Application of the Laplace Decomposition Method to Nonlinear Homogeneous and Non-Homogenous Advection Equations

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In this paper, we apply the Laplace decomposition method to obtain series solutions of nonlinear advection equations. The equations are Laplace transformed and the nonlinear terms are represented by Adomian polynomials. The results are in good agreement with those obtained by the Adomian decomposition method and the variational iteration method but the convergence is faster.

Key words: Laplace Decomposition Method; Nonlinear Advection Equation; Adomian Polynomials.

1. Introduction

The Laplace transform is an elementary but useful technique for solving linear ordinary differential equations that is widely used by scientists and engineers for tackling linearized models. In fact, the Laplace transform is one of only a few methods that can be applied to linear systems with periodic or discontinuous driving inputs. Despite its great usefulness in solving linear problems, however, the Laplace transform is totally incapable of handling nonlinear equations because of the difficulties caused by the nonlinear terms. Various ways have been proposed recently to deal with these nonlinearities such as the Adomian decomposition method [1] and the Laplace decomposition algorithm [2, 3]. Furthermore, the Laplace transformation method is also combined with the well-known homotopy perturbation method [4 – 7] and the variational iteration method [8 – 11] to produce a highly effective technique for handling many nonlinear problems (see, for example, [12, 13]). These new developments, together with a host of other transform methods such as the differential transform [14 – 16] and the Sumudu transform [17], form the bulk of the collection of methods that are currently available for the solution of nonlinear differential equations. This paper considers the effectiveness of the Laplace decomposition algorithm in solving nonlinear advection equations, both homogeneous and otherwise.

2. The Laplace Decomposition Method [1]

To illustrate the basic idea of this method, we consider a general nonlinear non-homogeneous partial differential equation with an initial condition of the form

$$\begin{aligned} &Du(x,t)+Ru(x,t)+Nu(x,t)=g(x,t), \\ &u(x,0)=h(x), \end{aligned} \tag{1}$$

where D is the linear differential operator $D=\partial/\partial t$, R the remaining linear operator of less order than D, N represents the nonlinear differential operator, and $g(x,t)$ is the source term. Taking the Laplace transform (L) of (1), we have

$$L[Du(x,t)]+L[Ru(x,t)]+L=L[g(x,t)], \tag{2}$$

which gives

$$\begin{aligned} L[u(x,t)] = \frac{h(x)}{s}-\frac{1}{s}L[Ru(x,t)]+\frac{1}{s}L[g(x,t)] \\ -\frac{1}{s}L[Nu(x,t)]. \end{aligned} \tag{3}$$

We now assume that the solution can be expressed in an infinite series in the form

$$u(x,t)=\sum_{n=0}^{\infty}u_n(x,t), \tag{4}$$

so that the nonlinear term can be decomposed as

$$Nu(x,t)=\sum_{n=0}^{\infty}A_n(u), \tag{5}$$

for some Adomian polynomials A_n (see [18]) that are given by

$$A_n = \frac{1}{n!}\frac{\mathrm{d}^n}{\mathrm{d}\lambda^n}\left[N\left(\sum_{n=0}^{\infty}\lambda^i u_i\right)\right]_{\lambda=0}, \quad n = 0,1,2,3\dots. \tag{6}$$

Substituting (5) and (4) in (3) we get

$$L\left[\sum_{n=0}^{\infty} u_n(x,t)\right] = \frac{h(x)}{s} - \frac{1}{s}L[Ru(x,t)] + \frac{1}{s}L[g(x,t)] - \frac{1}{s}L\left[\sum_{n=0}^{\infty} A_n(u)\right] \tag{7}$$

and

$$\sum_{n=0}^{\infty} L[u_n(x,t)] = \frac{h(x)}{s} - \frac{1}{s}L[Ru(x,t)] + \frac{1}{s}L[g(x,t)] - \frac{1}{s}L\left[\sum_{n=0}^{\infty} A_n(u)\right], \tag{8}$$

which gives, by comparing both sides of (8),

$$L[u_0(x,t)] = \frac{h(x)}{s} + \frac{1}{s}L[g(x,t)],$$

$$\begin{aligned} L[u_1(x,t)] &= -\frac{1}{s}L[Ru_0(x,t)] - \frac{1}{s}L[A_0(u)], \\ L[u_2(x,t)] &= -\frac{1}{s}L[Ru_1(x,t)] - \frac{1}{s}L[A_1(u)], \\ L[u_3(x,t)] &= -\frac{1}{s}L[Ru_2(x,t)] - \frac{1}{s}L[A_2(u)]. \end{aligned} \tag{9}$$

Clearly, the general recursive relation is given by

$$L[u_{n+1}(x,t)] = -\frac{1}{s}L[Ru_n(x,t)] - \frac{1}{s}L[A_n(u)], \quad n \geq 0, \tag{10}$$

so that on applying the inverse Laplace transform, we have

$$u_0(x,t) = H(x,t) \tag{11}$$

and

$$u_{n+1}(x,t) = -L^{-1}\left[\frac{1}{s}L[Ru_n(x,t)] + \frac{1}{s}L[A_n(u)]\right], \quad n \geq 0, \tag{12}$$

where the function $H(x,t)$ arises from the source term and the prescribed initial condition.

3. The Homogeneous Advection Problem

Example 1. Consider the following homogeneous advection problem [18]:

$$u_t + uu_x = 0, \quad u(x,0) = -x \tag{13}$$

and apply the Laplace transform to get

$$su(x,s) - u(x,0) = -L[uu_x]. \tag{14}$$

The initial condition now implies that

$$su(x,s) = -x - L[uu_x] \tag{15}$$

and

$$u(x,s) = -\frac{x}{s} - \frac{1}{s}L[uu_x], \tag{16}$$

so that by applying the inverse Laplace transform, we have

$$u(x,t) = -x - L^{-1}\left[\frac{1}{s}L[uu_x]\right]. \tag{17}$$

Since a series solution of the form

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t) \tag{18}$$

is assumed in the Laplace decomposition method, (18) is substituted into (17) to get

$$\sum_{n=0}^{\infty} u_n(x,t) = -x - L^{-1}\left[\frac{1}{s}L\left[\sum_{n=0}^{\infty} A_n(u)\right]\right], \tag{19}$$

where $A_n(u)$ are Adomian polynomials that represent the nonlinear terms and satisfy

$$\sum_{n=0}^{\infty} A_n(u) = uu_x. \tag{20}$$

The first few components of the Adomian polynomials, for example, are given by

$$\begin{aligned} A_0(u) &= u_0 u_{0x}, \\ A_1(u) &= u_0 u_{1x} + u_1 u_{0x}, \\ A_2(u) &= u_0 u_{2x} + u_1 u_{1x} + u_2 u_{0x}, \\ &\vdots \end{aligned} \tag{21}$$

It is clear from (19) that the recursive relation is

$$u_0(x,t) = -x, \tag{22}$$

$$u_{n+1}(x,t) = -L^{-1}\left[\frac{1}{s}L\left[\sum_{n=0}^{\infty} A_n(u)\right]\right], \quad n \geq 0, \tag{23}$$

so that for $n = 0$, we have

$$\begin{aligned}
u_1(x,t) &= -L^{-1}\left[\frac{1}{s}L[A_0(u)]\right],\\
u_1(x,t) &= -L^{-1}\left[\frac{1}{s}L[u_0u_{0x}]\right],\\
u_1(x,t) &= -L^{-1}\left[\frac{1}{s}L[x]\right],\\
u_1(x,t) &= -L^{-1}\left[\frac{x}{s^2}\right],\\
u_1(x,t) &= -xt,
\end{aligned} \tag{24}$$

and for $n = 1$, we have

$$\begin{aligned}
u_2(x,t) &= -L^{-1}\left[\frac{1}{s}L[A_1(u)]\right],\\
u_2(x,t) &= -L^{-1}\left[\frac{1}{s}L[u_0u_{1x}+u_1u_{0x}]\right],\\
u_2(x,t) &= -L^{-1}\left[\frac{1}{s}L[2xt]\right],\\
u_2(x,t) &= -L^{-1}\left[\frac{2x}{s^3}\right],\\
u_2(x,t) &= -xt^2.
\end{aligned} \tag{25}$$

Proceeding in a similar manner, we have

$$\begin{aligned}
&A_2(u) = 3xt^2,\\
&u_3(x,t) = -xt^3,\\
&A_3(u) = 4xt^3,\\
&u_4(x,t) = -xt^4,\\
&\vdots
\end{aligned} \tag{26}$$

so that the solution $u(x,t)$ is given by

$$u(x,t) = -x\left(1+t+t^2+t^3+t^4+\ldots\right) \tag{27}$$

in series form, and

$$u(x,t) = \frac{x}{t-1} \tag{28}$$

in closed form.

Example 2. Consider the following homogeneous advection problem:

$$u_t + uu_x = 0, \quad u(x,0) = 4x. \tag{29}$$

In a similar way as above we have

$$u(x,s) = 4x\frac{1}{s} - \frac{1}{s}L[uu_x]. \tag{30}$$

The inverse of the Laplace transform implies that

$$u(x,t) = 4x - L^{-1}\left[\frac{1}{s}L[uu_x]\right]. \tag{31}$$

Proceeding as before we obtain

$$\sum_{n=0}^{\infty} u_n(x,t) = 4x - L^{-1}\left[\frac{1}{s}L\left[\sum_{n=0}^{\infty} A_n(u)\right]\right], \tag{32}$$

$A_n(u)$ are Adomian polynomials that represent the nonlinear terms suggested in (21). Matching both sides of (32), the components of u can be obtained as follows:

$$u_0(x,t) = 4x, \tag{33}$$

$$\begin{aligned}
u_1(x,t) &= -L^{-1}\left[\frac{1}{s}L[A_0(u)]\right],\\
u_1(x,t) &= -L^{-1}\left[\frac{1}{s}L[u_0u_{0x}]\right],\\
u_1(x,t) &= -16xt,
\end{aligned} \tag{34}$$

$$\begin{aligned}
u_2(x,t) &= -L^{-1}\left[\frac{1}{s}L[A_1(u)]\right],\\
u_2(x,t) &= -L^{-1}\left[\frac{1}{s}L[u_0u_{1x}+u_1u_{0x}]\right],\\
u_2(x,t) &= 64xt\left(x+\frac{t}{2}\right).
\end{aligned} \tag{35}$$

Therefore a series solution is obtained, which reads

$$u(x,t) = 4x - 16xt + 64xt\left(x+\frac{t}{2}\right) - \ldots. \tag{36}$$

The series provide a closedform solution, that is

$$u(x,t) = \begin{cases} 4x & \text{for } t = 0,\\ \frac{1}{8t}\left(\sqrt{1+64xt}-1\right), & \text{for } t > 0. \end{cases} \tag{37}$$

4. The Non-Homogeneous Advection Problem

We now consider the non-homogeneous advection problem [18]:

$$u_t + uu_x = 2t + x + t^3 + xt^2, \quad u(x,0) = 0. \tag{38}$$

Applying the Laplace transform gives

$$u(x,s) = \frac{2!}{s^3} + \frac{x}{s^2} + \frac{3!}{s^5} + x\frac{2!}{s^4} - \frac{1}{s}L[uu_x], \tag{39}$$

which, after taking the inverse Laplace transform, gives

$$u(x,t) = t^2 + xt + \frac{t^4}{4} + x\frac{t^3}{3} - L^{-1}\left[\frac{1}{s}L[uu_x]\right]. \quad (40)$$

Assuming a series form of the solution function

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t), \quad (41)$$

we substitute (41) into (40) to get

$$\sum_{n=0}^{\infty} u_n(x,t) = t^2 + xt + \frac{t^4}{4} + x\frac{t^3}{3} - L^{-1}\left[\frac{1}{s}L\left[\sum_{n=0}^{\infty} A_n(u)\right]\right], \quad (42)$$

where $A_n(u)$ are Adomian polynomials that represent the nonlinear terms and are given by

$$\sum_{n=0}^{\infty} A_n(u) = uu_x. \quad (43)$$

From (42), the recursive relation is given by

$$u_0(x,t) = t^2 + xt + \frac{t^4}{4} + x\frac{t^3}{3}, \quad (44)$$

$$u_{n+1}(x,t) = -L^{-1}\left[\frac{1}{s}L\left[\sum_{n=0}^{\infty} A_n(u)\right]\right], \quad n \geq 0, \quad (45)$$

so that for $n = 0$, we have

$$\begin{aligned} u_1(x,t) &= -L^{-1}\left[\frac{1}{s}L[A_0(u)]\right], \\ u_1(x,t) &= -L^{-1}\left[\frac{1}{s}L[u_0u_{0x}]\right], \\ u_1(x,t) &= -\frac{1}{4}t^4 - \frac{1}{3}xt^3 - \frac{2}{15}xt^5 - \frac{7}{72}t^6 \\ &\quad - \frac{1}{63}xt^7 - \frac{1}{98}t^8, \end{aligned} \quad (46)$$

and for $n = 1$, we have

$$\begin{aligned} u_2(x,t) &= -L^{-1}\left[\frac{1}{s}L[A_1(u)]\right], \\ u_2(x,t) &= -L^{-1}\left[\frac{1}{s}L[u_0u_{1x} + u_1u_{0x}]\right], \\ u_2(x,t) &= \frac{5}{8064}t^{12} + \frac{2}{2079}xt^{11} + \frac{2783}{302400}t^{10} \\ &\quad + \frac{38}{2835}xt^9 + \frac{143}{2880}t^8 + \frac{22}{315}xt^7 \\ &\quad + \frac{7}{12}t^6 + \frac{2}{15}xt^5, \\ &\vdots \end{aligned} \quad (47)$$

It is important to recall here that the noise terms appear between the components $u_0(x,t)$ and $u_1(x,t)$, where the noise terms are those pairs of terms that are identical but carrying opposite signs. More precisely, the noise terms $\pm\frac{1}{4}t^4 \pm \frac{1}{3}xt^3$ between the components $u_0(x,t)$ and $u_1(x,t)$ can be cancelled and the remaining terms of $u_0(x,t)$ still satisfy the equation. The exact solution is therefore

$$u(x,t) = t^2 + xt. \quad (48)$$

5. Discussion and Conclusions

This paper uses Adomian polynomials to decompose the nonlinear terms in equations. Though the solution procedure is simple, but the calculation of Adomian polynomials is complex. To overcome the shortcoming, we will suggest an alternative approach using He's polynomials [19, 20].

In this paper, a series solution for the Advection equation is derived by using the Laplace decomposition method (LDM). Such an analysis does not exist in the literature and the results obtained are new. These results are in good agreement with those given in [15] as well as those obtained by the variational iteration method and the Adomian decomposition method. This analysis therefore provides further support for the validity of LDM.

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