Solution of Fractional Diffusion Equation with a Moving Boundary Condition by Variational Iteration Method and Adomian Decomposition Method

Subir Das and Rajeev

Department of Applied Mathematics, Institute of Technology, Banaras Hindu University, Varanasi -221 005, India

Reprint requests to S. D.; E-mail: subir_das08@hotmail.com

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In this paper, the approximate analytic solutions of the mathematical model of time fractional diffusion equation (FDE) with a moving boundary condition are obtained with the help of variational iteration method (VIM) and Adomian decomposition method (ADM). By using boundary conditions, the explicit solutions of the diffusion front and fractional releases in the dimensionless form have been derived. Both mathematical techniques used to solve the problem perform extremely well in terms of efficiency and simplicity. Numerical solutions of the problem show that only a few iterations are needed to obtain accurate approximate analytical solutions. The results obtained are presented graphically.

Key words: Fractional Diffusion Equation; Moving Boundary Problem; Variational Iteration Method; Adomian Decomposition Method; Caputo Derivative.

1. Introduction

Moving boundary problems are difficult to solve since the position of the moving interface is not known a priori and depends on the variables which are to be solved. The nonlinearity of the problem, a discontinuity in derivatives at the moving boundary and also the discontinuity of the temperature gradient across the interface due to the effect of latent heat are the key reasons for which mathematical complexities arise in moving boundary problems. Some mathematical techniques, both exact and approximate, are available for limited cases such as a single space variable and for a semi-infinite domain. The history and classical solutions to the moving boundary problems are well covered in the monographs of Crank [1] and Hill [2].

An interesting moving-boundary problem, related to the diffusional release of a solute from a polymer matrix in which the initial loading is higher/lower than the solubility, occurs in the field of science and technology. In 1976, Paul and McSpadden [3] presented a mathematical model of solute release from a planar polymer matrix. They used a diffusion equation in one dimension of integer order. It has been shown that the solute movement in the heterogeneity media is usually anomalous and cannot be described by the standard diffusion equation. Hence, in the last two decades fractional (space or time or both) diffusion equations have been widely used by the researchers, see [4 – 7]. However, moving boundary problems dealing with the fractional derivative are rare. In 2004, Liu and Xu [8] first presented a mathematical model of the moving-boundary problem with fractional anomalous diffusion in drug release devices. They used a time fractional diffusion equation and gave an exact solution. Li et al. [9] developed a space-time fractional diffusion equation to describe the process of a solute release from a polymer matrix in which the initial solute loading is higher than the solubility and presented the exact solution in term of the Fox-H function. Recently, Li et al. [10] presented a similarity solution of the partial differential equation of fractional-order with a moving-boundary condition in terms of a generalized Wright function. The exact solutions of the moving-boundary problems are difficult to obtained and restricted for specific cases. Hence, many approximate analytical methods have been used to solve moving boundary problems. But to the Author's knowledge, the problems of fractional diffusion equation with a moving boundary condition is not yet been solved by using VIM and ADM.

The idea of using fractional calculus as a mathematical description in the physical dynamical system is not new. The subject is nearly 300 years old. The great

Mathematicians like Leibnitz, Bernoulli, Liouville, and Abel already described the meaning of fractional-order derivative and integration in their scientific contributions. During the last two decades, applications of the subject have been increased significantly in the fields of electro-chemistry, seismic analysis, viscous damping, electric fractal network, noise, circuit and system, signal processing, control, robotics, and fractal structures. Recently, chaos in fractional-order system becomes one of the hottest areas of research. Hartley et al. [11] studied the effect of fractional derivative as the dynamical system through fractional-order Chua's system. The chaos in Chen's system can be obtained in the article of Li and Chen [12].

Synchronization of fractional-order chaotic systems is one of the potent research areas due to its extensive application in Communication theory and control processing. Recently, Xu et al. [13] have studied the chaos synchronization between two different fractional-order chaotic systems by using active control. Numerical algorithms for chaos synchronization of fractional-order systems based on the Laplace transform can be obtained from the articles of Wang et al. [14], Li et al. [15], Yu and Li [16]. Recently, Odibat [17] has studied the master-slave synchronization based on the stability results of linear fractional-order systems and of coupled fractional-order chaotic systems.

The variational iteration method (VIM) was first proposed by He ([18 – 22]) and subsequently implemented by He [23] to solve the fractional differential equations. The practice was successfully followed by scientists like Sweilam et al. [24], Odibat and Momani [25, 26], Abbasbandy [27], Momani and Odibat [28], Das [29] etc. to solve nonlinear systems of partial differential equations (PDEs) and nonlinear differential equations of fractional-orders. The main advantage of this method is the computational simplicity that it offers. The solution obtained by this method is expected to be a better approximation in a straight forward manner.

The decomposition method of Adomian (ADM) has been applied to solve a wide class of nonlinear differential and partial differential equations (Adomian [30 – 32], Adomian and Rach [33]) etc. ADM is a really new approache to provide an approximate analytical solution to linear and nonlinear problems, and is particularly valuable as a tool for scientists, engineers, and applied mathematicians, because these techniques provide immediate and visible symbolic terms of the analytical solution, as well as numerical approximate solutions to both linear and nonlinear differential equations without linearization or discretization.

In 2007, Wazwaz [34] has made a comparative study of both methods through their applications in solving homogeneous and non-homogeneous advection problems. According to Wazwaz, the advantages of the methods are that these can be applied for solving differential and integral equations, and also these are capable of reducing the size of computational work as well as give high degree of accuracy of the numerical solutions. He also pointed out that the evaluation of the Lagrangian multiplier for VIM and also evaluation of Adomian polynomials for ADM require tedious work.

Still there are some problems of using ADM specially for solving nonlinear problems. The solution procedure of Adomian polynomials is very difficult. In order to overcome the demerit, Gorbani [35] introduced an alternative approach, viz., He's polynomials based on the homotopy perturbation method (HPM). Gorbani and Saberi-Nadjafi [36] have used He's polynomials algorithm to calculate the Adomian polynomials. The effectiveness and convenience of the algorithm are shown through the solutions of three difficult problems. Recently, Mohyud-Din et al. [37] have successfully applied this algorithm to solve travelling solitary wave solutions of seventh-order generalized Korteweg-de Vries (KdV) equations.

Tari [38] has modified VIM by inserting some unknown parameters into the correction functional. The main advantage of the method is that it can avoid the shortcomings like uncontrollability problems of non-zero end point conditions encountered in traditional VIM. Later, Abassy et al. [39] in their article have shown that the proposed method overcome the time consuming effort for repeated computations for VIM. In 2008, Noor and Mohyud-Din [40] have applied VIM using He's polynomials for solving higher-order boundary value problems. The proposed algorithm is an elegant combination of VIM and HPM and is a quite efficient and reliable method for solving nonlinear problems.

In this paper, efficient mathematical techniques like VIM and ADM are successfully applied to obtain the approximate analytical solutions of the fractional diffusion equation governing the process of a solute release from a polymer matrix in which initial solute loading is higher than the solubility. The expressions of the diffusion front and fractional releases for different Brownian motions and for different values of the ratio of initial concentration of the solute and solubility of the solvent are calculated numerically and presented through graphs. The elegance of the methods can be attributed to its simplistic approach in seeking the approximate analytical solution of the problem.

2. Mathematical Formulation of the Problem

Here, the diffusion release of a solute from a planar polymer matrix into a perfect sink fluid is considered. The diffusion coefficient is assumed to be constant. The initial drug loading (C_0) is taken higher than the solubility (C_s) of the drug in the tissue fluid and only the early stage of loss before the diffusion front moves to R is considered (see Fig. 1).

The drug distribution in the plane after a certain time of drug release in shown in Figure 1, where R is the scale of the polymer matrix. The moving interface position $s(t)$ divides each matrix into two regions: the surface zone $0 < x < s(t)$, in which all solute is dissolved and $s(t) < x < R$, which contains undissolved solute. Here, we consider the model of time-fractional anomalous diffusion equation given by Liu and Xu [8] as

$$\frac{\partial^\alpha}{\partial t^\alpha} C(x,t) = \mathcal{D} \frac{\partial^2}{\partial x^2} C(x,t), \quad 0 < x < s(t), \quad 0 < \alpha \le 1, \tag{1}$$

$$C(x,t) = 0, \quad x = 0, \tag{2}$$

$$C(x,t) = C_s, \quad x = s(t), \tag{3}$$

$$(C_0 - C_s) \frac{\partial^\alpha}{\partial t^\alpha} s(t) = \mathcal{D} \frac{\partial}{\partial x} C(x,t), \quad x = s(t), \tag{4}$$

$$s(t) = 0, \quad t = 0, \tag{5}$$

where $C(x,t)$ and $\mathcal{D}$ are the concentration and diffusivity of the drug in the matrix. The operator $D_t^\alpha \equiv \frac{\partial^\alpha}{\partial t^\alpha}$ is the Caputo fractional derivative of order α.

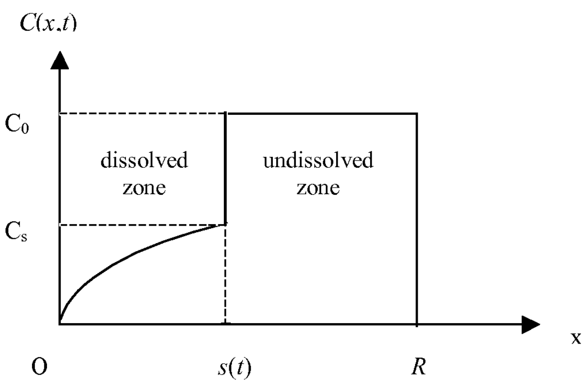

Fig. 1. Profile of concentration.

Using dimensionless variables defined as

$$\xi = \frac{x}{R}, \quad \tau = \left(\frac{\mathcal{D}}{R^2}\right)^{\frac{1}{\alpha}} t, \quad \theta = \frac{C}{C_s}, \quad S(\tau) = \frac{s(t)}{R}, \tag{6}$$

(1) – (5) become

$$\frac{\partial^\alpha}{\partial \tau^\alpha} \theta(\xi,\tau) = \frac{\partial^2}{\partial \xi^2} \theta(\xi,\tau), \quad 0 < \xi < S(\tau), \quad 0 < \alpha \le 1, \tag{7}$$

$$\theta(\xi,\tau) = 0, \quad \xi = 0, \tag{8}$$

$$\theta(\xi,\tau) = 1, \quad \xi = S(\tau), \tag{9}$$

$$\eta \frac{\partial^\alpha}{\partial \tau^\alpha} S(\tau) = \frac{\partial}{\partial \xi} \theta(\xi,\tau), \quad \xi = S(\tau), \tag{10}$$

$$S(\tau) = 0, \quad \tau = 0, \tag{11}$$

where $\eta = \frac{C_0}{C_s} - 1$.

If θ_0 and S_0 be the initial approximations, then we may easily obtain the differential equation

$$\frac{\partial^2 \theta_0}{\partial \xi^2} = 0 \tag{12}$$

with the boundary conditions

$$\theta_0(0,\tau) = 0, \tag{13}$$

$$\theta_0(S_0,\tau) = 1, \tag{14}$$

$$\frac{\partial \theta_0}{\partial \xi} = \eta \frac{\partial^\alpha S_0}{\partial \tau^\alpha} \text{ at } \xi = S_0, \tag{15}$$

$$S_0(0) = 0. \tag{16}$$

(12) with the aid of (13) and (14) gives rise to

$$\theta_0 = S_0^{-1} \xi. \tag{17}$$

(15) with the help of (16) and (17) gives

$$S_0 = a_0 \tau^{\alpha/2}, \tag{18}$$

where $a_0 = \left[\frac{\Gamma\left(1-\frac{\alpha}{2}\right)}{\eta \Gamma\left(1+\frac{\alpha}{2}\right)}\right]^{1/2}$.

3. Application of He's Variational Iteration Method

In this section we applied VIM for solving (7) with the boundary conditions. According to the VIM we consider the correction functional in ξ-direction in the following form:

$$\theta_{n+1}(\xi,\tau) = \theta_n(\xi,\tau) + \int_0^{\xi} \lambda(x) \left\{ \frac{\partial^2 \theta_n}{\partial x^2} - \frac{\partial^\alpha \tilde{\theta}_n}{\partial \tau^\alpha} \right\} \mathrm{d}x. \quad (19)$$

It is obvious that the successive approximation θ_j, $j \geq 0$, can be established by determining the Lagrange multiplier λ. The function $\tilde{\theta}_n$ is a restricted variation, which means $\delta \tilde{\theta}_n = 0$. The successive approximation $\theta_{n+1}(\xi,\tau)$, $n \geq 0$, of the solution $\theta(\xi,\tau)$ will be readily obtained upon using the Lagrange multiplier and by using any selective function θ_0. The initial values $\theta(0,\tau)$ and $\theta_\xi(0,\tau)$ are usually used for selecting the zeroth approximation θ_0. To find the optimal value of λ, we have

$$\delta\theta_{n+1}(\xi,\tau) = \delta\theta_n(\xi,\tau) + \delta \int_0^{\xi} \lambda(x) \frac{\partial^2 \theta_n}{\partial x^2} \mathrm{d}x = 0, \quad (20)$$

$$[1-\lambda'(x)]_{x=\xi}\,\delta\theta_n(\xi,\tau) + [\lambda(x)]_{x=\xi}\,\delta \frac{\partial \theta(\xi,\tau)}{\partial \tau} + \int_0^{\xi} [\lambda''(x)]_{x=\xi}\,\delta\theta_n(x,\tau)\mathrm{d}x = 0, \quad (21)$$

which yields

$$[1-\lambda'(x)]_{x=\xi} = 0, \quad (22)$$

$$[\lambda''(x)]_{x=\xi} = 0, \quad (23)$$

$$[\lambda(x)]_{x=\xi} = 0, \quad (24)$$

which gives $\lambda(x) = x - \xi$.

We therefore obtain the following iteration formula:

$$\theta_{n+1}(\xi,\tau) = \theta_n(\xi,\tau) + \int_0^{\xi} (x-\xi) \left\{ \frac{\partial^2 \theta_n}{\partial x^2} - \frac{\partial^\alpha \theta_n}{\partial \tau^\alpha} \right\} \mathrm{d}x. \quad (25)$$

Beginning with the initial approximation

$$\theta_0(\xi,\tau) = \theta(0,\tau) + \xi \frac{\partial}{\partial \xi}\theta(0,\tau) = {a_0}^{-1}\tau^{-\alpha/2}\xi, \quad (26)$$

we obtain the following successive approximations from (19):

$$\theta_1(\xi,\tau) = \frac{\Gamma\left(1-\frac{\alpha}{2}\right)}{a_0}\left[\frac{\tau^{-\frac{\alpha}{2}}\xi}{\Gamma\left(1-\frac{\alpha}{2}\right)} + \frac{\tau^{-\frac{3\alpha}{2}}\xi^3}{3!\Gamma\left(1-\frac{3\alpha}{2}\right)}\right],$$

$$\theta_2(\xi,\tau) = \frac{\Gamma\left(1-\frac{\alpha}{2}\right)}{a_0} \cdot \left[\frac{\tau^{-\frac{\alpha}{2}}\xi}{\Gamma\left(1-\frac{\alpha}{2}\right)} + \frac{\tau^{-\frac{3\alpha}{2}}\xi^3}{3!\Gamma\left(1-\frac{3\alpha}{2}\right)} + \frac{\tau^{-\frac{5\alpha}{2}}\xi^5}{5!\Gamma\left(1-\frac{5\alpha}{2}\right)}\right],$$

$$\theta_3(\xi,\tau) = \frac{\Gamma\left(1-\frac{\alpha}{2}\right)}{a_0}\left[\frac{\tau^{-\frac{\alpha}{2}}\xi}{\Gamma(1-\frac{\alpha}{2})} + \frac{\tau^{-\frac{3\alpha}{2}}\xi^3}{3!\Gamma(1-\frac{3\alpha}{2})} + \frac{\tau^{-\frac{5\alpha}{2}}\xi^5}{5!\Gamma(1-\frac{5\alpha}{2})} + \frac{\tau^{-\frac{7\alpha}{2}}\xi^7}{7!\Gamma(1-\frac{7\alpha}{2})}\right],$$

and so on. Using this procedure for a sufficiently large value of n, we get $\theta_n(\xi,\tau)$ as an approximation of the exact solution.

Thus, the approximate analytical solution may be obtained as

$$\theta(\xi,\tau) = \lim_{n\to\infty}\theta_n(\xi,\tau) = H \sum_{n=0}^{\infty} \frac{\left(\frac{\xi}{\tau^{\alpha/2}}\right)^{2n+1}}{(2n+1)!\Gamma\left(1-\frac{2n+1}{2}\alpha\right)}, \quad (27)$$

where $H = \frac{\Gamma\left(1-\frac{\alpha}{2}\right)}{a_0}$.

4. Application of the Adomian Decomposition Method

We consider (7) as

$$L_{\xi\xi}\theta(\xi,\tau) = \frac{\partial^\alpha \theta}{\partial \tau^\alpha}, \quad 0 < \xi < S(\tau), \quad \tau > 0, \quad (28)$$

where $L_{\xi\xi} \equiv \frac{\partial^2}{\partial \xi^2}$.

Applying the two-fold integration inverse operator $L_{\xi\xi}^{-1} = \int\int_0^{\xi}{}_0^{\xi}(.)\mathrm{d}\xi\mathrm{d}\xi$ to (28), we get

$$\theta(\xi,\tau) = \phi_\xi + L_{\xi\xi}^{-1}\left(\frac{\partial^\alpha}{\partial \tau^\alpha}\theta(\xi,\tau)\right), \quad (29)$$

where $\phi_\xi = \theta(0,\tau) + \xi\frac{\partial}{\partial \xi}\theta(0,\tau) = {a_0}^{-1}\tau^{-\alpha/2}\xi$.

The Adomian decomposition method [31] assumes an infinite series solution for the unknown function $\theta(\xi,\tau)$ given by

$$\theta(\xi,\tau) = \sum_{n=0}^{\infty}\theta_n(\xi,\tau), \quad (30)$$

where the components θ_0, θ_1, θ_2, ... are usually determined recursively by

$$\begin{aligned} \theta_0 &= \phi_\xi, \\ \theta_1 &= L_{\xi\xi}^{-1}\left(\frac{\partial^\alpha}{\partial \tau^\alpha}\theta_0(\xi,\tau)\right), \\ \theta_2 &= L_{\xi\xi}^{-1}\left(\frac{\partial^\alpha}{\partial \tau^\alpha}\theta_1(\xi,\tau)\right), \\ \theta_{n+1} &= L_{\xi\xi}^{-1}\left(\frac{\partial^\alpha}{\partial \tau^\alpha}\theta_n(\xi,\tau)\right), \quad n \geq 0. \end{aligned} \quad (31)$$

Now substituting ϕ_ξ, we get

$$\theta_0 = a_0^{-1}\tau^{-\alpha/2}\xi,$$

$$\theta_1 = a_0^{-1}\frac{\Gamma\left(1-\frac{\alpha}{2}\right)}{\Gamma\left(1-\frac{3\alpha}{2}\right)}\tau^{-3\alpha/2}\frac{\xi^3}{3!},$$

$$\theta_2 = a_0^{-1}\frac{\Gamma\left(1-\frac{\alpha}{2}\right)}{\Gamma\left(1-\frac{5\alpha}{2}\right)}\tau^{-5\alpha/2}\frac{\xi^5}{5!},$$

and so on. Thus,

$$\begin{aligned}\theta(\xi,\tau) &= \theta_0+\theta_1+\theta_2+\theta_3+\ldots\\ &= H\left[\frac{\tau^{-\alpha/2}\xi}{\Gamma\left(1-\frac{\alpha}{2}\right)}+\frac{\tau^{-3\alpha/2}}{\Gamma\left(1-\frac{3\alpha}{2}\right)}\frac{\xi^3}{3!}\right.\\ &\quad\left.+\frac{\tau^{-5\alpha/2}}{\Gamma\left(1-\frac{5\alpha}{2}\right)5!}\frac{\xi^5}{5!}+\ldots\right].\end{aligned} \tag{32}$$

This expression of $\theta(\xi,\tau)$ is similar to the expression given in (27) obtained by VIM.

5. Solution of the Problem

(10) can be written as

$$\eta\frac{\partial S(\tau)}{\partial\tau} = \frac{\partial^{1-\alpha}}{\partial\tau^{1-\alpha}}\left[\frac{\partial}{\partial\xi}\theta(\xi,\tau)\right]_{\xi=S(\tau)}. \tag{33}$$

Integrating (33), we get

$$\begin{aligned}S(\tau) &= S_0+\frac{1}{\eta}\mathrm{J}_t^\alpha\left[a^{-1}\left(\tau^{-\alpha/2}+\frac{\Gamma\left(1-\frac{\alpha}{2}\right)}{\Gamma\left(1-\frac{3\alpha}{2}\right)}\tau^{-\frac{3\alpha}{2}}\right.\right.\\ &\quad\left.\left.\cdot\frac{(S(\tau))^2}{2!}+\frac{\Gamma\left(1-\frac{\alpha}{2}\right)}{\Gamma\left(1-\frac{5\alpha}{2}\right)}\tau^{-\frac{5\alpha}{2}}\frac{(S(\tau))^4}{4!}+\ldots\right)\right],\end{aligned} \tag{34}$$

where J_t^α is the inverse operator of the Caputo derivative D_t^α.

Taking $S_0 = a_0\tau^{\alpha/2}$ and applying ADM, we get

$$S_1 = H_2\tau^{\alpha/2},\quad S_2 = H_4\tau^{\alpha/2},$$

where

$$H_2 = H_1\frac{\Gamma\left(1-\frac{\alpha}{2}\right)}{\Gamma\left(1+\frac{\alpha}{2}\right)},$$

$$H_1 = \frac{H}{\eta}\sum_{n=0}^{\infty}\frac{1}{\Gamma\left(1-\frac{2n+1}{2}\alpha\right)}\frac{a_0^{2n}}{2n!},$$

$$H_4 = H_3\frac{\Gamma\left(1-\frac{\alpha}{2}\right)}{\Gamma\left(1+\frac{\alpha}{2}\right)},$$

$$H_3 = \frac{H}{\eta}\sum_{n=0}^{\infty}\frac{1}{\Gamma\left(1-\frac{2n+1}{2}\alpha\right)}\frac{H_2^{2n}}{2n!}.$$

Finally, we get the analytical expression of $S(\tau)$ as

$$S(\tau) = \sum_{n=0}^{\infty}S_n = M\tau^{\alpha/2}, \tag{35}$$

where $M = a_0+H_2+H_4+\ldots$.

Equations (27) and (35) with the aid of (9) and (10) give rise to

$$H\sum_{n=0}^{\infty}\frac{M^{2n+1}}{(2n+1)!\Gamma\left(1-\frac{2n+1}{2}\alpha\right)} = 1 \tag{36}$$

and

$$H\sum_{n=0}^{\infty}\frac{M^{2n}}{2n!\Gamma\left(1-\frac{2n+1}{2}\alpha\right)} = M\eta\frac{\Gamma\left(1+\frac{\alpha}{2}\right)}{\Gamma\left(1-\frac{\alpha}{2}\right)}. \tag{37}$$

These are the exact solutions of (7)–(11) and are in complete agreement with the result of Li et al. [10].

Now, the amount of drug release per unit area at time t is given by

$$M_\mathrm{t} = C_0 s(t)-\int_0^{s(t)}C(x,t)\mathrm{d}x. \tag{38}$$

The dimensionless form of fractional release rate is obtained as

$$\begin{aligned}&\frac{M_\mathrm{t}}{M_\infty} = S(\tau)-\frac{S_s}{C_0}\int_0^{S(\tau)}\theta(\xi,\tau)\mathrm{d}\xi =\\ &\left(M-\frac{H}{\eta+1}\sum_{n=0}^{\infty}\frac{1}{\Gamma\left(1-\frac{2n+1}{2}\alpha\right)}\frac{M^{2n+2}}{(2n+2)!}\right)\tau^{\alpha/2},\end{aligned} \tag{39}$$

where $M_\infty = CR$ is the total amount of drug release per unit area at infinite time.

6. Numerical Results and Discussion

In this section, numerical results of the diffusion front position $S(\tau)$ and the fractional drug release for different Brownian motions $\alpha = \frac{1}{3}, \frac{1}{2}, \frac{2}{3}$, and also for the standard motion $\alpha = 1$ are calculated for various values of τ at different solute loading levels $\eta = 2.5$, 3.0, 5.0, and these results are depicted through Figures 2–7.

It is observed from Figures 2–4 that $S(\tau)$ increases with the increase in τ and decreases with the increase in α for all values of η. The rate of increase of $S(\tau)$ decreases with the increase of α which confirms the exponential decay of the regular Brownian motion. This result is in complete agreement with the model devel-

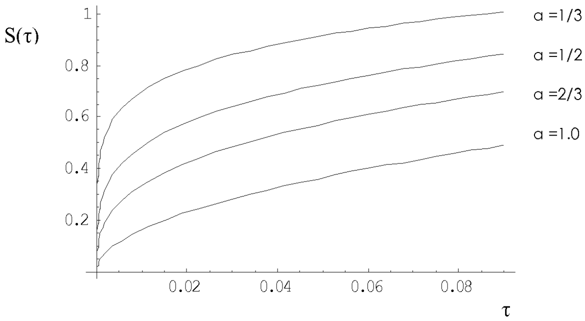

Fig. 2. Plot of $S(\tau)$ vs. τ for different values of α at $\eta = 2.5$.

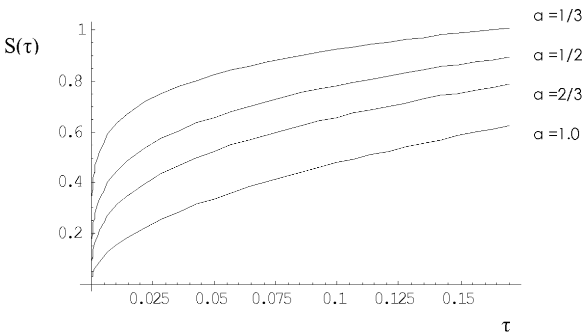

Fig. 3. Plot of $S(\tau)$ vs. τ for different values of α at $\eta = 3.5$.

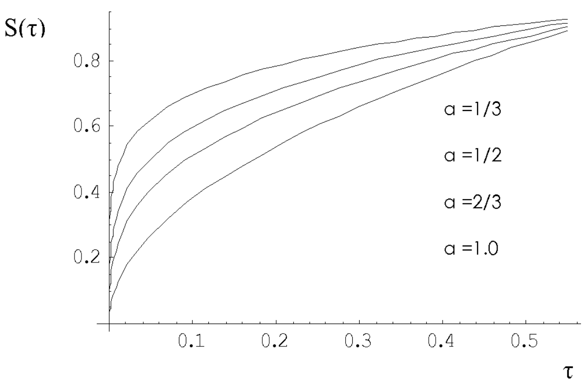

Fig. 4. Plot of $S(\tau)$ vs. τ for different values of α at $\eta = 5$.

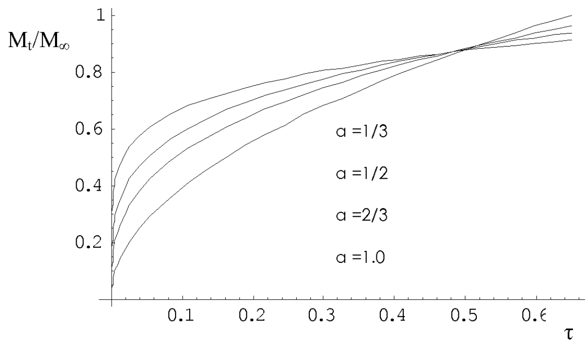

Fig. 5. Plot of M_t/M_∞ vs. τ for different values of α at $\eta = 2.5$.

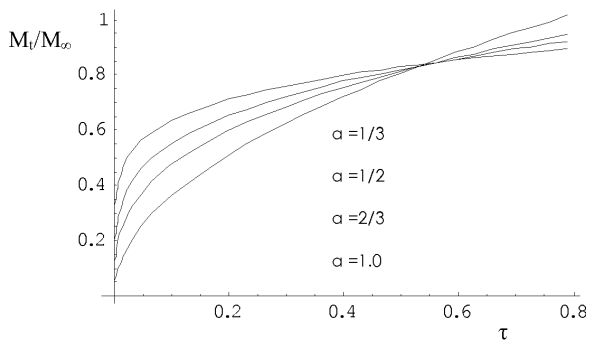

Fig. 6. Plot of M_t/M_∞ vs. τ for different values of α at $\eta = 3.5$.

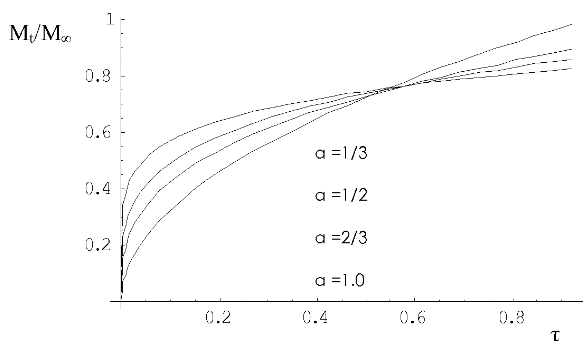

Fig. 7. Plot of M_t/M_∞ vs. τ for different values of α at $\eta = 5$.

oped by Das [7, 29]. It is also seen that as the values of η increase, it requires the longer time to reach R for the diffusion front $S(\tau)$. This result shows that this model is consistent and agrees with the model developed by Liu and Xu [8].

It is also seen from the Figures 5 – 7 that the fractional drug release takes more time with the increase of solute loading level for any value of α.

7. Conclusion

Both the variational iteration method and the Adomian decomposition method are very powerful in finding the solutions for various physical problems like moving boundary problems related to fractional derivative. It is seen that both methods are efficient for finding the solutions in higher degree of accuracy. The basic difference between the methods are that the variational iteration method is direct, straight forward, and it avoids the volume of calculations of requiring the Adomian polynomials for finding the solution by Adomian decomposition method. Again He's variational iteration method facilitates computational work for which it gives the required solution faster in compare with the Adomian decomposition method.

Another important part of the study is to present the decay of $S(\tau)$ with the increase of fractional time derivative α and the explanation of the increase of the moving front velocity with the decrease in solute loading which has been accomplished by the authors numerically.

The model and the analytical methods employed in this article are very much useful for the theoretical analyses of viscoelasitic fluid mechanics. The recent appearance of nonlinear FDE as models in some fields such as drug release in tissues, nutrition transport from capillaries to tissues, anomalous diffusion of water molecule in biological tissues, solute transport in ground water, solute release from a polymer matrix, and viscoplasticity makes it necessary to investigate the method of solutions for such equations (analytical and numerical), and we hope that this work is a step in this direction.

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[1] J. Crank, Free and moving boundary problem, Clarendon Press, Oxford 1984.
[2] J. M. Hill, One dimensional Stefan problems, An introduction, Longman Scientific and technical, New York 1987.
[3] D. R. Paul and S. K. McSpadden, J. Memb. Sci. **1**, 33 (1976).
[4] W. Wyss, J. Math. Phys. **27**, 2782 (1986).
[5] D. del-Castillo-Negrete, Phys. Plasmas **13**, 082308 (2006).
[6] A. S. Chaves, Phys. Lett. A **239**, 13 (1998).
[7] S. Das, Chaos, Solitons, and Fractals **42**, 2074 (2009).
[8] J. Y. Liu and M. Y. Xu, Z. Angew. Math. Mech. **84**, 22 (2004).
[9] X. C. Li, M. Y. Xu, and S. W. Wang, J. Phys. A. Math. Theor. **40**, 12131 (2007).
[10] X. C. Li, M. Y. Xu, and S. W. Wang, J. Phys. A. Math. Theor. **41**, 155 (2008).
[11] T. T. Hartley, C. F. Lorenzo, and H. R. Qammer, IEEE Trans. on Circuit System, **42**, 485 (1995).
[12] C. G. Li and G. Chen, Chaos, Solitons, and Fractals **22**, 549 (2004).
[13] C. Xu, G. Wu, J. W. Feng, and W. Zhang, Int. J. Nonlinear Sci. Numer. Simul. **9**, 89 (2008).
[14] J. Wang, X. Xiong, and Y. Zhang, Physica A **370**, 279 (2007).
[15] C. P. Li, W. A. Deng, and D. Xu, Physica A **360**, 171 (2006).
[16] Y. Yu and H. Li, Physica A **387**, 1393 (2008).
[17] Z. Odibat, ICCSA, Le Havre, Normandy, France, June 29 – July 2 (2009).
[18] J. H. He, Comput. Methods Appl. Mech. Eng. **167**, 57 (1998).
[19] J. H. He, Comput. Methods Appl. Mech. Eng. **167**, 69 (1998).
[20] J. H. He, Int. J. Nonlinear Mech. **34**, 699 (1999).
[21] J. H. He, Appl. Math. Comput. **114**, 115 (2000).
[22] J. H. He, Int. J. Modern Phys. B **20**, 1141 (2006).
[23] J. H. He, Comput. Methods Appl. Mech. Eng. **167**, 57 (1998).
[24] N. H. Sweilam, M. M. Khader, and R. F. Ali-Bar, Phys. Lett. A **371**, 26 (2007).
[25] Z. Odibat and S. Momani, Appl. Math. Model. **32**, 28 (2008).
[26] Z. Odibat and S. Momani, Int. J. Nonlinear Sci. Numer. Simul. **7**, 27 (2006).
[27] S. Abbasbandy, J. Comput. Appl. Math. **207**, 53 (2007).
[28] S. Momani and Z. Odibat, Chaos, Solitons, and Fractals **31**, 1248 (2007).
[29] S. Das, Int. J. Nonlinear Sci. Numer. Simul. **9**, 361 (2008).
[30] G. Adomian, J. Math. Anal. Appl. **135**, 501 (1988).
[31] G. Adomian, Solving frontier problems of physics: The Decomposition Method, Kluwer Academic Publishers, Boston 1994.
[32] G. Adomian, Appl. Math. Lett. **11**, 121 (1998).
[33] G. Adomian and R. Rach, Found Phys. **21**, 983 (1991).
[34] A. M. Wazwaz, J. Comput. Appl. Math. **207**, 129 (2007).
[35] A. Ghorbani, Chaos, Solitons, and Fractals **39**, 1486 (2009).
[36] A. Ghorbani and J. Saberi-Nadjafi, Int. J. Nonlinear Sci. Numer. Simul. **8**, 229 (2007).
[37] S. T. Mohyud-Din, M. A. Noor, and K. I. Noor, Int. J. Nonlinear Sci. Numer. Simul. **10**, 223 (2009).
[38] H. Tari, Physics Letters A **369**, 290 (2007).
[39] T. A. Abassy, M. A. El-Tawil, and H. El Zoheiry, J. Comput. Appl. Math. **207**, 137 (2007).
[40] M. A. Noor and S. T. Mohyud-Din, Int. J. Nonlinear Sci. Numer. Simul. **9**, 141 (2008).