Application of the Homotopy Analysis Method for Systems of Differential-Difference Equations

Qi Wang

Department of Applied Mathematics, Shanghai University of Finance and Economics, Shanghai 200433, P. R. China

Reprint requests to Q. W.; E-mail: wangqee@gmail.com

Z. Naturforsch. **65a,** 811 – 817 (2010); received August 13, 2009 / revised November 26, 2009

The homotopy analysis method is used for solving systems of differential-difference equations. To demonstrate the validity and applicability of the presented technique the Volterra lattice system is taken as example. Analysis results show that the method is very effective and yields very accurate results.

Key words: Homotopy Analysis Method; Relativistic Toda Lattice System.
PACS numbers: 02.30.Xx; 02.30.Mv; 02.60.Lj

1. Introduction

In recent years, considerable interest in differential-difference equations (DDEs) has been stimulated due to their numerous applications in the areas of physics and engineering. Many important phenomena and dynamical processes in many different fields, such as particle vibrations in lattices, currents in electrical networks, pulses in biological chains and so on, can be well described by DDEs. Unlike difference equations which are fully discretized, DDEs are semi-discretized with some (or all) of their spacial variables discretized while time is usually kept continuous. There is a vast body of work on DDEs [1 – 12].

For better understanding the meaning of partial differential equations (PDEs), it is of great significance to search for exact analytic solutions of them. In history, many powerful methods such as the inverse scattering method [13], the Lie group method [14], the Bäcklund transformation [15], the Darboux transformation [16], the Hirota method [17], etc, have been presented to construct solutions of PDEs. Among them, the homotopy analysis method (HAM), which was firstly proposed by Liao [18], based on the idea of homotopy in topology, is a general analytic method for nonlinear problems. Unlike the traditional methods (for example, perturbation techniques and so on), the HAM contains many auxiliary parameters which provide us with a simple way to adjust and control the convergence region and rate of convergence of the series solution and has been successfully employed to solve explicit analytic solutions for many types of nonlinear problems [11, 12, 19 – 25].

Motivated by above works, we would like to extend the applications of HAM to systems of differential-difference equations. For illustration, we apply it to the Volterra lattice system whose conservation laws have been studied by Wadati and Watanabe [26] and can be solved by the discrete tanh method [27]. The other properties of this lattice system are investigated in [28].

This paper is organized as follows. In Section 2, a brief outline of the generalized HAM for a system of DDEs with initial condition is presented. In Section 3, we apply the proposed method to the Volterra lattice system to verify the effectiveness of it and also give the proof of convergence theorem. In Section 4, a brief analysis of the obtained results is given. A short summary and discussion are presented in final Section 5.

2. HAM for a System of DDEs

For illustration, we consider the following system of DDEs:

$$N_i[u_{i,n}(t), u_{i,n-1}(t), u_{i,n+1}(t), \cdots] = 0, \quad (1)$$

where N_i are nonlinear differential operators that represent the whole equations, $n \in \mathbb{N}$ and t denote independent variables, and $u_{i,n}(t)$ are unknown functions, respectively. By means of HAM, we construct the so-called zero-order deformation equations

$$\begin{aligned}&(1-q)L_i[\phi_{i,n}(t;q) - u_{i,n,0}(t)] \\ &= qhN_i[\phi_{i,n}(t;q), \phi_{i,n-1}(t;q), \phi_{i,n+1}(t;q), \cdots],\end{aligned} \quad (2)$$

where $q \in [0,1]$ is an embedding parameter, h is a nonzero auxiliary parameter, L_i are auxiliary linear operators, $u_{i,n,0}(t)$ are initial guesses of $u_{i,n}(t)$, $\phi_{i,n}(t;q)$ are unknown functions on independent variables n, t, and q. It is important to note that one has great freedom to choose auxiliary parameters such as h in HAM. Obviously, when the embedding parameter q increases from 0 to 1, $\phi_{i,n}(t;q)$ vary (or deforms) continuously from the initial approximations $\phi_{i,n}(t;0) = u_{i,n,0}(t)$ to the exact solutions $\phi_{i,n}(t;1) = u_{i,n}(t)$ of the original system (1).

Defining the so-called mth-order deformation derivatives

$$u_{i,n,m}(t) = \frac{1}{m!} \left. \frac{\partial^m \phi_{i,n}(t;q)}{\partial q^m} \right|_{q=0} \quad (3)$$

and expanding $\phi_{i,n}(t;q)$ in Taylor series with respect to the embedding parameter q, we have

$$\phi_{i,n}(t;q) = u_{i,n,0}(t) + \sum_{m=1}^{+\infty} u_{i,n,m}(t) q^m. \quad (4)$$

Then, correspondingly, for $k \in \mathbb{N}$:

$$\phi_{i,n-k}(t;q) = u_{i,n-k,0}(t) + \sum_{m=1}^{+\infty} u_{i,n-k,m}(t) q^m, \quad (5)$$

$$\phi_{i,n+k}(t;q) = u_{i,n+k,0}(t) + \sum_{m=1}^{+\infty} u_{i,n+k,m}(t) q^m. \quad (6)$$

If the auxiliary linear operator, the initial guesses, and the auxiliary parameter h are properly chosen, then the series (4) converge at $q = 1$ and one has

$$u_{i,n}(t) = u_{i,n,0}(t) + \sum_{m=1}^{+\infty} u_{i,n,m}(t), \quad (7)$$

$$u_{i,n-k}(t) = u_{i,n-k,0}(t) + \sum_{m=1}^{+\infty} u_{i,n-k,m}(t), \quad (8)$$

$$u_{i,n+k}(t) = u_{i,n+k,0}(t) + \sum_{m=1}^{+\infty} u_{i,n+k,m}(t), \quad (9)$$

which must be one of the solutions of the original nonlinear equations, as proved by Liao [21]. As $h = -1$, (2) becomes

$$\begin{aligned}&(1-q)L_i[\phi_{i,n}(t;q) - u_{i,n,0}(t)] \\ &+ qN_i[\phi_{i,n}(t;q), \phi_{i,n-1}(t;q), \phi_{i,n+1}(t;q), \cdots] = 0,\end{aligned} \quad (10)$$

which is mostly used in the homotopy perturbation method [29].

For brevity, define the vectors

$$\boldsymbol{u}_{i,n,m}(t) = \{u_{i,n,0}(t), u_{i,n,1}(t), \cdots, u_{i,n,m}(t)\}, \quad (11)$$

$$\boldsymbol{u}_{i,n-k,m}(t) = \{u_{i,n-k,0}(t), u_{i,n-k,1}(t), \cdots, u_{i,n-k,m}(t)\}, \quad (12)$$

$$\boldsymbol{u}_{i,n+k,m}(t) = \{u_{i,n+k,0}(t), u_{i,n+k,1}(t), \cdots, u_{i,n+k,m}(t)\}. \quad (13)$$

Differentiating the zero-order deformation equations in (2) m times with respect to q and then dividing them by $m!$ and finally setting $q = 0$, we have the mth-order deformation equations

$$\begin{aligned}&L_i[u_{i,n,m}(t) - \chi_m u_{i,n,m-1}(t)] = \\ &hR_{i,m}[\boldsymbol{u}_{i,n,m-1}(t), \boldsymbol{u}_{i,n-1,m-1}(t), \boldsymbol{u}_{i,n+1,m-1}(t), \cdots],\end{aligned} \quad (14)$$

where

$$\begin{aligned}&R_{i,m}[\boldsymbol{u}_{i,n,m-1}(t), \boldsymbol{u}_{i,n-1,m-1}(t), \boldsymbol{u}_{i,n+1,m-1}(t), \cdots] \\ &= \frac{1}{(m-1)!} \left. \frac{\partial^{m-1} N_i[\phi_{i,n}(t;q), \cdots]}{\partial q^{m-1}} \right|_{q=0}\end{aligned} \quad (15)$$

and

$$\chi_m = \begin{cases} 0 & m = 1, \\ 1 & m > 1. \end{cases} \quad (16)$$

It should be emphasized that $u_{i,n,m}(t)$ $(m \geq 1)$ is governed by the linear equations (14) with the linear boundary conditions that come from the original problem. Thus we can gain $u_{i,n,1}(t)$, $u_{i,n,2}(t), \cdots$ by solving the linear high-order deformation equations (14) one after the other based on symbolic computation softwares such as Maple, Mathematica and so on.

3. Application to the Volterra Lattice System

In this section, to verify the validity and the effectiveness of HAM in solving system of DDEs, we apply it to the Volterra lattice system [27]:

$$\frac{\partial u_n}{\partial t} = u_n(v_n - v_{n-1}), \quad \frac{\partial v_n}{\partial t} = v_n(u_{n+1} - u_n), \quad (17)$$

subject to following initial conditions:

$$\begin{aligned} u_n(0) &= -c\coth(d) + c\tanh(dn+\delta), \\ v_n(0) &= -c\coth(d) - c\tanh(dn+\delta), \end{aligned} \tag{18}$$

whose exact solutions can be written as

$$\begin{aligned} u_n(t) &= -c\coth(d) + c\tanh(dn+ct+\delta), \\ v_n(t) &= -c\coth(d) - c\tanh(dn+ct+\delta). \end{aligned} \tag{19}$$

Here, $u_n(t)$ and $v_n(t)$ are functions of continuous time variable t and discrete variable n.

To solve system (17)–(18) by means of HAM, we choose the initial approximations

$$\begin{aligned} u_{n,0}(t) &= -c\coth(d) + c\tanh(dn+\delta), \\ v_{n,0}(t) &= -c\coth(d) - c\tanh(dn+\delta), \end{aligned} \tag{20}$$

and the auxiliary linear operator

$$L[\phi_{i,n}(t,q)] = \frac{\partial \phi_{i,n}(t,q)}{\partial t}, \quad i = 1,2, \tag{21}$$

with the property

$$L[c_i] = 0, \tag{22}$$

where c_i, $i = 1,2$, are integral constants. Furthermore, system (17) suggest that we define a system of nonlinear operators as

$$\begin{aligned} &N_1[\phi_{i,n}(t;q), \phi_{i,n-1}(t;q), \phi_{i,n+1}(t;q), \cdots] = \\ &\frac{\partial \phi_{1,n}(t;q)}{\partial t} - \phi_{1,n}(t;q)(\phi_{2,n}(t;q) - \phi_{2,n-1}(t;q)), \\ &N_2[\phi_{i,n}(t;q), \phi_{i,n-1}(t;q), \phi_{i,n+1}(t;q), \cdots] = \\ &\frac{\partial \phi_{2,n}(t;q)}{\partial t} - \phi_{2,n}(t;q)(\phi_{1,n+1}(t;q) - \phi_{1,n}(t;q)). \end{aligned} \tag{23}$$

Using above definitions, we construct the zeroth-order deformation equations

$$\begin{aligned} &(1-q)L[\phi_{1,n}(t;q) - u_{n,0}(t)] = \\ &qhN_1[\phi_{i,n}(t;q), \phi_{i,n-1}(t;q), \phi_{i,n+1}(t;q), \cdots], \\ &(1-q)L[\phi_{2,n}(t;q) - v_{n,0}(t)] = \\ &qhN_2[\phi_{i,n}(t;q), \phi_{i,n-1}(t;q), \phi_{i,n+1}(t;q), \cdots] \end{aligned} \tag{24}$$

with the initial conditions

$$\phi_{1,n}(0;q) = u_{n,0}(0), \quad \phi_{2,n}(0;q) = v_{n,0}(0), \tag{25}$$

where $q \in [0,1]$ denotes an embedding parameter and $h \neq 0$ is an auxiliary parameter. Obviously, when $q = 0$ and $q = 1$,

$$\begin{aligned} \phi_{1,n}(t;0) &= u_{n,0}(t), \quad \phi_{1,n}(t;1) = u_n(t), \\ \phi_{2,n}(t;0) &= v_{n,0}(t), \quad \phi_{2,n}(t;1) = v_n(t). \end{aligned} \tag{26}$$

Therefore, as the embedding parameter q increases continuously from 0 to 1, $\phi_{i,n}(t;q)$ vary from the initial guesses $u_{n,0}(t)$ and $v_{n,0}(t)$ to the solutions $u_n(t)$ and $v_n(t)$. Expanding $\phi_{i,n}(t;q)$ in Taylor series with respect to q one has

$$\begin{aligned} \phi_{1,n}(t;q) &= u_{n,0}(t) + \sum_{m=1}^{+\infty} u_{n,m}(t) q^m, \\ \phi_{2,n}(t;q) &= v_{n,0}(t) + \sum_{m=1}^{+\infty} v_{n,m}(t) q^m, \end{aligned} \tag{27}$$

where

$$\begin{aligned} u_{n,m}(t) &= \frac{1}{m!} \left. \frac{\partial^m \phi_{1,n}(t;q)}{\partial q^m} \right|_{q=0}, \\ v_{n,m}(t) &= \frac{1}{m!} \left. \frac{\partial^m \phi_{2,n}(t;q)}{\partial q^m} \right|_{q=0}. \end{aligned} \tag{28}$$

If the auxiliary parameter h is properly chosen, above series (27) are convergent at $q = 1$. Then one has

$$u_n(t) = \sum_{m=0}^{+\infty} u_{n,m}(t), \quad v_n(t) = \sum_{m=0}^{+\infty} v_{n,m}(t), \tag{29}$$

and we will prove at the end of this section that they must be solutions of the original system.

Now, we define the vectors

$$\begin{aligned} \boldsymbol{u}_{n,m}(t) &= \{u_{n,0}(t), u_{n,1}(t), \cdots, u_{n,m}(t)\}, \\ \boldsymbol{v}_{n,m}(t) &= \{v_{n,0}(t), v_{n,1}(t), \cdots, v_{n,m}(t)\}. \end{aligned} \tag{30}$$

So the mth-order deformation equations are

$$\begin{aligned} &L[u_{n,m}(t) - \chi_m u_{n,m-1}(t)] = \\ &\quad hR_1[\boldsymbol{u}_{n,m-1}(t), \boldsymbol{v}_{n,m-1}(t), \boldsymbol{u}_{n-1,m-1}(t), \\ &\quad \boldsymbol{v}_{n-1,m-1}(t), \boldsymbol{u}_{n+1,m-1}(t), \boldsymbol{v}_{n+1,m-1}(t), \cdots], \\ &L[v_{n,m}(t) - \chi_m v_{n,m-1}(t)] = \\ &\quad hR_2[\boldsymbol{u}_{n,m-1}(t), \boldsymbol{v}_{n,m-1}(t), \boldsymbol{u}_{n-1,m-1}(t), \\ &\quad \boldsymbol{v}_{n-1,m-1}(t), \boldsymbol{u}_{n+1,m-1}(t), \boldsymbol{v}_{n+1,m-1}(t), \cdots] \end{aligned} \tag{31}$$

with the initial conditions

$$u_{n,m}(0) = 0, \quad v_{n,m}(0) = 0, \quad m \geq 1, \tag{32}$$

where

$$\begin{aligned}&R_1[\boldsymbol{u}_{n,m-1}(t),\boldsymbol{v}_{n,m-1}(t),\boldsymbol{u}_{n-1,m-1}(t),\boldsymbol{v}_{n-1,m-1}(t),\\&\boldsymbol{u}_{n+1,m-1}(t),\boldsymbol{v}_{n+1,m-1}(t),\cdots]=\\&\frac{\partial u_{n,m-1}}{\partial t}-\sum_{j=0}^{m-1}\left(\sum_{i=0}^{j}u_{n,i}!\right)(v_{n,m-1-j}-v_{n-1,m-1-j}),\\&R_2[\boldsymbol{u}_{n,m-1}(t),\boldsymbol{v}_{n,m-1}(t),\boldsymbol{u}_{n-1,m-1}(t),\boldsymbol{v}_{n-1,m-1}(t),\\&\boldsymbol{u}_{n+1,m-1}(t),\boldsymbol{v}_{n+1,m-1}(t),\cdots]=\\&\frac{\partial v_{n,m-1}}{\partial t}-\sum_{j=0}^{m-1}\left(\sum_{i=0}^{j}v_{n,i}\right)(u_{n+1,m-1-j}-u_{n,m-1-j}),\end{aligned}\tag{33}$$

and χ_m satisfying (16).

It should be emphasized that $u_{n,m}(t)$ and $v_{n,m}(t)$ ($m \geq 1$) are governed by the linear equations (32) with the linear initial conditions (33). Thus we can get all $u_{n,m}(t)$ and $v_{n,m}(t)$ ($m \geq 1$) easily and then according to (29), we can get the solutions of system (17) and (18).

Then, HAM for the system of DDEs provides us with a family of solution expression in the auxiliary parameter h. The convergence region of the solution series depend upon the value of h. Next we will illustrate the convergence theorem and prove it.

Theorem 3.1 *Convergence Theorem*

The series (30) are exact solutions of system (17) and (18) as long as they are convergent.

Proof. Since $u_n(t) = \sum_{m=0}^{+\infty} u_{n,m}(t)$ and $v_n(t) = \sum_{m=0}^{+\infty} v_{n,m}(t)$ are convergent, we must have

$$\lim_{m\to+\infty} u_{n,m}(t)=0, \quad \lim_{m\to+\infty} v_{n,m}(t)=0, \tag{34}$$

Due to the definitions (16) of χ_m and the mth-order deformation equations (31), it holds

$$\begin{aligned}&h\sum_{m=1}^{+\infty}R_1[\boldsymbol{u}_{n,m-1}(t),\boldsymbol{v}_{n,m-1}(t),\boldsymbol{u}_{n-1,m-1}(t),\\&\boldsymbol{v}_{n-1,m-1}(t),\boldsymbol{u}_{n+1,m-1}(t),\boldsymbol{v}_{n+1,m-1}(t),\cdots]=\\&\lim_{m\to+\infty}L[u_{n,m}(t)]=L[\lim_{m\to+\infty}u_{n,m}(t)]=0,\\&h\sum_{m=1}^{+\infty}R_2[\boldsymbol{u}_{n,m-1}(t),\boldsymbol{v}_{n,m-1}(t),\boldsymbol{u}_{n-1,m-1}(t),\\&\boldsymbol{v}_{n-1,m-1}(t),\boldsymbol{u}_{n+1,m-1}(t),\boldsymbol{v}_{n+1,m-1}(t),\cdots]=\\&\lim_{m\to+\infty}L[v_{n,m}(t)]=L[\lim_{m\to+\infty}v_{n,m}(t)]=0,\end{aligned}\tag{35}$$

which give

$$\begin{aligned}&\sum_{m=1}^{+\infty}R_1[\boldsymbol{u}_{n,m-1}(t),\boldsymbol{v}_{n,m-1}(t),\boldsymbol{u}_{n-1,m-1}(t),\\&\boldsymbol{v}_{n-1,m-1}(t),\boldsymbol{u}_{n+1,m-1}(t),\boldsymbol{v}_{n+1,m-1}(t),\cdots]=0,\\&\sum_{m=1}^{+\infty}R_2[\boldsymbol{u}_{n,m-1}(t),\boldsymbol{v}_{n,m-1}(t),\boldsymbol{u}_{n-1,m-1}(t),\\&\boldsymbol{v}_{n-1,m-1}(t),\boldsymbol{u}_{n+1,m-1}(t),\boldsymbol{v}_{n+1,m-1}(t),\cdots]=0,\end{aligned}\tag{36}$$

because the auxiliary parameter h is nonzero. Substituting the definitions (33) of R_i into above expressions, we have

$$\begin{aligned}&\sum_{m=1}^{+\infty}\frac{\partial u_{n,m-1}}{\partial t}-\sum_{m=1}^{+\infty}\sum_{j=0}^{m-1}\left(\sum_{i=0}^{j}u_{n,i}\right)\\&\qquad\cdot(v_{n,m-1-j}-v_{n-1,m-1-j})=\\&\frac{\partial}{\partial t}\sum_{m=0}^{+\infty}u_{n,m}-\sum_{m=0}^{+\infty}\sum_{j=0}^{m}\left(\sum_{i=0}^{j}u_{n,i}\right)(v_{n,m-j}-v_{n-1,m-j})\\&=\frac{\partial}{\partial t}\sum_{m=0}^{+\infty}u_{n,m}-\sum_{m=0}^{+\infty}u_{n,m}\left(\sum_{m=0}^{+\infty}v_{n,m}-\sum_{m=0}^{+\infty}v_{n-1,m}\right),\end{aligned}\tag{37}$$

and also

$$\begin{aligned}&\sum_{m=1}^{+\infty}\frac{\partial v_{n,m-1}}{\partial t}-\sum_{m=1}^{+\infty}\sum_{j=0}^{m-1}\left(\sum_{i=0}^{j}v_{n,i}\right)\\&\qquad\cdot(u_{n+1,m-1-j}-u_{n,m-1-j})=\\&\frac{\partial}{\partial t}\sum_{m=0}^{+\infty}v_{n,m}-\sum_{m=0}^{+\infty}v_{n,m}\left(\sum_{m=0}^{+\infty}u_{n+1,m}-\sum_{m=0}^{+\infty}u_{n,m}\right).\end{aligned}\tag{38}$$

Besides, using the initial conditions (33) and the definitions (18) of the initial guesses we have

$$\begin{aligned}&\sum_{m=0}^{+\infty}u_{n,m}(0)=u_{n,0}(0)=u_n(0)\\&\qquad=-c\coth(d)+c\tanh(dn+\delta),\\&\sum_{m=0}^{+\infty}v_{n,m}(0)=v_{n,0}(0)=v_n(0)\\&\qquad=-c\coth(d)-c\tanh(dn+\delta).\end{aligned}\tag{39}$$

Thus, due to (37)–(39), the series $\sum_{m=0}^{+\infty} u_{n,m}(t)$ and $\sum_{m=0}^{+\infty} v_{n,m}(t)$ must be exact solutions of systems (17) and (18). This ends the proof.

4. Results Analysis

It has been proved that, as long as a series solution given by HAM converges, it must be one of the exact solutions. So the validity of HAM is based on such

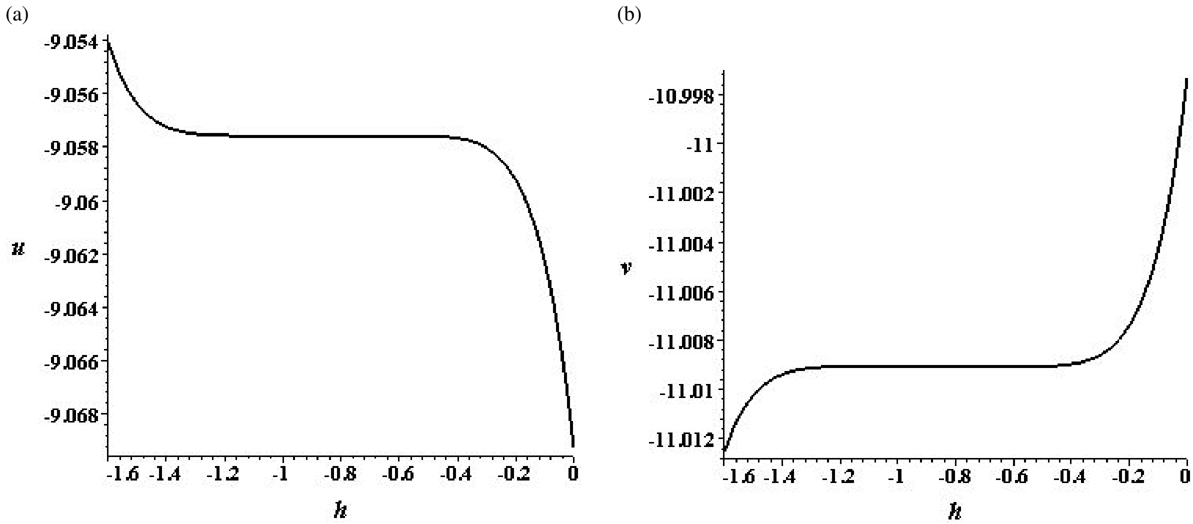

Fig. 1. h-curve for 7th-order HAM approximations (30): (a) h-curve for 7th-order HAM approximation $u_n(t)$ and (b) h-curve for 7th-order HAM approximation $v_n(t)$, when $c = \delta = 1$, $d = 0.1$, $n = 10$, and $t = 0.2$.

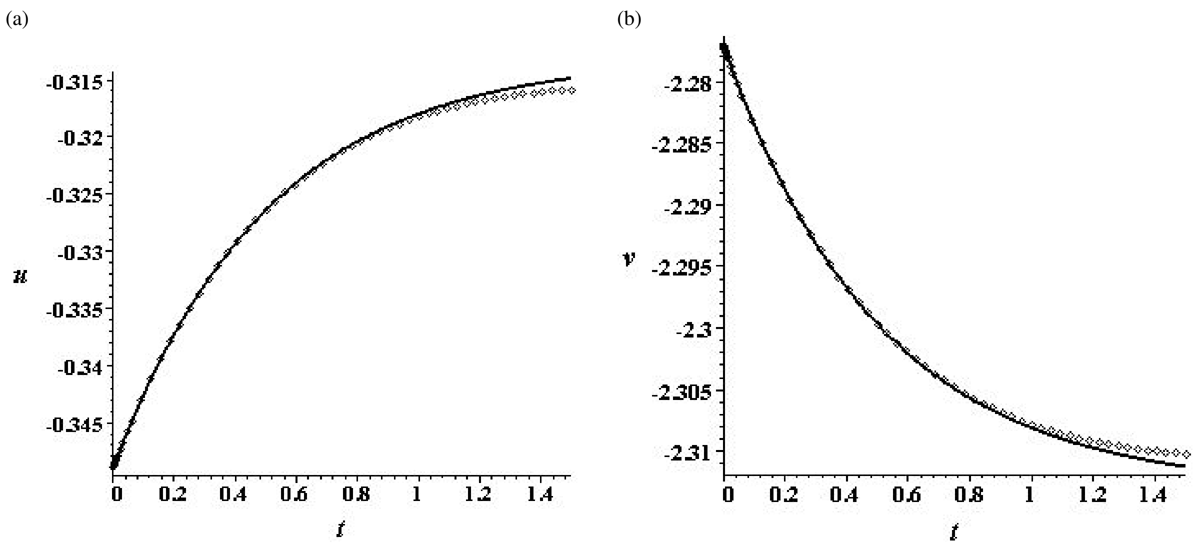

Fig. 2. Comparisons of the exact solutions with the homotopy-Padé (HP) approximations: (a) between the exact solution $u_n(t)$ and [2,2] HP approximations; (b) between the exact solution $v_n(t)$ and [2,2] HP approximations, when $c = d = n = \delta = 1$, and $h = -1.1$. Dotted line: [2,2] HP approximations; solid line: exact solutions.

an assumption that the series (4) converge at $q = 1$ which can be ensured by the properly chosen auxiliary parameter h. In general, by means of the so-called h-curve [22], it is straightforward to choose a proper value of h.

In Figure 1, we plotted the h-curve for the 7th-order HAM approximations (30) at $c = \delta = 1, d = 0.1$, $n = 10$, and $t = 0.2$. By HAM, it is easy to discover the valid region of h, which corresponds to the line segments nearly parallel to the horizontal axis. Then from Figure 1, we could find that if h is about in area $[-1.4, -0.4]$ the results are convergent.

To increase the accuracy and convergence of the solution, Liao [22] has developed a new technique, namely the homotopy-Padé (HP) method. Here comparisons are made between the [2,2] HP approximations and the exact solutions, when $c = d = \delta = n = 1$ and $h = -1.1$, as shown in Figure 2. From this fig-

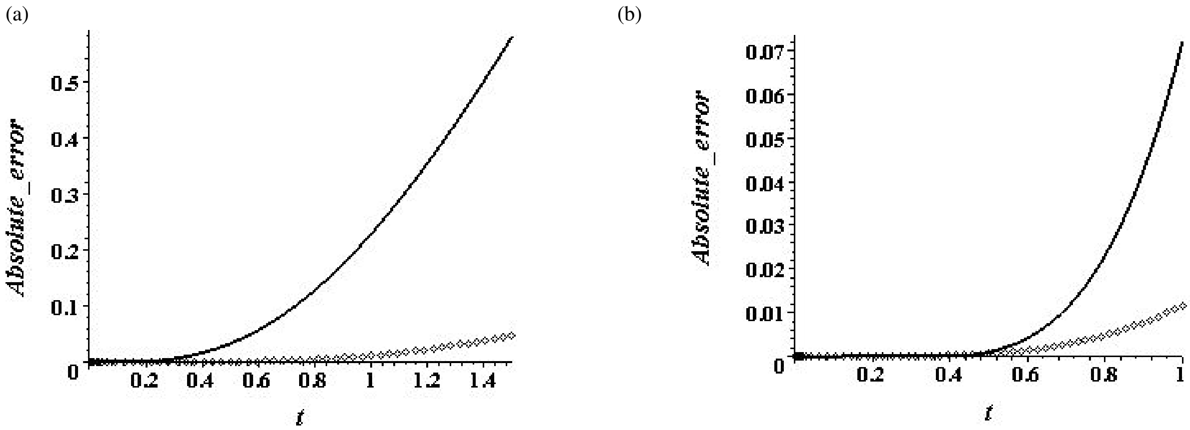

Fig. 3. Comparisons between absolute errors of [2,2] HP approximations and 4th HAM approximations: (a) for solution $u_n(t)$; (b) for solution$v_n(t)$, when $n = 10$, $d = 0.1$, $\delta = -1$, $c = 1$, and $h = -1.1$. Dotted line: absolute error of $[2,2]$ HP approximations; solid line: absolute error of 4th HAM approximations.

ure, we can conclude that the approximations obtained by the HP method agree well with the exact solutions when t tends to $t = 0$.

In Figure 3, to verify the effectiveness of the HP method, comparisons are made between absolute errors of the [2,2] HP and 4th HAM approximations. One can easily draw the conclusion that the HP method is an effective method to accelerate the convergence of the result and enlarge the convergence field.

5. Conclusions

In this paper, we successfully extend HAM to solve a system of DDEs: Volterra lattice system. The advantage of HAM is the auxiliary parameter which provides a convenient way of controlling the convergence region of the series solutions. The results obtained here show that the HAM is a very effective method and a promising tool for solving a system of DDEs. The power series has often finite radius of convergence. So, one must apply the HP technique to enlarge the convergence-region. Actually, it would be much better to use exponential functions as base functions. And I will try other auxiliary linear operators in following works.

Acknowledgements

This work was supported by Leading Academic Discipline Program, 211 Project for Shanghai University of Finance and Economics (the 3rd phase). The author would like to thank the City University of Hong Kong for warm hospitality. He is also grateful to Professor Benny Y. C. Hon and Professor Yufeng Zhang for stimulating discussions and helpful suggestions. And the author would like to thank the reviewers for their valuable comments and suggestions.

[1] E. Fermi, J. Pasta, and S. Ulam, Collected Papers of Enrico Fermi II, Univ. of Chicago Press, Chicago 1965.
[2] M. Wadati and M. Toda, J. Phys. Soc. Jpn. **39**, 1196 (1975).
[3] D. Levi and O. Ragnisco, Lett. Nuovo. Cimento. **22**, 691 (1978).
[4] S. I. Svinolupov and R. I. Yamilov, Phys. Lett. A **160**, 548 (1991).
[5] R. I. Yamilov, J. Phys. A: Math. Gen. **27**, 6839 (1994).
[6] V. E. Adler, S. I. Svinolupov, and R. I. Yamilov, Phys. Lett. A **254**, 24 (1999).
[7] A. B. Shabat and R. I. Yamilov, Leningrad Math. J. **2**, 377 (1991).
[8] E. G. Fan and H. H. Dai, Phys. Lett. A **372**, 4578 (2008).
[9] Y. C. Hon and E. G. Fan, Chaos, Solitons, and Fractals **40**, 1297 (2009).
[10] Q. Wang and Y. X. Yu, Chaos, Solitons, and Fractals **29**, 904 (2006).
[11] L. Zou, Z. Zong, Z. Wang, and L. He, Phys. Lett. A **370**, 287 (2007).
[12] L. Zou, Z. Zong, and G. H. Dong, Comput. Math. Appl. **56**, 2289 (2008).
[13] M. J. Ablowitz and P. A. Clarkson, Soliton, Nonlinear Evolution Equations and Inverse Scattering, Cambridge University Press, New York 1991.

[14] P. J. Olver, Applications of Lie groups to differential equations, Springer, New York 1993.
[15] C. Rogers and W. K. Schief, Bäcklund and Darboux transformations, geometry and modern applications in soliton theory, Cambridge University Press, Cambridge 2002.
[16] C. H. Gu, H. S. Hu, and Z. X. Zhou, Darboux transformations in integrable systems: theory and their applications to geometry, Springer, Berlin 1991.
[17] R. Hirota, Direct methods in soliton theory, Cambridge University Press, Cambridge 1980.
[18] S. J. Liao, Ph.D. Thesis, Shanghai Jiao Tong University, 1992.
[19] S. J. Liao, Int. J. Nonlinear Mech. **30**, 371 (1995).
[20] S. J. Liao, Int. J. Nonlinear Mech. **32**, 815 (1997).
[21] S. J. Liao, Int. J. Nonlinear Mech. **34**, 759 (1999).
[22] S. J. Liao, Beyond Perturbation: Introduction to the Homotopy Analysis Method, Chapman & Hall/CRC Press, Boca Raton 2003.
[23] S. Abbasbandy, Phys. Lett. A **360**, 109 (2006).
[24] S. Abbasbandy, Phys. Lett. A **361**, 478 (2007).
[25] S. Abbasbandy, T. Hayat, R. Ellahi, and S. Asghar, Z. Naturforsch. **64a**, 59 (2009).
[26] M. Wadati and M. Watanabe, Prog. Theor. Phys. **57**, 808 (1977).
[27] D. Baldwin, Ü. Göktaş, and W. Hereman, Comput. Phys. Commun. **162**, 203 (2004).
[28] Y. B. Suris, in: Miura transformation for Toda-type integrable system, with applications to the problem of integrable discretizations, solv-int/9902003.
[29] J. H. He, Perturbation Methods: Basic and Beyond, Elsevier, Amsterdam 2006.