad l = l, \quad \omega = -\frac{\lambda l^2 + k^2 a_1 - \gamma k^3}{k}, \quad (16)$$

where

$$A = a_1^3 b_0^2 + a_1 a_0^2 - 4b_0^2 \gamma k a_1^2 + 6b_0 a_1 a_0 \gamma k - 2a_1^2 b_0 a_0 - 2\gamma k a_0^2 + 4b_0^2 a_1 \gamma^2 k^2 - 4b_0 \gamma^2 k^2 a_0 \tag{17}$$

and

$$\tilde{A} = a_1^2 b_0^2 + a_0^2 - 2\gamma k a_1 b_0^2 + 2\gamma k a_0 b_0 - 2a_1 b_0 a_0. \tag{18}$$

Substituting (15) and (16) in (14), we obtain the following soliton solutions of (1):

$$u_1(x,y,t) = \frac{(4\gamma k b_{-1} + a_{-1})\exp(\eta) + a_{-1} b_{-1} \exp(-\eta)}{b_{-1}\exp(\eta) + b_{-1}^2 \exp(-\eta)}, \tag{19}$$

where $\eta = kx + ly - \left(\frac{\lambda l^2 b_{-1} + k^2 a_{-1} + 2\gamma k^3 b_{-1}}{k b_{-1}}\right) t$, and

$$u_2(x,y,t) = \frac{4\gamma^2 k^2 a_1 \exp(\eta) + 4\gamma^2 k^2 a_0 - A\exp(-\eta)}{4\gamma^2 k^2 \exp(\eta) + 4\gamma^2 k^2 b_0 - \tilde{A}\exp(-\eta)}, \tag{20}$$

where $\eta = kx + ly - \frac{\lambda l^2 + k^2 a_1 - \gamma k^3}{k} t$ and $A, \tilde{A}$ are defined as in (17) and (18).

Further, since a_{-1} and b_{-1} are free parameters, we take $a_{-1} = -4k\gamma b_{-1}$ and $b_{-1} = 1$, then (19) admits a new soliton solution as follows:

$$u(x,y,t) = \frac{-4k\gamma \exp(-\eta)}{\exp(\eta) + \exp(-\eta)} = -2k\gamma(1 - \tanh\eta), \quad \text{where } \eta = kx + ly - \frac{\lambda l^2 - 2\gamma k^3}{k} t. \tag{21}$$

Moreover, if we take $a_{-1} = -4k\gamma b_{-1}$ and $b_{-1} = -1$, then (19) admits the following solution:

$$u(x,y,t) = 2k\gamma(1 - \coth\eta), \quad \text{where } \eta = kx + ly + \frac{-\lambda l^2 + 2\gamma k^3}{k} t. \tag{22}$$

The solutions (21) and (22) are similar to the solution obtained by the tanh-function method [13].

Case 2. $p = c = 2, d = q = 2$:

As mentioned above the values of c and d can be freely chosen, therefore we set $p = c = 2$, and $d = q = 2$. Also there are some free parameters, we set $b_2 = 1, b_1 = b_{-1} = 0$ for simplicity, then (5) becomes

$$v(\eta) = \left[a_2 \exp(2\eta) + a_1 \exp(\eta) + a_0 + a_{-1}\exp(-\eta) + a_{-2}\exp(-2\eta)\right] \cdot \left[\exp(2\eta) + b_0 + b_{-2}\exp(-2\eta)\right]^{-1}. \tag{23}$$

By the same calculation as illustrated above, we obtain

$$a_2 = a_2, \quad a_1 = 0, \quad a_0 = a_0, \quad a_{-1} = 0, \quad a_{-2} = -\frac{B}{16\gamma^2 k^2}, \quad b_0 = b_0, \quad b_{-2} = -\frac{\tilde{B}}{16\gamma^2 k^2}, \quad k = k, \quad l = l, \quad \omega = -\frac{-2\gamma k^3 + k^2 a_2 + \lambda l^2}{k}, \tag{24}$$

where

$$B = a_2 a_0^2 + 12 b_0 a_0 \gamma k a_2 + a_2^3 b_0^2 - 2a_2^2 b_0 a_0 - 8b_0^2 \gamma k a_2^2 - 4\gamma k a_0^2 - 16 b_0 \gamma^2 k^2 a_0 + 16 b_0^2 a_2 \gamma^2 k^2 \tag{25}$$

and

$$\tilde{B} = a_0^2 + 4\gamma k a_0 b_0 + a_2^2 b_0^2 - 2a_2 b_0 a_0 - 4\gamma k a_2 b_0^2, \tag{26}$$

$$a_2 = a_2, \quad a_1 = a_1, \quad a_0 = -\frac{C}{4\gamma^2 k^2 a_1^2}, \quad a_{-1} = -\frac{4\gamma^2 k^2 b_{-2}}{a_1}, \quad a_{-2} = -2\gamma k b_{-2} + a_2 b_{-2}, \quad b_0 = -\frac{16\gamma^4 k^4 b_{-2} + a_1^4}{4\gamma^2 k^2 a_1^2}, \quad b_{-2} = b_{-2}, \quad k = k, \quad l = l, \quad \omega = -\frac{k^2 a_2 + \lambda l^2 - \gamma k^3}{k}, \tag{27}$$

where

$$C = 16 a_2 \gamma^4 k^4 b_{-2} + a_2 a_1^4 - 2\gamma k a_1^4. \tag{28}$$

Substituting (24) and (27) in (23), we obtain the following soliton solutions of (1):

$$u_1(x,y,t) = \frac{16\gamma^2 k^2 a_2 \exp(2\eta) + 16\gamma^2 k^2 a_0 - B\exp(-2\eta)}{16\gamma^2 k^2 \exp(2\eta) + 16\gamma^2 k^2 b_0 - \tilde{B}\exp(-2\eta)}, \tag{29}$$

where $\eta = kx + ly - \frac{-2\gamma k^3 + k^2 a_2 + \lambda l^2}{k} t$ and $B, \tilde{B}$ are defined as in (25) and (26),

$$u_2(x,y,t) = \left[4\gamma^2 k^2 a_1^2 (a_2 \exp(2\eta) + a_1 \exp(\eta)) - C - 16\gamma^4 k^4 a_1 b_{-2} \exp(-\eta) + \tilde{C}\exp(-2\eta)\right] \cdot \left[4\gamma^2 k^2 a_1^2 \exp(2\eta) - (16\gamma^4 k^4 b_{-2} + a_1^4) + 4\gamma^2 k^2 a_1^2 b_{-2} \exp(-2\eta)\right]^{-1}, \tag{30}$$

where $\tilde{C} = -8\gamma^3 k^3 a_1^2 b_{-2} + 4\gamma^2 k^2 a_1^2 a_2 b_{-2}$, $\eta = kx + ly - \frac{k^2 a_2 + \lambda l^2 - \gamma k^3}{k} t$ and C is defined as in (28).

Case 3. *$p = c = 3$, $d = q = 3$:*

Now consider the case $p = c = 3$, and $d = q = 3$ with $b_2 = 1$, $b_3 = b_1 = b_{-1} = 0$, under this case (5) can be expressed as

$$v(\eta) = \big[a_3 \exp(3\eta) + a_2 \exp(2\eta) + a_1 \exp(\eta) + a_0 + a_{-1} \exp(-\eta) + a_{-2} \exp(-2\eta) + a_{-3} \exp(-3\eta)\big] \cdot \big[\exp(2\eta) + b_0 + b_{-2} \exp(-2\eta) + b_{-3} \exp(-3\eta)\big]^{-1}. \tag{31}$$

By a simple calculation, we obtain

$$\begin{aligned} &a_3 = 0, \quad a_2 = \frac{2\gamma k b_{-3} + a_{-3}}{b_{-3}}, \quad a_1 = a_1, \\ &a_0 = -\frac{D}{4\gamma^2 k^2 a_1^3 b_{-3}}, \\ &a_{-1} = -\frac{4k^2\gamma^2(a_1 b_{-2} + 2\gamma k b_{-3})}{a_1^2} \\ &a_{-2} = -\frac{4\gamma^2 k^2 b_{-3}^2 - a_{-3} b_{-2} a_1}{a_1 b_{-3}}, \\ &a_{-3} = a_{-3}, \quad b_0 = -\frac{\tilde{D}}{4\gamma^2 k^2 a_1^3}, \quad b_{-2} = b_{-2}, \\ &b_{-3} = b_{-3}, \quad k = k, \\ &l = l, \quad \omega = -\frac{\lambda l^2 b_{-3} + \gamma k^3 b_{-3} + a_{-3} k^2}{k b_{-3}}, \end{aligned} \tag{32}$$

where

$$D = 32\gamma^5 k^5 b_{-3} a_1 b_{-2} + 64\gamma^6 k^6 b_{-3}^2 + a_{-3} a_1^5 + 16\gamma^4 k^4 a_1 b_{-2} a_{-3} + 32\gamma^5 k^5 b_{-3} a_{-3} \tag{33}$$

and

$$\tilde{D} = a_1^5 + 16\gamma^4 k^4 a_1 b_{-2} + 32 b_{-3} \gamma^5 k^5. \tag{34}$$

Substituting (32) in (31), we obtain the following soliton solution of (1):

$$u(x,y,t) = \big[N \exp(2\eta) + N_1 a_1 \exp(\eta) - D - N_2 \exp(-\eta) - N_3 \exp(-2\eta) + N_1 a_{-3} \exp(-3\eta)\big] \cdot \big[4\gamma^2 k^2 a_1^3 b_{-3} \exp(2\eta) + \tilde{D} b_{-3} + 4\gamma^2 k^2 a_1^3 b_{-2} \cdot b_{-3} \exp(-2\eta) + 4\gamma^2 k^2 a_1^3 b_{-3}^2 \exp(-3\eta)\big]^{-1}, \tag{35}$$

where $N = 8\gamma^3 k^3 a_1^3 b_{-3} + 4\gamma^2 k^2 a_1^3 a_{-3}$, $N_1 = 4\gamma^2 k^2 a_1^3 b_{-3}$, $N_2 = 16k^4\gamma^4 a_1^2 b_{-2} b_{-3} - 32k^5\gamma^5 a_1 b_{-3}^2$, $N_3 = 16\gamma^4 k^4 a_1^2 b_{-3}^2 - 4\gamma^2 k^2 a_1^3 a_{-3} b_{-2}$, $\eta = kx + ly - \frac{\lambda l^2 b_{-3} + \gamma k^3 b_{-3} + a_{-3} k^2}{k b_{-3}} t$ and D, $\tilde{D}$ are defined as in (33) and (34).

3. Solution of the Vakhnenko Equation

In this section in order to obtain the solution of the Vakhnenko equation (2), we consider the transformation $u = v(\eta)$, $\eta = kx + \omega t$, which converts (2) into an ordinary differential equation of the form

$$k^2 \omega v v''' - k^2 \omega v' v'' + \omega v^2 v' = 0, \tag{36}$$

where the prime denotes the derivation with respect to η. By the same procedure as illustrated in Section 2, we can determine values of c and p by balancing vv''' and $v^2 v'$ in (36). We get

$$v v''' = \frac{c_1 \exp[(7p + 2c)\eta] + \cdots}{c_2 \exp(9p\eta) + \cdots} \tag{37}$$

and

$$v^2 v' = \frac{c_3 \exp[(6p + 3c)\eta] + \cdots}{c_4 \exp(9p\eta) + \cdots}. \tag{38}$$

Balancing highest order of exp-function in (37) and (38), we obtain

$$7p + 2c = 6p + 3c \tag{39}$$

which gives

$$p = c. \tag{40}$$

By a similar derivation, balancing lowest-order of exp-function in (37) and (38), we obtain

$$-(7q + 2d) = -(6q + 3d) \tag{41}$$

which gives

$$q = d. \tag{42}$$

Case 1. *$p = c = 1$, $d = q = 1$*:

As mentioned in the previous section, the values of c and d can be freely chosen. For simplicity, we choose $p = c = 1, b_1 = 1$, and $d = q = 1$, then the trail function (5) becomes

$$v(\eta) = \frac{a_1 \exp(\eta) + a_0 + a_{-1} \exp(-\eta)}{\exp(\eta) + b_0 + b_{-1} \exp(-\eta)}. \tag{43}$$

Substituting (43) in (36) and equating to zero the coefficients of all powers of $\exp(n\eta)$ yields a set of algebraic equations for $a_1, a_0, a_{-1}, b_0, b_{-1}, k$, and ω. Solving this systems of algebraic equations using Maple, we obtain

$$\begin{aligned} &a_1 = 0, \quad a_0 = 3k^2 b_0, \quad a_{-1} = 0, \quad b_0 = b_0, \\ &b_{-1} = \frac{b_0^2}{4}, \quad k = k, \omega = \omega, \end{aligned} \tag{44}$$

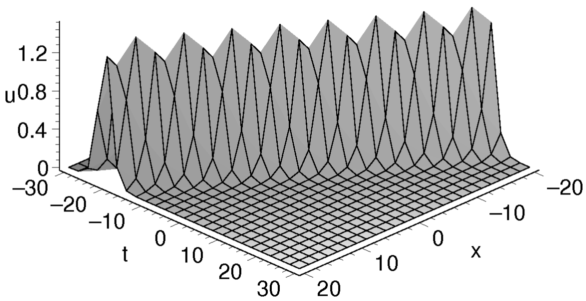

Fig. 1. Solution of (46) with $b_0 = 1$, $k = 1$, $\omega = 1$.

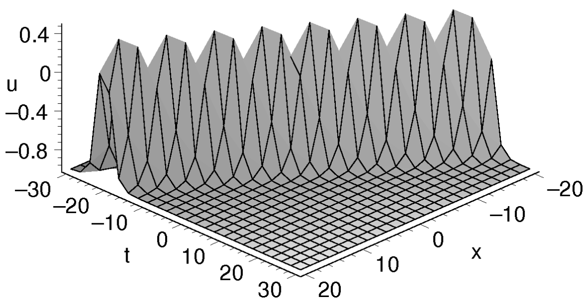

Fig. 2. Solution of (47) with $a_0 = 1$, $k = 1$, $\omega = 1$.

$$\begin{aligned} &a_1 = -k^2, \quad a_0 = a_0, \quad a_{-1} = -\frac{a_0^2}{16k^2}, \\ &b_0 = \frac{a_0}{2k^2}, \quad b_{-1} = \frac{a_0^2}{16k^4}, \quad k = k, \quad \omega = \omega. \end{aligned} \tag{45}$$

Substituting (44) and (45) in (43), we obtain the following soliton solutions of (2):

$$u_1(x,t) = \frac{12k^2 b_0}{4\exp(\eta) + 4b_0 + b_0^2 \exp(-\eta)}, \tag{46}$$
where $\eta = kx + \omega t$,

$$u_2(x,t) = \frac{-16k^6 \exp(\eta) + 16k^4 a_0 - a_0^2 k^2 \exp(-\eta)}{16k^4 \exp(\eta) + 8k^2 a_0 + a_0^2 \exp(-\eta)}, \tag{47}$$
where $\eta = kx + \omega t$.

The behaviour of the obtained exact solutions (46) and (47) is shown graphically in Figure 1 and 2.

In addition, when k and ω are imaginary numbers, the obtained soliton solutions can be converted into periodic solutions. We write $k = \mathrm{i}K$, $\omega = \mathrm{i}\Omega$ to obtain $\eta = \mathrm{i}(Kx + \Omega t)$. Using the transformation

$$\exp(\eta) = \cos(Kx + \Omega t) + \mathrm{i}\sin(Kx + \Omega t)$$

and

$$\exp(\eta) = \cos(Kx + \Omega t) - \mathrm{i}\sin(Kx + \Omega t).$$

Then (46) becomes

$$\begin{aligned} u(x,t) = -12K^2 b_0 \big[&(4 + b_0^2)\cos(Kx + \Omega t) + 4b_0 \\ &+ \mathrm{i}(4 - b_0^2)\sin(Kx + \Omega t)\big]^{-1}. \end{aligned} \tag{48}$$

If we search for a periodic solution, the imaginary part in (48) must be zero, which requires that

$$b_0 = \pm 2. \tag{49}$$

Substituting (49) in (48) yields two periodic solutions

$$u(x,t) = \frac{-3K^2}{1 + \cos(Kx + \Omega t)} \tag{50}$$

and

$$u(x,t) = \frac{3K^2}{-1 + \cos(Kx + \Omega t)}. \tag{51}$$

Since $k = \mathrm{i}K$, $\omega = \mathrm{i}\Omega$, we write $K = -\mathrm{i}k$ and $\Omega = -\mathrm{i}\omega$, then from (50) we obtain

$$\begin{aligned} u(x,t) &= \frac{3k^2}{2\cos^2(\frac{-\mathrm{i}kx}{2} - \frac{\mathrm{i}\omega t}{2})} \\ &= \frac{3k^2}{2\cos^2(-\mathrm{i}(\frac{kx}{2} + \frac{\omega t}{2}))} \\ &= \frac{3}{2}k^2 \sec h^2(\frac{kx}{2} + \frac{\omega t}{2}). \end{aligned} \tag{52}$$

The solution (52) was similar to the solution obtained in [33].

Case 2. *$p = c = 1$, $d = q = 2$*:

For simplicity, we set $p = c = 1, b_1 = 1$, and $d = q = 2$, then by the same procedure as illustrated in Section 2, we obtain the following equation:

$$v(\eta) = \frac{a_1 \exp(\eta) + a_0 + a_{-1}\exp(-\eta) + a_{-2}\exp(-2\eta)}{\exp(\eta) + b_0 + b_{-1}\exp(-\eta) + b_{-2}\exp(-2\eta)}. \tag{53}$$

By the same calculation as illustrated in the previous subsection, we obtain

$$\begin{aligned} &a_1 = 0, \quad a_0 = a_0, \quad a_{-1} = a_{-1}, \quad a_{-2} = 0, \\ &b_0 = \frac{3k^2 a_{-1} + a_0^2}{3k^2 a_0}, \quad b_{-1} = \frac{12k^2 a_{-1} + a_0^2}{36k^4}, \\ &b_{-2} = \frac{a_0 a_{-1}}{36k^4}, \quad k = k, \quad \omega = \omega, \end{aligned} \tag{54}$$

$$a_1=-k^2,\quad a_0=a_0,$$
$$a_{-1}=\frac{5k^4b_0^2+2k^2a_0b_0-3a_0^2}{12k^2},$$
$$a_{-2}=-\frac{(-a_0^3+3k^4a_0b_0^2+2b_0^3k^6)}{108k^4},\tag{55}$$
$$b_0=b_0,\quad b_{-1}=\frac{3k^4b_0^2+2k^2a_0b_0-a_0^2}{12k^4},$$
$$b_{-2}=\frac{-a_0^3+3k^4a_0b_0^2+2b_0^3k^6}{108k^6},\ k=k,\ \omega=\omega.$$

Substituting (54) and (55) in (53), we obtain the following soliton solutions of (2):

$$u_1(x,t)=\left[36k^4a_0(a_0+a_{-1}\exp(-\eta))\right]\cdot\left[36k^4a_0\exp(\eta)+12k^2\tilde{H}\right.\tag{56}$$
$$\left.+a_0(12k^2a_{-1}+a_0^2)\exp(-\eta)+a_0^2a_{-1}\exp(-2\eta)\right]^{-1},$$

where $\tilde{H}=3k^2a_{-1}+a_0^2$ and $\eta=kx+\omega t$,

$$u_2(x,t)=\left[-108k^8\exp(\eta)+108k^6b_0+9k^4H_1\exp(-\eta)-k^2H_2\exp(-2\eta)\right]$$
$$\cdot\left[-108k^6\exp(\eta)+108k^6b_0+9k^2H_3\exp(-\eta)+H_2\exp(-2\eta)\right]^{-1},\tag{57}$$

where $H_1=5k^4b_0^2+2k^2a_0b_0-3a_0^2$, $H_2=-a_0^3+3k^4a_0b_0^2+2b_0^3k^6$, $H_3=3k^4b_0^2+2k^2a_0b_0-a_0^2$, and $\eta=kx+\omega t$.

4. Conclusion

In this paper, we have applied the exp-function method to obtain both the new generalized solitonary and periodic solutions of Burgers-Kadomtsev-Petviashvili and Vakhnenko equations. The result reveals that the exp-function method is a promising tool since it can provide a variety of new solutions of distinct physical structures when compared with existing methods. This indicates the validity and great potential of the exp-function method in solving complicated solitary wave problems.

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