Heat Transfer for Flow of a Third-Grade Fluid between Two Porous Plates

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Z. Naturforsch. **65a,** 837 – 843 (2010); received January 5, 2009 / revised December 10, 2009

This article concentrates on the analytic solution for the heat transfer analysis of a third-grade fluid between two porous plates. The nonlinear problem for velocity profile is solved by employing the homotopy analysis method (HAM). Using the velocity profile, the energy equation with dissipation effects is solved for the series solution. The present solution demonstrates the dependency of the viscoelastic parameters. The obtained results are also sketched and discussed.

Key words: Series Solutions; Third-Grade Fluid; Homotopy Analysis Method.

1. Introduction

The non-Newtonian fluids have attracted considerable attention because these are encountered in industry and technology. The flows of such fluids provide the benchmark problems in computational fluid mechanics and heat transfer. The resulting systems define the ideal problems for testing different numerical methods as well as the validity of constitutive equations used to characterize the rheological properties of non-Newtonian fluids. As a result a wealth of literature exists on such flow covering a wide range of fluids and governing parameters. Recently, there has been an increasing interest in the study of flow and heat transfer of the differential-type fluids (a category of non-Newtonian fluids). The flows and heat transfer of such fluids have wide applications in heat exchangers, the screw extrusion process, electronics cooling, and many others. Extensive studies describing the flows of second-grade fluids (the simplest subclass of differential-type fluids) have been undertaken in various flow geometries and under several assumptions. Few fundamental analytical studies on the topic may be mentioned by the investigations [1 – 15].

In case of second-grade fluids, the normal stress effects can be predicted only in steady flow whereas the shear-thinning/shear-thickening properties cannot be taken into account. The third-grade fluid model [16 – 20] can explain such properties. In view of this fact the model in the present investigation is third-grade. Besides this it is an established fact that the governing equations of non-Newtonian fluids in general are of higher order than the non-linear Navier-Stokes equations. Therefore the extra initial/boundary conditions are necessary to determine a unique solution [21 – 23].

The present work concentrates on the heat transfer effects of a third-grade fluid bounded between two plates. In other words, the aim here is to extend the flow analysis of our very recent study [20] for the heat transfer analysis. The modelling is based to assess the role of viscous dissipation in the thermal development of the flow field. The main intent here is to construct the series solution of the temperature profile by using the homotopy analysis method (HAM) [24 – 34]. The convergence of the obtained solution is discussed and the effects of several interesting parameters entering into the problem is studied.

2. Problem Statement

Let us discuss the flow of a thermodynamic third-grade fluid filling the space between two plates distant b apart. The plates are porous and there is cross flow of the fluid with uniform velocity v_0. Here $v_0 < 0$ corresponds to the suction velocity and $v_0 > 0$ holds for injection or blowing velocity. We select the Cartesian coordinate system in such a manner that x and y-axes are parallel and perpendicular to the plates. The expressions of the Cauchy stress tensor in a thermodynamic third-grade fluid is [16 – 20]

$$\mathbf{T} = -p^*\mathbf{I} + \mu\mathbf{A}_1 + \alpha_1\mathbf{A}_2 + \alpha_2\mathbf{A}_1^2 + \beta_3\left(\mathrm{tr}\mathbf{A}_1^2\right)\mathbf{A}_1, \quad (1)$$

in which the first two Rivlin-Erickson tensors are given by

$$\mathbf{A}_1 = \nabla\mathbf{V} + (\nabla\mathbf{V})^t, \tag{2}$$

$$\mathbf{A}_2 = \frac{\mathrm{d}\mathbf{A}_1}{\mathrm{d}t} + \mathbf{A}_1(\nabla\mathbf{V}) + (\nabla\mathbf{V})^t\mathbf{A}_1, \tag{3}$$

where $\mathbf{V}$ and $\mathrm{d}/\mathrm{d}t$ are the velocity and material derivative, respectively, and μ, α_1, α_2, and β_3 are the material constants [16].

The velocity field for the problem under consideration is

$$\mathbf{V} = [u(y), v_0, 0]. \tag{4}$$

Upon making use of (1) – (4) into continuity, momentum, and energy equations, one can write

$$\frac{\partial v}{\partial y} = 0, \tag{5}$$

$$\rho v_0 \frac{\mathrm{d}u}{\mathrm{d}y} = \mu \frac{\mathrm{d}^2u}{\mathrm{d}y^2} + \alpha_1 v_0 \frac{\mathrm{d}^3u}{\mathrm{d}y^3} + 6\beta_3 \frac{\mathrm{d}^2u}{\mathrm{d}y^2}\left(\frac{\mathrm{d}u}{\mathrm{d}y}\right)^2 - \frac{\mathrm{d}\widehat{p}}{\mathrm{d}x}, \tag{6}$$

$$\rho c_p v_0 \frac{\mathrm{d}T^*}{\mathrm{d}y} = \mu\left(\frac{\mathrm{d}u}{\mathrm{d}y}\right)^2 + \alpha_1 v_0 \frac{\mathrm{d}^2u}{\mathrm{d}y^2}\frac{\mathrm{d}u}{\mathrm{d}y} + 2\beta_3\left(\frac{\mathrm{d}u}{\mathrm{d}y}\right)^4 + \overline{K}\frac{\mathrm{d}^2T^*}{\mathrm{d}y^2}. \tag{7}$$

The appropriate boundary conditions for u and T^* are

$$u(0) = 0, \quad u(b) = 0, \tag{8}$$

$$T^*(0) = T_\mathrm{l}, \quad T^*(b) = T_\mathrm{u}. \tag{9}$$

In above equations $\widehat{p}$ is the modified pressure, T^* is the temperature, T_l and T_u are the respective temperatures at the lower and upper plates, $\overline{K}$ is the thermal conductivity of fluid, and c_p is the specific heat. Note that the radiation effects are excluded in (7).

The problems in dimensionless variables become

$$KRU''' + U'' - RU' + TU'^2U'' = -1, \tag{10}$$

$$U(0) = U(1) = 0, \tag{11}$$

$$\theta'' - PR\theta' + PE\left[U'^2 + KRU''U' + \frac{T}{3}U'^4\right] = 0, \tag{12}$$

$$\theta(0) = 0, \quad \theta(1) = 1. \tag{13}$$

In above equations

$$\eta = \frac{y}{b},\ U(\eta) = -\frac{\mu u(y)}{b^2}\left(\frac{\mathrm{d}\widehat{p}}{\mathrm{d}x}\right)^{-1},$$

$$R = \frac{\rho v_0 b}{\mu},\ K = \frac{\alpha_1}{\rho b^2},\ T = \frac{6\beta_3 b^2(\mathrm{d}\widehat{p}/\mathrm{d}x)^2}{\mu^3},$$

$$\theta(\eta) = \frac{T^* - T_\mathrm{l}}{T_\mathrm{u} - T_\mathrm{l}},\ P = \frac{\mu c_\mathrm{p}}{\overline{K}},\ E = \frac{b^4(\mathrm{d}\widehat{p}/\mathrm{d}x)^2}{(T_\mathrm{u} - T_\mathrm{l})\mu^2 c_\mathrm{p}}.$$

Here P and E are the Prandtl and Eckert numbers, respectively.

Writing

$$U(\eta) = \frac{\eta}{R} + f(\eta), \tag{14}$$

the resulting problems are given by

$$KRf''' + f'' - Rf' + T\left[\frac{f''}{R^2} + f'^2 f'' + \frac{2}{R}f'f''\right] = 0, \tag{15}$$

$$f(0) = 0,\ f(1) = -\frac{1}{R}, \tag{16}$$

$$\theta'' - PR\theta' + PE\left[\left(\frac{1}{R^2} + f'^2 + \frac{2}{R}f'\right) + KR\left(\frac{1}{R}f'' + f'f''\right) + \frac{T}{3}\left(f'^4 + \frac{4}{R}f'^3 + \frac{4}{R^3}f' + \frac{1}{R^4}\right)\right] = 0, \tag{17}$$

$$\theta(0) = 0,\ \theta(1) = 1. \tag{18}$$

3. Solution for $f(\eta)$ by Homotopy Analysis Method

In view of (15) and (16) we express

$$f(\eta) = \sum_{k=0}^{\infty}\sum_{m=0}^{\infty} a_{n,m}^k \eta^k \mathrm{e}^{m\eta}, \tag{19}$$

where $a_{n,m}^k$ are the coefficients to be determined. By rule of solution expression (19) and the boundary conditions (16), we select the following initial guess f_0 and auxiliary linear operator $\mathcal{L}_f$

$$f_0(\eta) = -\frac{1 - \mathrm{e}^\eta}{R(1 - \mathrm{e})}, \tag{20}$$

$$\mathcal{L}_f\left(\widehat{f}\right) = \left(\frac{\partial^2}{\partial\eta^2} - \frac{\partial}{\partial\eta}\right)\widehat{f}. \tag{21}$$

The above operator satisfies

$$\mathcal{L}_f\left(C_1+C_2\mathrm{e}^{\eta}\right)=0, \tag{22}$$

where C_1 and C_2 are arbitrary constants. Now we define a nonlinear operator in the form

$$\begin{aligned} N_f\Big[\widehat{f}(\eta;p)\Big] &= KR\frac{\partial^3\widehat{f}(\eta;p)}{\partial\eta^3}+\frac{\partial^2\widehat{f}(\eta;p)}{\partial\eta^2}-R\frac{\partial\widehat{f}(\eta;p)}{\partial\eta} \\ &+T\left[\frac{1}{R^2}\frac{\partial^2\widehat{f}(\eta;p)}{\partial\eta^2}+\frac{\partial^2\widehat{f}(\eta;p)}{\partial\eta^2}\left(\frac{\partial\widehat{f}(\eta;p)}{\partial\eta}\right)^2\right. \\ &\left.+\frac{2}{R}\frac{\partial^2\widehat{f}(\eta;p)}{\partial\eta^2}\frac{\partial\widehat{f}(\eta;p)}{\partial\eta}\right]. \end{aligned} \tag{23}$$

The problem for f at the zeroth-order deformation is

$$(1-p)\mathcal{L}_f\Big[\widehat{f}(\eta;p)-f_0(\eta)\Big]=p\hbar_f N_f\Big[\widehat{f}(\eta;p)\Big], \tag{24}$$

$$\widehat{f}(0;p)=0,\quad \widehat{f}(1;p)=-\frac{1}{R}, \tag{25}$$

in which $\hbar_f$ is a non-zero auxiliary parameter and $p\in[0,1]$ is an embedding parameter. When $p=0$ and $p=1$, we have

$$\widehat{f}(\eta;0)=f_0(\eta),\quad \widehat{f}(\eta;1)=f(\eta). \tag{26}$$

When p increases from 0 to 1, the solution $\widehat{f}(\eta;p)$ varies from $f_0(\eta)$ to $f(\eta)$. If this continuous variation is smooth enough, the Maclaurin series with respect to p can be constructed for $\widehat{f}(\eta;p)$, and further, if the series is convergent at $p=1$, then we have

$$f(\eta)=f_0(\eta)+\sum_{n=1}^{\infty}f_n(\eta),$$

$$f_n(\eta)=\frac{1}{n!}\frac{\partial^n\widehat{f}(\eta;p)}{\partial p^n}\bigg|_{p=0}.$$

The problems at the nth-order deformation are

$$\begin{aligned} &\mathcal{L}_f\left[f_n(\eta)-\chi_n f_{n-1}(\eta)\right]=\hbar_f R_f^n(\eta), \\ &(n=1,2,3,\ldots), \end{aligned} \tag{27}$$

$$f_n(0)=0,\quad f_n(1)=0, \tag{28}$$

where

$$\chi_n=\begin{cases}0, & n\le 1,\\ 1, & n>1,\end{cases}$$

$$\begin{aligned} R_f^n(\eta) &= KRf'''_{n-1}+f''_{n-1}-Rf'_{n-1}+T\left[\frac{1}{R^2}f''_{n-1}\right. \\ &\left.+\sum_{i=0}^{n-1}\left(f'_{n-i-1}\sum_{j=0}^{i}f'_j f''_{i-j}\right)+\frac{2}{R}\sum_{i=0}^{n-1}f'_i f''_{n-i-1}\right]. \end{aligned} \tag{29}$$

(27) is solved up to 10th-order of approximations with the help of the software Mathematica. The solution obtained for $f(\eta)$ is of the form

$$\begin{aligned} f(\eta) &= \sum_{n=0}^{\infty}f_n(\eta) \\ &= \lim_{N\to\infty}\left[a_{0,0}^0+\sum_{m=1}^{2N+1}\mathrm{e}^{m\eta}\left(\sum_{n=m-1}^{2N}\sum_{k=0}^{2n+1-m}a_{n,m}^k\eta^k\right)\right], \end{aligned} \tag{30}$$

where $a_{0,0}^0=-\mathrm{e}/R(\mathrm{e}-1)$ and $a_{0,1}^0=1/R(\mathrm{e}-1)$.

4. Solution for Temperature $\theta(\eta)$ by Homotopy Analysis Method

In view of (17) and (18), we assume

$$\theta(\eta)=\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}b_{n,m}^k\eta^k\mathrm{e}^{m\eta}, \tag{31}$$

where $b_{n,m}^k$ are the constants to be determined. For the series solution of $\theta(\eta)$, we choose the initial guess

$$\theta_0(\eta)=\frac{1-\mathrm{e}^{PR\eta}}{1-\mathrm{e}^{PR}}, \tag{32}$$

and an auxiliary linear operator

$$\mathcal{L}_\theta(\theta)=\left(\frac{\partial^2}{\partial\eta^2}-PR\frac{\partial}{\partial\eta}\right)\theta. \tag{33}$$

Note that

$$\mathcal{L}_\theta(D_1+D_2\mathrm{e}^{PR\eta})=0, \tag{34}$$

where D_1 and D_2 are arbitrary constants. The zeroth-order deformation problem is given by

$$\begin{aligned} (1-p)\mathcal{L}_\theta&\left[\widehat{\theta}(\eta;p)-\theta_0(\eta)\right] \\ &=p\hbar_\theta N_\theta\left[\widehat{\theta}(\eta;p),\widehat{f}(\eta;p)\right], \end{aligned} \tag{35}$$

$$\widehat{\theta}(0;p)=0,\quad \widehat{\theta}(1;p)=1. \tag{36}$$

Here $\hbar_\theta$ indicates an auxiliary non-zero parameter and the nonlinear differential operator N_θ is

$$\begin{aligned} N_\theta\Big[\widehat{\theta}(\eta;p),\widehat{f}(\eta;p)\Big] &= \frac{\partial^2\widehat{\theta}(\eta;p)}{\partial\eta^2}-PR\frac{\partial\widehat{\theta}(\eta;p)}{\partial\eta} \\ &+PE\left[\left(\widehat{f}'^2(\eta;p)+\frac{2}{R}\widehat{f}'(\eta;p)+\frac{1}{R^2}\right)\right. \\ &+KR\left(\frac{1}{R}\widehat{f}''(\eta;p)+\widehat{f}''(\eta;p)\widehat{f}'(\eta;p)\right)+\frac{T}{3}\left(\widehat{f}'^4(\eta;p)\right. \\ &\left.\left.+\frac{4}{R}\widehat{f}'^3(\eta;p)+\frac{1}{R^4}+\frac{6}{R^2}\widehat{f}'^2(\eta;p)+\frac{4}{R^3}\widehat{f}'(\eta;p)\right)\right]. \end{aligned} \tag{37}$$

When $p = 0$ and $p = 1$, we may write

$$\widehat{\theta}(\eta;0) = \theta_0(\eta), \quad \widehat{\theta}(\eta;1) = \theta(\eta). \tag{38}$$

Obviously, when p increases from 0 to 1, $\widehat{\theta}(\eta;p)$ varies from $\theta_0(\eta)$ to $\theta(\eta)$. By Maclaurin's series and (38) we can express that

$$\widehat{\theta}(\eta;p) = \theta_0(\eta) + \sum_{n=1}^{\infty} \theta_n(\eta),$$
$$\theta_n(\eta) = \frac{1}{n!} \frac{\partial^n \widehat{\theta}(\eta;p)}{\partial p^n}\bigg|_{p=0}.$$

The nth-order deformation problems are expressed by the following equations:

$$\mathcal{L}_\theta\left[\theta_n(\eta) - \chi_n \theta_{n-1}(\eta)\right] = \hbar_\theta R_\theta^n(\eta), \tag{39}$$

$$\theta_n(0) = 0, \quad \theta_n(1) = 0, \tag{40}$$

$$\begin{aligned} R_\theta^n(\eta) = {} & \theta''_{n-1} - PR\theta'_{n-1} + PE\Bigg\{\sum_{i=0}^{n-1} f'_i f'_{n-i-1} \\ & + \frac{2}{R} f'_{n-1} + \frac{1}{R^2}(1-\chi_n) + KR\left(\frac{1}{R} f''_{n-1} + \sum_{i=0}^{n-1} f'_i f''_{n-i-1}\right) \\ & + \frac{T}{3}\Bigg(\sum_{i=0}^{n-1}\Bigg[f'_{n-i-1} \sum_{j=0}^{i}\left(f'_{i-j} \sum_{r=0}^{j} f'_r f'_{j-r}\right)\Bigg) \\ & + \frac{4}{R^3} f'_{n-1} + \frac{4}{R}\sum_{i=0}^{n-1}\left(f'_{n-i-1} \sum_{j=0}^{i} f'_j f'_{i-j}\right) \\ & + \frac{6}{R^2}\sum_{i=0}^{n-1} f'_i f'_{n-i-1} + \frac{1}{R^4}(1-\chi_n)\Bigg]\Bigg\}. \end{aligned} \tag{41}$$

The analytic solution of above problem is

$$\begin{aligned} \theta(\eta) &= \sum_{n=0}^{\infty} \theta_n(\eta) \\ &= \lim_{N\to\infty}\left[\sum_{m=1}^{2N+2} \mathrm{e}^{m\eta}\left(\sum_{n=m-1}^{2N+1}\ \sum_{k=0}^{2n+2-m} b_{n,m}^k \eta^k\right)\right], \end{aligned} \tag{42}$$

where $b_{0,0}^0 = 1/(1-\mathrm{e})$, $b_{0,1}^0 = -1/(1-\mathrm{e})$.

5. Convergence of the HAM Solution

Note that (30) and (42) contain two auxiliary parameters $\hbar_f$ and $\hbar_\theta$. The convergence region and rate of approximation of the homotopy analysis method strongly depend upon these auxiliary parameters. In view of this fact the $\hbar$-curves have been plotted. In Figure 1, the range for the admissible values of $\hbar_f$ is $-1.22 \le \hbar_f \le -0.23$, and in Figure 2, the suitable range for $\hbar_\theta$ is $-1.4 \le \hbar_\theta \le -0.6$. Figures 3 and 4 describe the influence of physical parameters P and E on the $\hbar_\theta$-curves. In both figures the effects of P and E on the variations of $\hbar_\theta$ are quite opposite. It is also observed that the series for $f(\eta)$ converges faster than that of the series for $\theta(\eta)$.

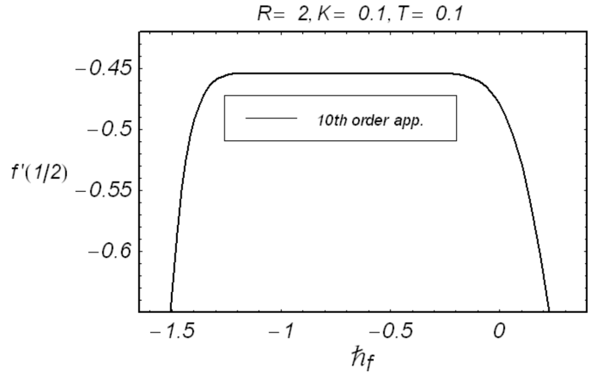

Fig. 1. $\hbar_f$-curve for the 10th-order of approximation for the dimensionless velocity profile $f(\eta)$.

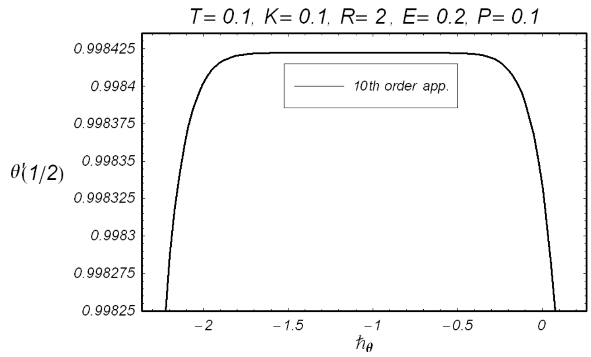

Fig. 2. $\hbar_\theta$-curve for the 10th-order of approximation for the dimensionless temperature profile $\theta(\eta)$.

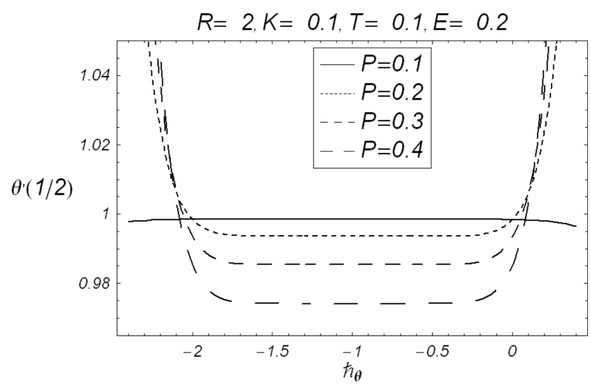

Fig. 3. Variation in $\hbar_\theta$-curve with an increase in P.

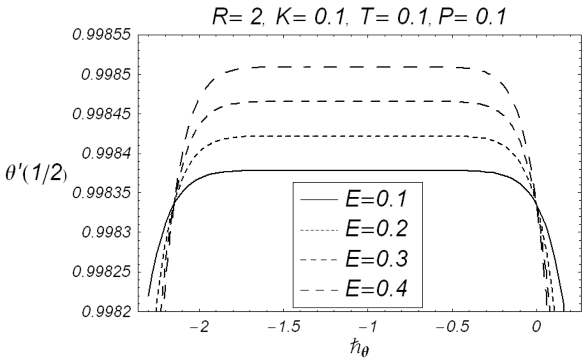

Fig. 4. Variation in $\hbar_\theta$-curve with an increase in E.

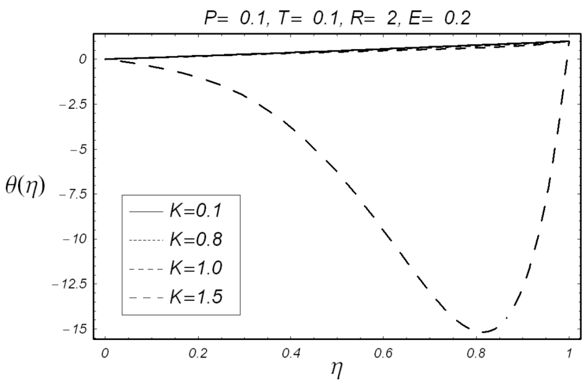

Fig. 5. Variation of the dimensionless temperature profile $\theta(\eta)$ with increasing parameter K.

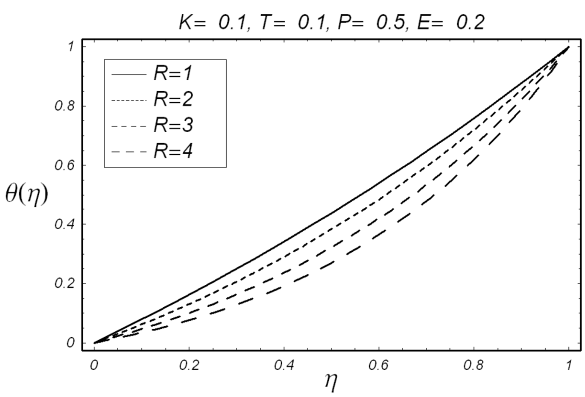

Fig. 6. Variation of the dimensionless temperature profile $\theta(\eta)$ with increasing parameter R.

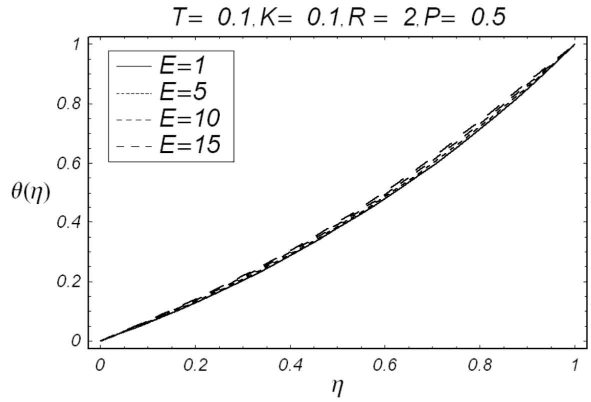

Fig. 7. Variation of the dimensionless temperature profile $\theta(\eta)$ with increasing parameter E.

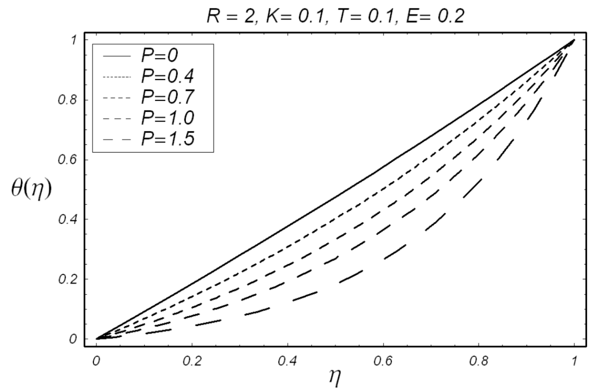

Fig. 8. Variation of the dimensionless temperature profile $\theta(\eta)$ with increasing parameter P.

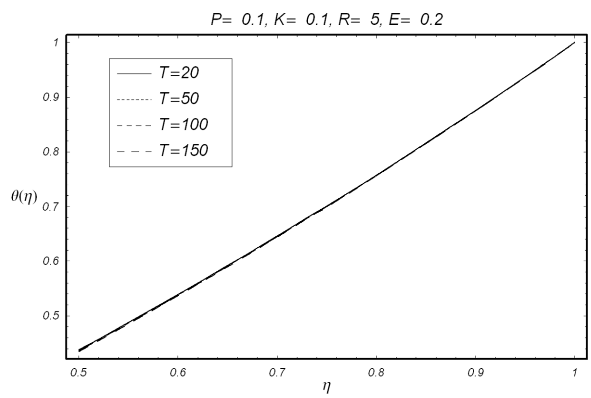

Fig. 9. Variation of the dimensionless temperature profile $\theta(\eta)$ with increasing parameter T.

6. Results and Discussion

The aim of this section is to discuss the variations of viscoelastic parameter K, third-grade parameter T, Reynolds number R, Prandtl number P, and Eckert number E on θ in the middle of the channel. In view of this fact in mind, we draw Figures 5 – 9 and present Table 1.

The effect of K on the temperature θ is displayed in Figure 5. It has been observed that an increase in the

Table 1. Values of $\theta(\eta)$ in the middle of the channel.

Reynolds No. (R)	Viscoelastic parameter (K)	Third-grade parameter (T)	Prandtl No. (P)	Eckert No. (E)	$\theta(\eta)$, at $\eta = 1/2$
1	0.1	0.1	0.1	0.2	0.487591
2					0.475086
3					0.462618
4					0.450202
5					0.437852
2	0.1				0.475086
	0.2				0.475073
	0.3				0.475065
	0.4				0.475060
	0.5				0.475056
	0.1	0			0.475086
		1.0			0.475085
		5.0			0.475083
		10			0.475081
		20			0.475079
		0.1	0		0.500000
			0.1		0.475086
			0.2		0.450293
			0.3		0.425744
			0.4		0.401554
			0.5		0.377835
			1.0		0.269446
			0.1	0	0.475021
				0.1	0.475053
				0.2	0.475086
				0.3	0.475118
				0.4	0.475151
				0.5	0.475183
				1.0	0.475346

value of K decreases the value of θ when $K < 1$ and $K > 1$. However, it is noticed that θ decreases much for $K > 1$ when compared with $K < 1$. Such features are encountered due to viscoelastic properties of the fluid in terms of normal stress.

Figure 6 depicts that the value of θ decreases with an increase of R. This results in terms of small values of viscous effects. The variation of θ with the Eckert number E is presented in Figure 7. It is noticed that the value of θ increases when E is increased. Figure 8 describes the effect of the Prandtl number P on θ for $P < 1$ and $P > 1$. Interestingly, θ is a decreasing function of P. This corresponds to the situation that the order of thermal conductivity is small when compared with that of viscosity and specific heat. Furthermore, θ in case of $P < 1$ is greater than for $P > 1$. It is found that the increasing of P reduces the value of θ. The variation of T on θ is given in Figure 9. The examination of this figure shows that the value of θ decreases with the increase in value of T. This decrease is, however, very small.

Conclusions

This study investigates the heat transfer effects on the flow of a third-grade fluid. Series solutions of velocity and temperature are constructed. The tabular values (see Table 1) here indicated that

- θ is a decreasing function of K,
- the influence of R and T on θ are same,
- the series solutions corresponding to viscous fluids can be deduced by taking $T = K = 0$.

Acknowledgement

We are grateful to the referees for their valuable suggestions regarding an earlier version of this paper.

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