Effect of an Induced Magnetic Field on the Peristaltic Motion of Phan-Thien-Tanner (PTT) Fluid

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Z. Naturforsch. **65a,** 665 – 676 (2010); received May 5, 2009 / revised December 5, 2009

This article looks at the influence of an induced magnetic field on peristaltic motion of an incompressible fluid in a planar channel with non-conductive walls. Peristaltic flow is generated by a sinusoidal wave travelling down its walls. The problem formulation in a wave frame of reference moving with velocity of wave is established. Mathematical relations for the stream function, pressure gradient, magnetic force function, and axial induced magnetic field are constructed. The pressure rise and frictional force are discussed by performing numerical integration. Effects of many sundry parameters entering into the governing problem are examined by plotting graphs.

Key words: Induced Magnetic Field; PTT Fluid; Long Wavelength.

1. Introduction

The phenomenon of peristalsis in channels or tubes has received the attention of researchers in recent times. This is due to its indispensable role in transport of many physiological and industrial fluids. In physiology examples include urine transport from kidney to bladder, movement of ovum in the fallopian tubes, transport of spermatozoa in the ductus efferentes of the male reproductive tract, vasomotion of small blood vessels and many others. In industrial applications it is common in sanitary fluid transport, transport of corrosive fluids, and blood pumps in heart lung machine. The phenomenon of peristalsis also exists during the operation of roller and finger pumps. Latham [1] initiated an experimental study for the peristaltic transport of viscous fluid. Shapiro et al. [2] made a theoretical attempt for the peristaltic motion of viscous fluid under long wavelength and low Reynolds number assumption. An agreement between these two investigations is shown. After the works of Latham [1] and Shapiro et al. [2] there have been many inroads on the peristaltic flows under different reliable assumptions. Even now some of these are Mekheimer and Elmaboud [3, 4], Haroun [5], Srinivas and Kothandapani [6], Kothandapani and Srinivas [7, 8], Hayat and Ali [9, 10], and Hayat et al. [11].

Radhakrishnamacharya [12] studied the peristaltic motion of a power law fluid using long wavelength approximation. Hayat et al. [13] considered the peristaltic transport of a Johnson-Segalman fluid in an asymmetric channel. In another attempt, Hayat et al. [14] studied the endoscopic effects on the magnetohydrodynamic (MHD) peristaltic transport of a Jeffery fluid. Nadeem and Akbar [15] examined the effects of heat transfer on the peristaltic motion of an Herschel-Bulkley fluid in a non-uniform tube. The peristaltic flow of a fourth-grade fluid has been presented by Haroun [16]. Muthu et al. [17] discussed the effects of wall properties on the peristaltic motion of a micropolar fluid in circular cylindrical tubes.

Less attention has been focused on the peristaltic flows with an induced magnetic field. The investigations of peristaltic flow of couple stress and micropolar fluids under the influence of an induced magnetic field have been carried out very recently by Mekheimer [18, 19]. Hayat et al. in their study [20, 21] appear to be the first to look at the peristaltic flow of a third-order and Carreau fluid in the presence of an induced magnetic field. The purpose of the present investigation is to venture further in the regime of peristaltic flows of non-Newtonian fluids which exhibit viscoelastic and shear thinning properties. The next section consists of fundamental equations. Section 3 describes the statement of problem. Series solution is derived in Section 4. Last section discusses the salient features of the obtained expressions.

2. Basic Equations

The constitutive equations for the Phan-Thien-Tanner (PTT) model are

$$\mathbf{T} = -p\mathbf{I} + \boldsymbol{\tau}, \tag{1}$$

$$f(\mathrm{tr}(\boldsymbol{\tau}))\boldsymbol{\tau} + \kappa\boldsymbol{\tau}^{\nabla} = 2\mu\mathbf{D}, \tag{2}$$

$$\boldsymbol{\tau}^{\nabla} = \frac{\mathrm{d}\boldsymbol{\tau}}{\mathrm{d}t} - \boldsymbol{\tau}\cdot\mathbf{L}^{*} - \mathbf{L}\cdot\boldsymbol{\tau}, \tag{3}$$

$$\mathbf{L} = \mathrm{grad}\,\mathbf{V},$$

where p is the pressure, $\mathbf{I}$ is the identity tensor, $\mathbf{V}$ is the velocity tensor, $\mathbf{T}$ is the Cauchy stress tensor, μ is dynamic viscosity, $\boldsymbol{\tau}$ is an extra-stress tensor, $\mathbf{D}$ is the deformation-rate tensor, κ is the relaxation time, and $\boldsymbol{\tau}^{\nabla}$ denotes Oldroyd's upper-convected derivative, $\mathrm{d}/\mathrm{d}t$ is the material derivative, tr is the trace, and the asterik denotes the transpose.

The function f in the linearized PTT model satisfies the following expression:

$$f(\mathrm{tr}(\boldsymbol{\tau})) = 1 + \frac{\varepsilon\kappa}{\mu}\mathrm{tr}(\boldsymbol{\tau}). \tag{4}$$

Note that the PTT model reduces to an upper convected Maxwell (UCM) model when ε is zero.

In the absence of displacement current, the Maxwell's equations are

$$\boldsymbol{\nabla}\cdot\mathbf{E} = 0, \quad \boldsymbol{\nabla}\cdot\mathbf{H} = 0, \tag{5}$$

$$\boldsymbol{\nabla}\times\mathbf{E} = -\mu_{\mathrm{e}}\frac{\partial\mathbf{H}}{\partial t}, \quad \boldsymbol{\nabla}\times\mathbf{H} = \mathbf{J}, \tag{6}$$

and

$$\mathbf{J} = \sigma(\mathbf{E} + \mu_{\mathrm{e}}(\mathbf{V}\times\mathbf{H})). \tag{7}$$

In the above equations μ_{e} is the magnetic permeability, $\mathbf{J}$ is the electric current density, σ is the electrical conductivity, $\mathbf{E}$ is the electric field, and $\mathbf{H}$ the magnetic field.

3. Formulation of the Problem

Consider the magnetohydrodynamic (MHD) flow of an incompressible PTT fluid in a two-dimensional channel of uniform thickness $2a$. A sinusoidal wave of velocity c propagates on the channel walls. We choose rectangular coordinates $(\bar{X},\bar{Y})$ with $\bar{X}$ in the direction of wave propagation and $\bar{Y}$ transverse to it. A constant magnetic field of strength H_0 acts in the transverse direction. This results in an induced magnetic field $\mathbf{H}(\bar{h}_{\bar{x}}(\bar{X},\bar{Y},\bar{t}),\bar{h}_{\bar{y}}(\bar{X},\bar{Y},\bar{t}),0)$ and therefore the total magnetic field is $\mathbf{H}^{+}(\bar{h}_{\bar{x}}(\bar{X},\bar{Y},\bar{t}),H_0 + \bar{h}_{\bar{y}}(\bar{X},\bar{Y},\bar{t}),0)$. The geometry of the wall surface is

$$\bar{h}(\bar{X},\bar{t}) = a + b\sin\left(\frac{2\pi}{\lambda}(\bar{X} - c\bar{t})\right) \tag{8}$$

in which a is the channel half width, b is the wave amplitude, λ is the wavelength, c is the wave speed, and $\bar{t}$ is the time. The velocity field for the two-dimensional flow is written as

$$\mathbf{V} = [\bar{U}(\bar{X},\bar{Y},\bar{t}),\bar{V}(\bar{X},\bar{Y},\bar{t}),0]. \tag{9}$$

In laboratory frame $(\bar{X},\bar{Y})$ the flow is unsteady. However, if observed in a coordinate system moving at the wave speed c (waveframe, $\bar{x},\bar{y}$) it can be treated as steady. If $(\bar{U},\bar{V})$ and $(\bar{u},\bar{v})$ are the velocity components in the laboratory and wave frames, respectively, then

$$\begin{aligned} &\bar{x} = \bar{X} - c\bar{t}, \quad \bar{y} = \bar{Y}, \\ &\bar{u}(\bar{x},\bar{y}) = \bar{U} - c, \quad \bar{v}(\bar{x},\bar{y}) = \bar{V}, \end{aligned} \tag{10}$$

where $\bar{x}$ and $\bar{y}$ indicate the coordinates in the wave frame.

The fundamental equations which can govern the flow are

(i) the continuity equation

$$\boldsymbol{\nabla}\cdot\mathbf{V} = 0, \tag{11}$$

(ii) the momentum equation

$$\begin{aligned} \rho\frac{\mathrm{d}\mathbf{V}}{\mathrm{d}t} &= \mathrm{div}\,\mathbf{T} + \mu_{\mathrm{e}}(\boldsymbol{\nabla}\times\mathbf{H}^{+})\times\mathbf{H}^{+} \\ &= \mathrm{div}\,\mathbf{T} + \mu_{\mathrm{e}}\left[(\mathbf{H}^{+}\cdot\boldsymbol{\nabla})\mathbf{H}^{+} - \frac{\boldsymbol{\nabla}\mathbf{H}^{+^2}}{2}\right], \end{aligned} \tag{12}$$

(iii) the induction equation

$$\frac{\mathrm{d}\mathbf{H}^{+}}{\mathrm{d}t} = \boldsymbol{\nabla}\times(\mathbf{V}\times\mathbf{H}^{+}) + \frac{1}{\varsigma}\boldsymbol{\nabla}^2\mathbf{H}^{+}, \tag{13}$$

where $\varsigma = \sigma\mu_{\mathrm{e}}$ is the magnetic diffusivity.

From (2)–(13) one can write the following scalar equations:

$$\frac{\partial\bar{u}}{\partial\bar{x}} + \frac{\partial\bar{v}}{\partial\bar{y}} = 0, \tag{14}$$

$$\rho\left(\bar{u}\frac{\partial}{\partial\bar{x}}+\bar{v}\frac{\partial}{\partial\bar{y}}\right)\bar{u}+\frac{\partial\bar{p}}{\partial\bar{x}}=\frac{\partial\bar{\tau}_{xx}}{\partial\bar{x}}+\frac{\partial\bar{\tau}_{xy}}{\partial\bar{y}}$$
$$-\frac{\mu_e}{2}\left(\frac{\partial H^{+^2}}{\partial\bar{x}}\right)+\mu_e\left(\bar{h}_{\bar{x}}\frac{\partial\bar{h}_x}{\partial\bar{x}}+\bar{h}_{\bar{y}}\frac{\partial\bar{h}_x}{\partial\bar{y}}+H_0\frac{\partial\bar{h}_x}{\partial\bar{y}}\right), \tag{15}$$

$$\rho\left(\bar{u}\frac{\partial}{\partial\bar{x}}+\bar{v}\frac{\partial}{\partial\bar{y}}\right)\bar{v}+\frac{\partial\bar{p}}{\partial\bar{y}}=\frac{\partial\bar{\tau}_{yx}}{\partial\bar{x}}+\frac{\partial\bar{\tau}_{yy}}{\partial\bar{y}}$$
$$-\frac{\mu_e}{2}\left(\frac{\partial H^{+^2}}{\partial\bar{y}}\right)+\mu_e\left(\bar{h}_{\bar{x}}\frac{\partial\bar{h}_{\bar{y}}}{\partial\bar{x}}+\bar{h}_{\bar{y}}\frac{\partial\bar{h}_{\bar{y}}}{\partial\bar{y}}+H_0\frac{\partial\bar{h}_{\bar{y}}}{\partial\bar{y}}\right), \tag{16}$$

$$f\bar{\tau}_{xx}+\kappa\left(\bar{u}\frac{\partial\bar{\tau}_{xx}}{\partial\bar{x}}+\bar{v}\frac{\partial\bar{\tau}_{xx}}{\partial\bar{y}}-2\frac{\partial\bar{u}}{\partial\bar{x}}\bar{\tau}_{xx}-2\frac{\partial\bar{u}}{\partial\bar{y}}\bar{\tau}_{xy}\right)$$
$$=2\mu\frac{\partial\bar{u}}{\partial\bar{x}}, \tag{17}$$

$$f\bar{\tau}_{yy}+\kappa\left(\bar{u}\frac{\partial\bar{\tau}_{yy}}{\partial\bar{x}}+\bar{v}\frac{\partial\bar{\tau}_{yy}}{\partial\bar{y}}-2\frac{\partial\bar{v}}{\partial\bar{x}}\bar{\tau}_{yx}-2\frac{\partial\bar{v}}{\partial\bar{y}}\bar{\tau}_{yy}\right)$$
$$=2\mu\frac{\partial\bar{v}}{\partial\bar{y}}, \tag{18}$$

$$f\bar{\tau}_{zz}+\kappa\left(u\frac{\partial\bar{\tau}_{zz}}{\partial\bar{x}}+v\frac{\partial\bar{\tau}_{zz}}{\partial\bar{y}}\right)=0, \tag{19}$$

$$f\bar{\tau}_{xy}+\kappa\left(u\frac{\partial\bar{\tau}_{xy}}{\partial\bar{x}}+v\frac{\partial\bar{\tau}_{xy}}{\partial\bar{y}}-\frac{\partial\bar{v}}{\partial\bar{x}}\bar{\tau}_{xx}-\frac{\partial\bar{v}}{\partial\bar{y}}\bar{\tau}_{xy}\right.$$
$$\left.-\frac{\partial\bar{u}}{\partial\bar{x}}\bar{\tau}_{xy}-\frac{\partial\bar{u}}{\partial\bar{y}}\bar{\tau}_{yy}\right)=\mu\left(\frac{\partial\bar{u}}{\partial\bar{y}}+\frac{\partial\bar{v}}{\partial\bar{x}}\right), \tag{20}$$

$$f=1+\frac{\varepsilon\kappa}{\mu}(\bar{\tau}_{xx}+\bar{\tau}_{xy}+\bar{\tau}_{zz}). \tag{21}$$

We define the following dimensionless quantities:

$$\begin{aligned}&We=\frac{kc}{a},\quad x=\frac{\bar{x}}{\lambda},\quad y=\frac{\bar{y}}{a},\quad t=\frac{c\bar{t}}{\lambda},\\&p=\frac{a^2\bar{p}}{c\lambda\mu},\quad M^2=ReS^2R_m,\quad \delta=\frac{a}{\lambda},\\&\tau_{ij}=\frac{a\bar{\tau}_{ij}}{\mu c}(\text{ for } i,j=1,2,3),\quad u=\frac{\bar{u}}{c},\\&v=\frac{\lambda\bar{v}}{ac},\quad Re=\frac{ca\rho}{\mu},\quad R_m=\sigma\mu_e ac,\\&S=\frac{H_0}{c}\sqrt{\frac{\mu_e}{\rho}},\quad \phi=\frac{\bar{\phi}}{H_0a},\quad \Psi=\frac{\bar{\Psi}}{ac},\\&\bar{h}_{\bar{x}}=\bar{\phi}_{\bar{y}},\quad \bar{h}_{\bar{y}}=-\bar{\phi}_{\bar{x}},\\&p_m=p+\frac{1}{2}Re\delta\frac{\mu_e(H^+)^2}{\rho c^2},\quad E=\frac{-E}{cH_0\mu_e}\end{aligned} \tag{22}$$

in which δ, We, Re, R_m, S, M are wave, Weissenberg, Reynolds, magnetic Reynolds, Strommer, and Hartman numbers, respectively, p_m is the total pressure which is sum of ordinary and magnetic pressures, E is the electric field strength, Ψ is the stream function, and ϕ is the magnetic force function.

Now (8) becomes

$$h=\frac{\bar{h}}{a}=1+\alpha\sin(2\pi x), \tag{23}$$

where $\alpha=b/a$ is the amplitude ratio.

Writing

$$u=\frac{\partial\Psi}{\partial y},\ v=-\delta\frac{\partial\Psi}{\partial x},\ h_x=\frac{\partial\phi}{\partial y},\ h_y=-\delta\frac{\partial\phi}{\partial x}, \tag{24}$$

(14) is automatically satisfied and equations (15) – (21) yield

$$Re\delta\left(\frac{\partial\Psi}{\partial y}\frac{\partial}{\partial x}-\frac{\partial\Psi}{\partial x}\frac{\partial}{\partial y}\right)\frac{\partial\Psi}{\partial y}+\frac{\partial p_m}{\partial x}=\delta\frac{\partial\tau_{xx}}{\partial x}+\frac{\partial\tau_{xy}}{\partial y}$$
$$+\delta ReS^2\left(\frac{\partial\phi}{\partial y}\frac{\partial}{\partial x}-\frac{\partial\phi}{\partial x}\frac{\partial}{\partial y}\right)\frac{\partial\phi}{\partial y}+ReS^2\frac{\partial^2\phi}{\partial y^2}, \tag{25}$$

$$-Re\delta^3\left(\frac{\partial\Psi}{\partial y}\frac{\partial}{\partial x}-\frac{\partial\Psi}{\partial x}\frac{\partial}{\partial y}\right)\frac{\partial\Psi}{\partial x}+\frac{\partial p_m}{\partial y}=$$
$$\delta\left(\delta\frac{\partial\tau_{yx}}{\partial x}+\frac{\partial\tau_{yy}}{\partial y}\right)-\delta^3ReS^2\left(\frac{\partial\phi}{\partial y}\frac{\partial}{\partial x}-\frac{\partial\phi}{\partial x}\frac{\partial}{\partial y}\right)\frac{\partial\phi}{\partial x}$$
$$-Re\delta^2S^2\frac{\partial^2\phi}{\partial x\partial y}, \tag{26}$$

$$E=\frac{\partial\Psi}{\partial y}-\delta\left(\frac{\partial\Psi}{\partial y}\frac{\partial\phi}{\partial x}-\frac{\partial\Psi}{\partial x}\frac{\partial\phi}{\partial y}\right)$$
$$+\frac{1}{R_m}\left(\delta^2\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\phi, \tag{27}$$

$$f\tau_{xx}+We\left[\delta\left(\frac{\partial\Psi}{\partial y}\frac{\partial\tau_{xx}}{\partial x}-\frac{\partial\Psi}{\partial x}\frac{\partial\tau_{xx}}{\partial y}-2\frac{\partial^2\Psi}{\partial x\partial y}\tau_{xx}\right)\right.$$
$$\left.-2\frac{\partial^2\Psi}{\partial y^2}\tau_{xy}\right]=2\delta\frac{\partial^2\Psi}{\partial x\partial y}, \tag{28}$$

$$f\tau_{yy}+We\left[\delta\left(\frac{\partial\Psi}{\partial y}\frac{\partial\tau_{yy}}{\partial x}-\frac{\partial\Psi}{\partial x}\frac{\partial\tau_{yy}}{\partial y}-\delta\frac{\partial^2\Psi}{\partial x^2}\tau_{yx}\right.\right.$$
$$\left.\left.-\delta\frac{\partial^2\Psi}{\partial x\partial y}\tau_{yy}\right)-\delta\frac{\partial^2\Psi}{\partial x^2}\tau_{xy}-\frac{\partial^2\Psi}{\partial x\partial y}\tau_{yy}\right]=2\delta\frac{\partial v}{\partial y}, \tag{29}$$

$$f\tau_{zz}+We\left[\delta\left(\frac{\partial\Psi}{\partial y}\frac{\partial\tau_{zz}}{\partial x}-\frac{\partial\Psi}{\partial x}\frac{\partial\tau_{zz}}{\partial y}\right)\right]=0, \tag{30}$$

$$f\tau_{xy}+We\left[\delta\left(\frac{\partial\Psi}{\partial y}\frac{\partial\tau_{xy}}{\partial x}-\delta\frac{\partial\Psi}{\partial x}\frac{\partial\tau_{xy}}{\partial y}-\delta\frac{\partial^2\Psi}{\partial x^2}\tau_{xx}\right.\right.$$
$$\left.\left.-\frac{\partial^2\Psi}{\partial x\partial y}\tau_{xy}-\frac{\partial^2\Psi}{\partial x\partial y}\tau_{xy}\right)-\frac{\partial^2\Psi}{\partial y^2}\tau_{yy}\right]=\frac{\partial^2\Psi}{\partial y^2}-\delta^2\frac{\partial^2\Psi}{\partial x^2}, \tag{31}$$

$$f = 1 + \varepsilon We(\tau_{xx} + \tau_{yy} + \tau_{zz}). \tag{32}$$

By long wavelength approximation, the above equations take the form

$$\frac{\partial p}{\partial x} = \frac{\partial \tau_{xy}}{\partial y} + ReS^2 \frac{\partial^2 \phi}{\partial y^2}, \tag{33}$$

$$\frac{\partial p}{\partial y} = 0, \tag{34}$$

$$E = \frac{\partial \Psi}{\partial y} + \frac{1}{R_\mathrm{m}} \frac{\partial^2 \phi}{\partial y^2}, \tag{35}$$

$$f\tau_{xx} = 2We \frac{\partial^2 \Psi}{\partial y^2} \tau_{xy}, \tag{36}$$

$$f\tau_{yy} = 0 = f\tau_{zz} = 0, \tag{37}$$

$$f\tau_{xy} = -We \frac{\partial^2 \Psi}{\partial y^2} \tau_{yy} + \frac{\partial^2 \Psi}{\partial y^2}, \tag{38}$$

and (34) shows that $p \neq p(y)$ and therefore $p = p(x)$.

The subjected dimensionless boundary conditions are

$$\begin{aligned} &\Psi = 0, \quad \frac{\partial^2 \Psi}{\partial y^2} = 0, \quad \frac{\partial \phi}{\partial y} = 0 \ \text{ at } \ y = 0, \\ &\Psi = F, \quad \frac{\partial \Psi}{\partial y} = -1, \quad \phi = 0 \ \text{ at } \ y = h(x), \end{aligned} \tag{39}$$

$$F = \int_0^h \frac{\partial \Psi}{\partial y} \mathrm{d}y. \tag{40}$$

(37) depicts that $\tau_{yy} = \tau_{zz} = 0$ and the thus the trace of the stress tensor becomes τ_{xx}. Integration of (33) after using (35) subjected to the boundary condition $\tau_{xy} = 0$ at $y = 0$ (the symmetry line) gives

$$\tau_{xy} = y \frac{\mathrm{d}p}{\mathrm{d}x} - M^2(Ey - \Psi). \tag{41}$$

With the help of (37) and (38) we can write

$$\tau_{xx} = 2We\tau_{xy}^2. \tag{42}$$

From (32), (37), and (42) one obtains

$$\frac{\partial^2 \Psi}{\partial y^2} = \tau_{xy} + 2\varepsilon We^2 \tau_{xy}^3. \tag{43}$$

Substituting (41) into (43) we have

$$\begin{aligned} \frac{\partial^2 \Psi}{\partial y^2} = {} & y \frac{\mathrm{d}p}{\mathrm{d}x} - M^2(Ey - \Psi) \\ & + 2\varepsilon We^2 \left(y \frac{\mathrm{d}p}{\mathrm{d}x} - M^2(Ey - \Psi) \right)^3. \end{aligned} \tag{44}$$

4. Perturbation Solution

For the perturbation solution it is reasonable to expand the flow quantities as follows:

$$\Psi = \Psi_0 + We^2 \Psi_1 + O(We)^4, \tag{45}$$

$$F = F_0 + We^2 F_1 + O(We)^4, \tag{46}$$

$$p = p_0 + We^2 p_1 + O(We)^4, \tag{47}$$

$$\phi = \phi_0 + We^2 \phi_1 + O(We)^4. \tag{48}$$

Substituting above expressions into (39) and (44) and then comparing the coefficients of like powers of We^2, we obtain the two system.

4.1. System of Order We^0

$$\begin{aligned} \frac{\partial^2 \Psi_0}{\partial y^2} &= y \frac{\mathrm{d}p_0}{\mathrm{d}x} - M^2(Ey - \Psi_0), \\ \frac{\partial \Psi_0}{\partial y} &= -1 \ \text{ at } \ y = h, \\ \Psi_0 &= F_0 \ \text{ at } \ y = h, \end{aligned} \tag{49}$$

$$\begin{aligned} \frac{\partial^2 \phi_0}{\partial y^2} &= R_\mathrm{m}(Ey - \Psi_0), \\ \frac{\partial \phi_0}{\partial y} &= 0 \ \text{ at } \ y = 0, \\ \phi_0 &= 0 \ \text{ at } \ y = h. \end{aligned} \tag{50}$$

4.2. System of Order We^2

$$\begin{aligned} \frac{\partial^2 \Psi_1}{\partial y^2} &= y \frac{\mathrm{d}p_1}{\mathrm{d}x} + M^2 \Psi_1 \\ &\quad + 2\varepsilon \left(y \frac{\mathrm{d}p_0}{\mathrm{d}x} - M^2(Ey - \Psi_0) \right)^3, \\ \frac{\partial \Psi_1}{\partial y} &= 0 \ \text{ at } \ y = h. \\ \Psi_1 &= F_1 \ \text{ at } \ y = h, \end{aligned} \tag{51}$$

$$\begin{aligned} \frac{\partial^2 \phi_1}{\partial y^2} &= R_\mathrm{m}(-\Psi_1), \\ \frac{\partial \phi_1}{\partial y} &= 0 \ \text{ at } \ y = 0, \\ \phi_1 &= 0 \ \text{ at } \ y = h. \end{aligned} \tag{52}$$

4.3. Solution for System of Order We^0

The expressions of Ψ_0, $\frac{dp_0}{dx}$, and ϕ_0 at this order are

$$\Psi_0 = A_6\left[(F_0+h)A_1 + (-1+F_0M)yA_2+(1+F_0M)yA_3 - (F_0+h)A_4\right]\left[1+hM-1+hMA_5\right]^{-1}, \quad (53)$$

$$\frac{dp_0}{dx} = -M^2\left[(F_0-hE)M\cosh(Mh) + (1+E)\sinh(Mh)\right]\left[\cosh(Mh)-\sinh(Mh)\right]^{-1}, \quad (54)$$

$$\phi_0 = R_m A_6\left[-2(F_0+h)A_1 + (2h+h^2M-My^2 + F_0(2-h^2M^2+y^2M^2))A_2 + (2h-h^2M+My^2 + F_0(2-h^2M^2+y^2M^2))A_3 - 2(F_0+h)A_4 + M(1+hM)(h^2-y^2)A_7 + M(-1+hM)(h^2-y^2)A_8\right] \cdot\left[2M(1+hM+(-1+hM)A_9)\right]^{-1}. \quad (55)$$

4.4. Solution for System of Order We^2

The expressions at this order are

$$\Psi_1 = A_{10}\left[(F_0+h)^3M^7\varepsilon A_{11}(A_{24})A_{13} + 12M(-1+hM)(1+hM)^2(EM^2-P_1)yA_{14} - 6(A_{25})+A_{15}-6(A_{26})A_{16}+6(A_{27})A_{17} - (A_{28})A_{18} - 6(A_{29})A_{19} + 4M(1+hM)^3 \cdot(EM^2-P_1)yA_{10} + 12M(-1+hM)^2(1+hM) \cdot(EM^2-P_1)yA_{20} + 4M(-1+hM)^3(EM^2-P_1)yA_{21} + 2(A_{30})A_{22} - (F_0+h)^3M^7\varepsilon A_{23}\right] \cdot\left[4M^3(1+hM+(-1+hM)A_9)^4\right]^{-1}, \quad (56)$$

$$\frac{dp_1}{dx} = M^2\left[2E(1+hM+(-1+hM)A_9)^4 + M\{-2F_1(1+A_9)(1+hM+(-1+hM)A_9)^3 + 2(F_0+h)^3M^4\varepsilon A_{11}(12hM-8\sinh(2Mh) + \sinh(4Mh))\}\right]\left[2(1+hM+(-1+hM)A_9)^4\right]^{-1}, \quad (57)$$

$$\begin{aligned}\phi_1 = R_m\big[&(\cosh(3My)+\sinh(3My))L_1(\cosh(3Mh) + \sinh(3Mh)) + L_2(\cosh(5Mh)+\sinh(5Mh)) \\ &+ L_3(\cosh(3My)+\sinh(3My)) \\ &+ L_4(\cosh(3M(2h+y))+\sinh(3M(2h+y))) \\ &+ L_5(\cosh(M(h+2y))+\sinh(M(h+2y))) \\ &+ L_1(\cosh(3M(h+2y))+\sinh(3M(h+2y))) \\ &+ L_6(\cosh(M(3h+2y))+\sinh(M(3h+2y))) \\ &+ L_7(\cosh(M(5h+2y))+\sinh(M(5h+2y))) \\ &+ L_8(\cosh(M(7h+2y))+\sinh(M(7h+2y))) \\ &+ L_9(\cosh((M(2h+3y)))+\sinh((M(2h+3y)))) \\ &+ L_{10}(\cosh((M(4h+3y)))+\sinh(M(4h+3y))) \\ &+ L_{11}(\cosh(M(8h+3y))+\sinh(M(8h+3y))) \\ &+ L_{12}(\cosh(M(h+4y))+\sinh(M(h+4y))) \\ &+ L_{13}(\cosh(M(3h+4y))+\sinh(M(3h+4y))) \\ &+ L_{14}(\cosh(M(5h+4y))+\sinh(M(5h+4y))) \\ &+ L_8(\cosh(M(7h+4y))+\sinh(M(7h+4y))) \\ &+ L_2(\cosh(M(5h+4y))+\sinh(M(5h+4y))) \\ &+ L_{15}(\cosh(1+3My)+\sinh(1+3My)) \\ &+ L_{16}(\cosh(1+2hM+3My)+\sinh(1+2hM+3My)) \\ &+ L_{17}(\cosh(1+4hM+3My)+\sinh(1+4hM+3My)) \\ &+ L_{18}(\cosh(1+6hM+3My)+\sinh(1+6hM+3My)) \\ &+ L_{19}(\cosh(1+8hM+3My)+\sinh(1+8hM+3My))\big] \\ &\cdot\left[(L_{20}(\cosh(2Mh)+\sinh(2Mh)))^4\right]^{-1}. \end{aligned} \quad (58)$$

Putting (53) – (58) into (45) – (48) and using

$$F_0 = F - We^2F_1, \quad (59)$$

and then neglecting the terms greater than $O(We^2)$ we obtain

$$\Psi = (\cosh(3My)-\sinh(3My))(L_{21}(y)+L_{22}(y)) \cdot[(1+hM+(-1+hM)(\cosh(2Mh)+\sinh(2Mh)))]^{-1} + We^2\left[\{L_{23}(y)+L_{24}(y)-L_{25}(y)-L_{26}(y)+L_{27}(y) - L_{28}(y)-L_{29}(y)+L_{30}(y)+L_{31}(y)+L_{32}(y)+L_{33}(y) + L_{34}(y)\}\{(F+h)^3M^4\varepsilon(\cosh(3My)+\sinh(3My))\}\right] \cdot\left[4(1+hM+(-1+hM)(\cosh(2Mh)+\sinh(2Mh)))^4\right]^{-1}, \quad (60)$$

$$\frac{dp}{dx} = -M^2((F-hE)M\cosh(Mh)+(1+E)\sinh(Mh)) \cdot[hM\cosh(Mh)-\sinh(Mh)]^{-1} + We^2\left[M^2\{2L(1+hM+(-1+hM)A_9)^4 + 2(F+h)^3 \cdot M^5\varepsilon(\cosh(4Mh)+\sinh(4Mh))(12hM-8\sinh(2Mh) + \sinh(4Mh))\}\right]\left[2(1+hM+(-1+hM)A_9)^4\right]^{-1}, \quad (61)$$

$$\begin{aligned}\phi = B_1\big[&(F+h)^3M^4(1+hM)We^2\varepsilon\{(\cosh(3Mh) - \sinh(3Mh)) + (\cosh(3M(h+2y)) \\ &- \sinh(3M(h+2y)))\} + (F+h)^3M^4(-1+hM)We^2 \cdot\varepsilon(\cosh(5Mh)+\sinh(5Mh)) + B_2(\cosh(3My) \\ &+ \sinh(3My)) + 4B_3(\cosh(3M(2h+y)) + \sinh(3M(2h+y))) - 3(F+h)B_4(\cosh(M(h+2y)) \\ &+ \sinh(M(h+2y))) - 36(F+h)B_5(\cosh(M(3h+2y)) + \sinh(M(3h+2y))) + 36(F+h)B_6(\cosh(M(5h+2y)) \\ &+ \sinh(M(5h+2y))) - 3(F+h)B_7\cosh(M(7h+2y))\end{aligned}$$

$$+\sinh(M(7h+2y))-4B_9(\cosh(M(2h+3y)) + \sinh(M(2h+3y)))-36MB_{10}(\cosh(M(4h+3y)) + \sinh(M(4h+3y)))+B_{11}(\cosh(M(8h+3y)) + \sinh(M(8h+3y)))-3(F+h)B_{12}(\cosh(M(h+4y)) + \sinh(M(h+4y)))-36(F+h)B_{13}(\cosh(M(3h+4y)) + \sinh(M(3h+4y)))+36(F+h)B_{14}(\cosh(M(5h+4y)) + \sinh(M(5h+4y)))-3(F+h)B_{15}(\cosh(M(7h+4y)) + \sinh(M(7h+4y)))+M^4We^2\varepsilon(-1+hM)(F+h)^3 \cdot(\cosh(M(5h+6y))+\sinh(M(5h+6y))) +6M(1+hM)^4(-h^2+y^2)^3(\cosh(1+3My) + \sinh(1+3My))-24M(-1+hM)(1+hM)^3(h^2-y^2) \cdot(\cosh(1+M(2h+3y))+\sinh(1+M(2h+3y))) -24M(1+hM)(-1+hM)^3(h^2-y^2)(\cosh 1+3M \cdot(2h+y)+\sinh(1+3M(2h+y)))-6M(-1+hM)^4 \cdot(h^2-y^2)(\cosh(1+M(8h+3y)) + \sinh(1+M(8h+3y)))\Big], \quad (62)$$

where the involved A_i $(i=1-30)$, L_i $(i=1-34)$, and B_i $(i=1-15)$ are given in the Appendix.

The dimensionless axial induced magnetic field h_x, current density J_z, pressure rise ΔP_λ, and friction force F_λ, are given by

$$h_x=\frac{\partial\phi}{\partial y}, \quad (63)$$

$$J_z=-\frac{\partial^2\phi}{\partial y^2}, \quad (64)$$

$$\Delta P_\lambda=\int_0^1\frac{\mathrm{d}p}{\mathrm{d}x}\mathrm{d}x, \quad (65)$$

$$F_\lambda=\int_0^1 h\left(-\frac{\mathrm{d}p}{\mathrm{d}x}\right)\mathrm{d}x. \quad (66)$$

5. Discussion

In this section graphical results are presented and discussed for pressure gradient, pressure rise, frictional force, induced magnetic field, and the current density for various values of emerging parameters.

Figure 1 is plotted to show the variation of pressure rise ΔP_λ against mean flow rate θ for different values of Hartman number M. This figure shows that an increase in M causes an increase in the pressure rise. Pumping action is due to the dynamic pressure exerted by the walls on the fluid trapped between the

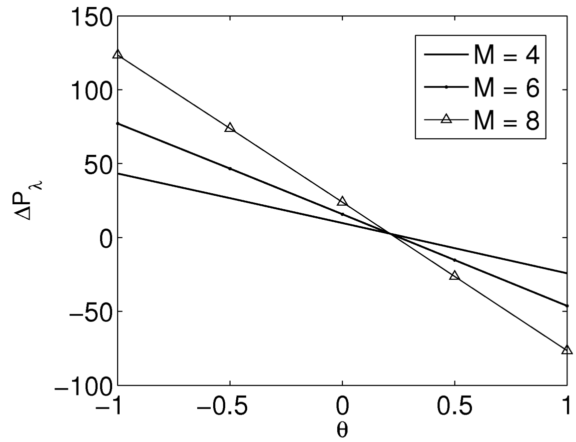

Fig. 1. Pressure rise ΔP_λ versus flow rate θ for $\alpha=0.6$, $\varepsilon=0.4$, $We=0.01$, and $E=-1$.

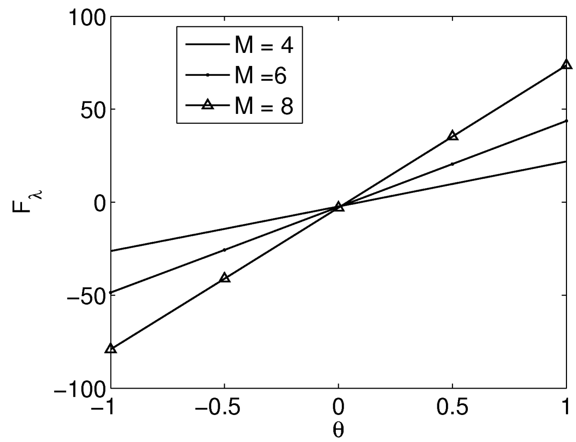

Fig. 2. Frictional force F_λ versus flow rate θ for $\alpha=0.6$, $\varepsilon=0.4$, $We=0.01$, and $E=-1$.

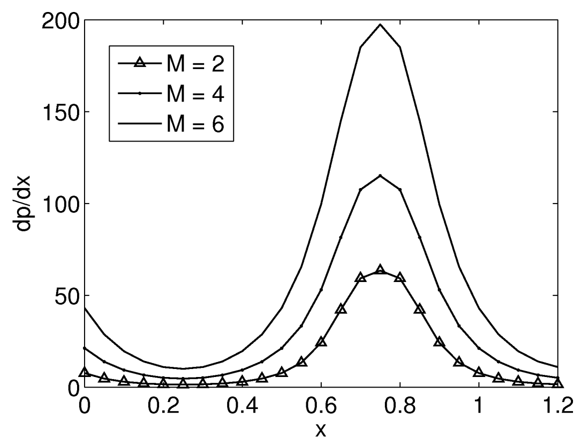

Fig. 3. Pressure gradient $\mathrm{d}p/\mathrm{d}x$ versus X for $\alpha=0.6$, $We=0.03$, $\phi=0.6$, $\varepsilon=0.4$, and $\theta=-1$.

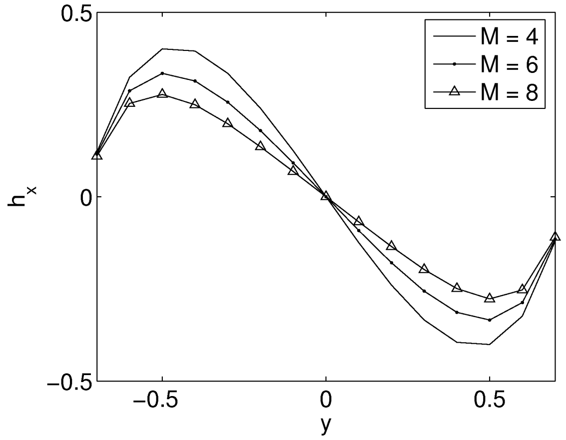

Fig. 4. Axial induced magnetic field versus Y for $\phi = 0.6$, $We = 0.01$, $M = 4$, $L = 0.8$, $R_\mathrm{m} = 1$, $\varepsilon = 0.3$, $x = 0.2$, and $\theta = 3$.

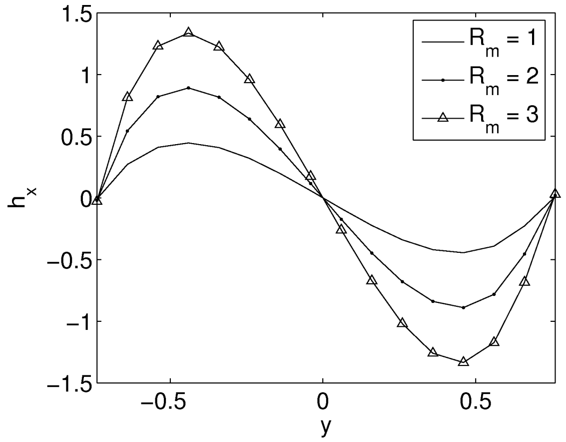

Fig. 5. Axial induced magnetic field versus Y for $\alpha = 0.6$, $We = 0.01$, $M = 3$, $\varepsilon = 0.3$, $x = 0.2$, and $\theta = 3$.

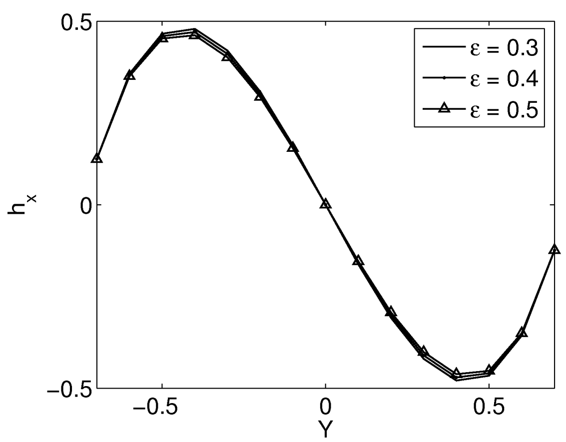

Fig. 6. Axial induced magnetic field versus Y for $\alpha = 0.6$, $We = 0.04$, $M = 1$, $R_\mathrm{m} = 1$, $x = 0.2$, and $\theta = 3$.

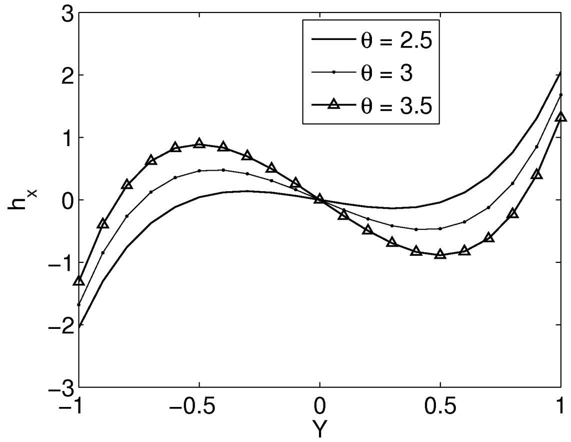

Fig. 7. Axial induced magnetic field versus Y for $\alpha = 0.6$, $We = 0.01$, $M = 4$, $R_\mathrm{m} = 1$, $\varepsilon = 0.3$, and $x = 0.2$.

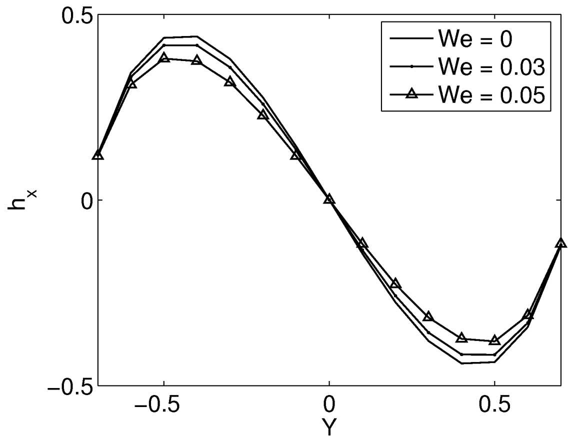

Fig. 8. Axial induced magnetic field versus Y for $\alpha = 0.6$, $M = 3$, $R_\mathrm{m} = 1$, $\varepsilon = 0.3$, $x = 0.2$, and $\theta = 3$.

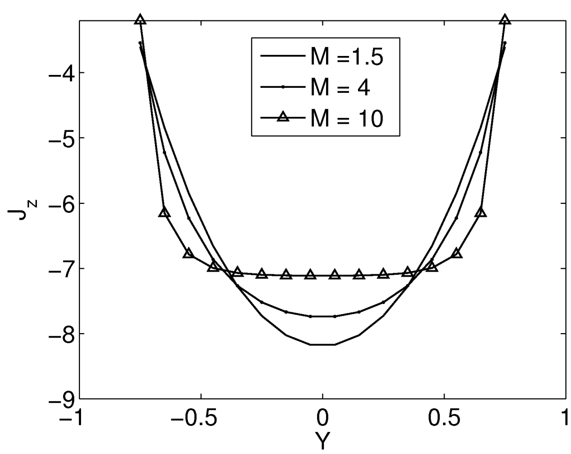

Fig. 9. Current density versus Y for $\alpha = 0.6$, $We = 0.01$, $R_\mathrm{m} = 1$, $\varepsilon = 0.4$, $x = 0.2$, and $\theta = -2$.

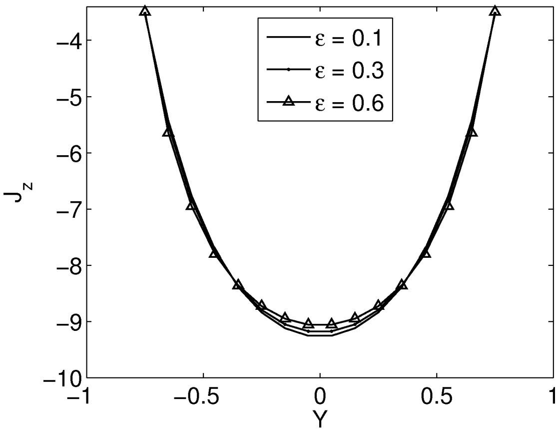

Fig. 10. Current density versus Y for $\alpha = 0.6$, $We = 0.01$, $R_m = 1$, $M = 3$, $x = 0.2$, and $\theta = -2.7$.

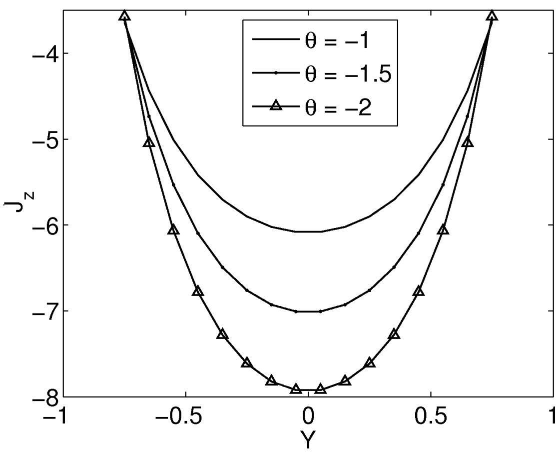

Fig. 11. Current density versus Y for $\alpha = 0.6$, $We = 0.03$, $R_m = 1$, $\varepsilon = 0.3$, $x = 0.2$, and $M = 3$.

contraction region. In the pumping region ($\Delta p_\lambda > 0$) the pumping rate increases by increasing M. The free pumping rate ($\Delta p_\lambda = 0$) increases by increasing M. In copumping ($\Delta p_\lambda < 0$) the pumping rate decreases by increasing M.

Figure 2 depicts the influence of M on the frictional force F_λ plotted against the flow rate θ. It can be seen that magnitude of frictional force decreases by increasing M.

Figure 3 represents the influence of Hartman number M on the pressure gradient for various values of M. We observe that the pressure gradient is small in the wider part of channel and large in the narrow part of channel.

Figures 4 – 8 are made to see the variation of axial induced magnetic field h_x against y. From these figures

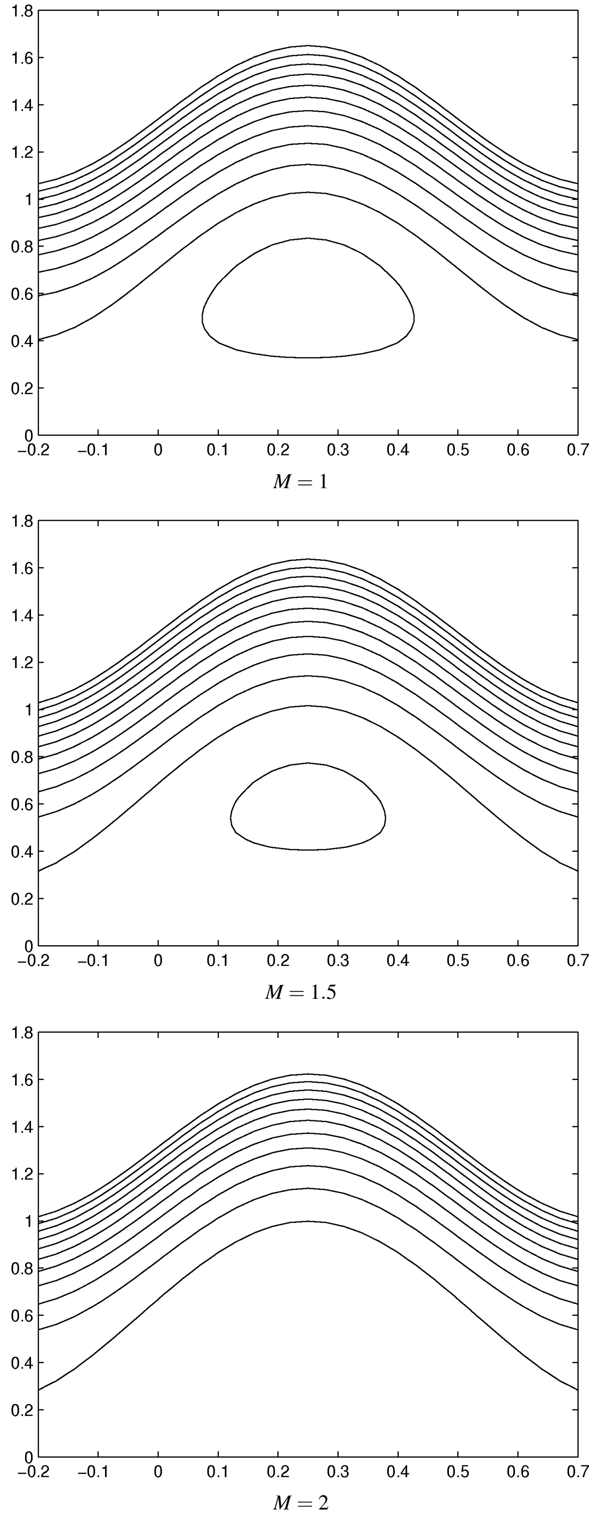

Fig. 12. Streamlines for different values of M.

the following observations are worth mentioning:

- In the half region the induced magnetic field is in one direction and in the other half it is in the opposite direction.
- The magnitude of h_x decreases as M increases.
- h_x is zero at $y = 0$.
- The magnitude of h_x increases as R_m increases.
- The magnitude of h_x decreases as the extensional parameter ε increases.
- The magnitude of the axial induced magnetic field h_x increases for increasing values of the flow rate θ.
- An increase in the value of the perturbation parameter We causes decrease in h_x.

The current density distribution J_z for different values of M, ε, and θ is plotted in Figures 9–11. The following conclusions can be extracted from these figures:

- The curves of J_z are parabolic in nature.
- The magnitude of the current density J_z increases by increasing M, ε, and θ.

Trapping is described as the bolus of fluid that moves as a whole with the wavefront. This phenomena can be seen by plotting the streamlines. Streamlines may split to trap the bolus of the fluid which then moves as a whole with wave speed. The plot of Figure 12 reflects the streamlines for different values of Hartman number M. It is observed that the volume of the bolus decreases with an increase of M and slowly disappears for large values of M.

6. Concluding Remarks

Mathematical modelling and analytical solution are presented for the peristaltic flow of MHD PTT fluid in a symmetric channel. The effect of an induced magnetic field is explicitly discussed. The salient features of peristaltic motion such as pumping against the pressure rise and trapping phenomena are discussed in detail. In addition the axial induced magnetic field and current density are briefly analyzed. The literature regarding effects of an induced magnetic field on peristalsis is scarce. The present study is therefore significant contribution in this direction.

Acknowledgement

Financial support of Higher Education Commission (HEC) of Pakistan is gratefully acknowledged.

Appendix

Here we provide the values appearing in the solutions of Ψ, $\frac{\mathrm{d}p}{\mathrm{d}x}$, and ϕ.

$$A_1 = \cosh(Mh) + \sinh(Mh),$$
$$A_2 = \cosh(My) + \sinh(My),$$
$$A_3 = \cosh(M(2h+y)) + \sinh(M(2h+y)),$$
$$A_4 = \cosh(M(h+2y)) + \sinh(M(h+2y)),$$
$$A_5 = \cosh(2Mh) - \sinh(2Mh),$$
$$A_6 = \cosh(My) - \sinh(My),$$
$$A_7 = \cosh(1+My) + \sinh(1+My),$$
$$A_8 = \cosh(1+2hM+My) + \sinh(1+2hM+My),$$
$$A_9 = \cosh(2Mh) + \sinh(2Mh),$$
$$A_{10} = \cosh(M(h+3y)) + \sinh(M(h+3y)),$$
$$A_{11} = \cosh(4Mh) + \sinh(4Mh),$$
$$A_{12} = \cosh(4My) + \sinh(4My),$$
$$A_{13} = \cosh(2M(h+y)) + \sinh(2M(h+y)),$$
$$A_{14} = \cosh(3M(h+y)) + \sinh(3M(h+y)),$$
$$A_{15} = \cosh(4M(h+y)) + \sinh(4M(h+y)),$$
$$A_{16} = \cosh(2M(2h+y)) + \sinh(2M(2h+y)),$$
$$A_{17} = \cosh(2M(3h+y)) + \sinh(2M(3h+y)),$$
$$A_{18} = \cosh(2M(4h+y)) + \sinh(2M(4h+y)),$$
$$A_{19} = \cosh(2M(h+2y)) + \sinh(2M(h+2y)),$$
$$A_{20} = \cosh(M(5h+3y)) + \sinh(M(5h+3y)),$$
$$A_{21} = \cosh(M(7h+3y)) + \sinh(M(7h+3y)),$$
$$A_{22} = \cosh(M(6h+4y)) + \sinh(M(6h+4y)),$$
$$A_{23} = \cosh(M(4h+6y)) + \sinh(M(4h+6y)),$$
$$\begin{aligned} A_{24} = {} & 2F_1M(1+hM)^3 - 2EM^2(1+hM)^4 + 2P_1 \\ & + 8hMP_1 + 12h^2M^2P_1 + 8h^3M^3P_1 + 2h^4M^4P_1 \\ & + F_0^3M^7 + 3F_0^2hM^7\varepsilon + 3F_0^2h^2M^7\varepsilon + h^3M^7\varepsilon, \end{aligned}$$
$$\begin{aligned} A_{25} = {} & -F_1M^3(-1+hM)^2(1+hM) \\ & + EM^2(-1+h^2M^2)^2 - P_1 + 2h^2M^2P_1 - h^4M^4P_1 \\ & + F_0^3M^7\varepsilon + 3F_0^2hM^7\varepsilon + 3F_0h^2M^7\varepsilon + h^3M^7\varepsilon \\ & + 2F_0^3hM^8\varepsilon + 6F_0^2h^2M^8\varepsilon + 2F_0^3hM^8\varepsilon \\ & + 6F_0^2h^2M^8\varepsilon + 2h^4M^8\varepsilon - 2F_0^3M^8y\varepsilon \\ & - 6F_0^2hM^8y\varepsilon - 6F_0^2M^8yh^2\varepsilon - 2h^3M^8y\varepsilon, \end{aligned}$$

$$A_{26} = -F_1M^3(-1+hM)(1+hM)^2 + EM^2(-1+h^2M^2)^2 - P_1 + 2h^2M^2P_1 - h^4M^4P_1 - F_0^3M^7\varepsilon - 3F_0^2hM^7\varepsilon - 3F_0h^2M^7\varepsilon - h^3M^7\varepsilon + 2F_0^3hM^8\varepsilon + 6F_0^2h^2M^8\varepsilon + 2F_0^3hM^8\varepsilon + 6F_0h^3M^8\varepsilon + 2h^4M^8\varepsilon - 2F_0^3M^8y\varepsilon - 6F_0^2hM^8y\varepsilon - 2h^3M^8y\varepsilon,$$

$$A_{27} = -F_1M^3(-1+hM)^2(1+hM) - EM^2(-1+hM)^3(1+hM) - P_1 + 2hMP_1 - 2h^3M^3P_1 + h^4M^4P_1 + F_0^3M^7\varepsilon + 3F_0^2hM^7\varepsilon + 3F_0h^2M^7\varepsilon + h^3M^7\varepsilon,$$

$$A_{28} = -2F_1M^3(-1+hM)^3 + 2EM^2(-1+hM)^4 - P_1 + 8hMP_1 - 12h^2M^2P_1 + 8h^3M^3P_1 - 2h^4M^4P_1 + F_0^3M^7\varepsilon + 3F_0^2hM^7\varepsilon + 3F_0h^2M^7\varepsilon + h^3M^7\varepsilon,$$

$$A_{29} = -F_1M^3(-1+hM)(1+hM)^2 + EM^2(-1+hM)(1+hM)^3 + P_1 + 2hMP_1 - 2h^3M^3P_1 - h^4M^4P_1 + F_0^3M^7\varepsilon + 3F_0^2hM^7\varepsilon + 3F_0h^2M^7\varepsilon + h^3M^7\varepsilon,$$

$$A_{30} = F_1M^3(-1+hM)^3 - EM^2(-1+hM)^3(1+hM) - P_1 + 2hMP_1 - 2h^3M^3P_1 + h^4M^4P_1 + F_0^3M^7\varepsilon + 3F_0^2hM^7\varepsilon + 3F_0h^2M^7\varepsilon + h^3M^7\varepsilon,$$

$$L_1 = (F_0+h)^3M^4(1+hM)\varepsilon,$$

$$L_2 = (F_0+h)^3M^4(-1+hM)\varepsilon,$$

$$L_3(y) = 6F_1(1+hM)^3(-2+h^2M^2-y^2M^2)+(F_0+h)^3 \cdot M^4(2+8hM+3h^2M^2-3y^2M^2)\varepsilon,$$

$$L_4(y) = 12F_1(-1+hM)^2(1+2hM)(-2+h^2M^2 - y^2M^2)+8(F_0+h)^3M^4(4-8hM+3h^2M^2-3y^2M^2)\varepsilon,$$

$$L_5 = 3(4F_1(1+hM)^3 - (F_0+h)^3M^4(1+3hM)\varepsilon),$$

$$L_6(y) = 36F_1(-1+hM)(1+hM)^2 - 36(F_0+h)^3M^4 \cdot(-1+h^2M^2-My-hM(1+My))\varepsilon,$$

$$L_7(y) = 36F_1(-1+hM)^2(1+hM) + 36(F_0+h)^3M^4 \cdot(-1+h^2M^2-My+hM(1+My))\varepsilon,$$

$$L_8 = 3(4F_1(-1+hM)^3 - (F_0+h)^3M^4(-1+3hM)\varepsilon),$$

$$L_9(y) = 12F_1(1+hM)^2(-1+2hM)(-2+h^2M^2- y^2M^2) - 8(F_0+h)^3M^4(4+8hM+3h^2M^2-3y^2M^2)\varepsilon,$$

$$L_{10}(y) = 36hM[-2F_1 + 3F_1h^2M^2 - F_1h^4M^4 - F_1h^2M^2 + F_1h^2M^4y^2 + 2(F_0+h)^3M^6(h^2-y^2)\varepsilon],$$

$$L_{11}(y) = 6[F_1(-1+hM)^3(-2+h^2M^2-y^2M^2) - (F_0+h)^3M^4(2-8hM+3h^2M^2-3y^2M^2)\varepsilon],$$

$$L_{12} = 3(4F_1(1+hM)^3 - (F_0+h)^3M^4(1+3hM)\varepsilon),$$

$$L_{13} = 36[F_1(-1+hM)(1+hM)^2(F_0+h)^3M^4 \cdot(-1+h^2M^2+My+hM(-1+My))\varepsilon],$$

$$L_{14}(y) = 36[F_1(-1+hM)^2(1+hM) + (F_0+h)^3M^4 \cdot(-1+h^2M^2+My+hM(1-My))\varepsilon],$$

$$L_{15}(y) = 6M(1+hM)^4(-h^2+y^2),$$

$$L_{16}(y) = 24M(-1+hM)(h^2-y^2)(1+hM)^3,$$

$$L_{17}(y) = 36M(-1+h^2M^2)^2(h^2-y^2),$$

$$L_{18}(y) = 24M(-1+hM)^3(1+hM)(h^2-y^2),$$

$$L_{19}(y) = 6M(-1+hM)^4(h^2-y^2),$$

$$L_{20} = 12M(1+hM)(-1+hM),$$

$$L_{21}(y) = (F+h)(\cosh(Mh)+\sinh(Mh)) +(-1+FM)y(\cosh(My)+\sinh(My)),$$

$$L_{22}(y) = (1+FM)y(\cosh(M(2h+y))+\sinh(M(2h +y))) - (F+h)(\cosh(M(h+2y))+\sinh(M(h+2y))),$$

$$L_{23}(y) = (1+hM)(\cosh(3Mh)+\sinh(3Mh)) +(-1+hM)(\cosh(5Mh)+\sinh(5Mh)),$$

$$L_{24}(y) = 2My(\cosh(3My)+\sinh(3My)) +16My(\cosh(3M(2h+y))+\sinh(3M(2h+y))),$$

$$L_{25}(y) = (1+3hM)(\cosh(M(h+2y)) +\sinh(M(h+2y))) + (1+hM)(\cosh(3M(h+2y)) +\sinh(3M(h+2y))),$$

$$L_{26}(y) = 12M(h^2M-y-hMy)(\cosh(M(3h+2y)) +\sinh(M(3h+2y))),$$

$$L_{27}(y) = 12M(h^2M-y-hMy)(\cosh(M(5h+2y)) +\sinh(M(5h+2y))),$$

$$L_{28}(y) = (-1+3hM)(\cosh(M(7h+2y)) +\sinh(M(7h+2y))) + 16My(\cosh(M(2h+3y)) +\sinh(M(2h+3y))),$$

$$L_{29}(y) = 16My(\cosh(M(2h+3y))+\sinh(M(2h+3y))) +48hM^2y(\cosh(M(4h+3y))+\sinh(M(4h+3y))),$$

$$L_{30}(y) = -2My(\cosh(M(8h+3y))+\sinh(M(8h+3y))) +(1+3hM)(\cosh(M(h+4y))+\sinh(M(h+4y))),$$

$L_{31}(y) = 12M(h^2M + y + hMy)(\cosh(M(3h+4y)) + \sinh(M(3h+4y))),$

$L_{32}(y) = -12M(h^2M + y - hMy)(\cosh(M(5h+4y)) + \sinh(M(5h+4y))),$

$L_{33}(y) = (-1+3hM)(\cosh(M(7h+4y)) + \sinh(M(7h+4y))),$

$L_{34}(y) = -(-1+hM)(\cosh(M(5h+6y)) + \sinh(M(5h+6y))),$

$B_1(y) = R_\mathrm{m}(\cosh(3My) - \sinh(3My))[12M(1+hM+ (-1+hM)(\cosh(2hM)+\sinh(2hM)))^4]^{-1},$

$B_2(y) = 6My^2 + 6h^2M(-7+3M^2y^2) + 6h(-2+3M^2y^2) +(F^3+3hF^2)M^4We^2\varepsilon(2+8hM+3h^2M^2-3M^2y^2) +3h^5M^4(-2+M^2We^2\varepsilon) + h^4(-30M^3+8M^5We^2\varepsilon) +h^3(-54M^2-3M^6We^2y^2\varepsilon+2M^4(3y^2+We^2\varepsilon)) +3F\{2h^5M^5-(2+6hM)(2+M^2y^2)+3h^4M^42 +M^2We^2\varepsilon+2h^3(M^3-M^5(y^2-4We^2\varepsilon)) -h^2M^2(10+3M^4We^2y^2\varepsilon+M^2(6y^2-2We^2\varepsilon))\},$

$B_3(y) = 6M(h^2-y^2)+h(-6+9M^2y^2)2(F^3+3hF^2) \cdot M^4We^2\varepsilon(4-8hM+3h^2M^2-3M^2y^2)-4h^4M^3(3 +4M^2We^2\varepsilon)+3M^4h^5(1+2M^2We^2\varepsilon)+h^3(9M^2 -6M^6y^2We^2\varepsilon+M^4(-3y^2+8We^2\varepsilon))+3F\{-2 +2h^5M^5-M^2y^2+3M^4h^4(-1+2M^2We^2\varepsilon)-2h^3M^3 \cdot(2+M^2(y^2+8We^2\varepsilon))h^2(7M^2-6M^6We^2y^2\varepsilon +M^4(3y^2+8We^2\varepsilon))\},$

$B_4(y) = -4+M^4We^2F^2\varepsilon+M^3h^3(-4+3M^2We^2\varepsilon) +M^2h^2(-12+M^2We^2\varepsilon+6FM^3e^2\varepsilon) +Mh(-12+2FM^3We^2\varepsilon+3F^2M^4We^2\varepsilon),$

$B_5(y) = 1+M^6h^4We^2\varepsilon-We^2\varepsilon M^4F^2(1+My) +M^3h^3(-1-M^2We^2\varepsilon+M^3We^2\varepsilon(2F-y)) +M^2h^2(-1-M^2We^2\varepsilon+FM^4We^2\varepsilon(1+My) +M^3We^2\varepsilon(2F+y))-hM(-1+2FM^3We^2\varepsilon(1+My) +F^2M^4We^2\varepsilon(1+My)),$

$B_6(y) = 1+M^6h^4We^2\varepsilon-F^2h^4We^2\varepsilon(1+My) +h^3(M^3+M^5We^2\varepsilon+M^6We^2\varepsilon(2F+y)) +M^2h^2(-1-M^2We^2\varepsilon+M^3We^2\varepsilon(2F-y) +FM^4We^2\varepsilon(2F+y))+hM(-1-2FM^3We^2\varepsilon(1+My) +F^2M^4We^2\varepsilon(1+My)),$

$B_7(y) = 4-F^2M^4We^2\varepsilon+M^3h^3(-4+3M^2We^2\varepsilon) +M^2h^2(12-M^2We^2\varepsilon+6FM^3We^2\varepsilon) +hM(-12-2FM^3We^2\varepsilon+2F^2M^4We^2\varepsilon),$

$B_8(y) = 2+2M^5h^5+M^2y^2+h^4(3M^4-6M^6We^2\varepsilon) -2M^3h^3(2+M^2(y^2+8We^2\varepsilon))+M^2h^2(-7+6M^4y^2 \cdot We^2\varepsilon-M^2(3y^2+8We^2\varepsilon)),$

$B_9(y) = h(-6+9M^2y^2)+2(F^3+3hF^2)M^4We^2\varepsilon \cdot(4+8hM+3M^2(h^2-y^2))6M(y^2-h^2)+4h^4M^3 \cdot(3+4M^2We^2\varepsilon)+3h^5M^4(1+2M^2We^2\varepsilon)+h^3(9M^2 -6y^2M^6We^2\varepsilon+M^4(-3y^2+8We^2\varepsilon))-3FB_8,$

$B_{10}(y) = h^2(-1+M^2y^2)+2M^6h^6We^2\varepsilon-(2F^3h +6F^2h^2)M^6We^2\varepsilon(y^2-h^2)+h^4(M^2-2M^6y^2We^2\varepsilon) -y^2+Fh\{-2-M^2y^2+M^4h^4(-1+6M^2We^2\varepsilon) +h^2(3M^2+M^4y^2-6y^2M^6We^2\varepsilon)\},$

$B_{11}(y) = 6My^2+6Mh^2(-7+3M^2y^2)-6h(-2+3M^2y^2) +(F^3+3hF^2)M^4We^2\varepsilon(-2+8hM+3M^2(y^2-h^2)) +h^4(-30M^3+8M^5We^2\varepsilon)+h^5(6M^4-3M^6We^2\varepsilon) +h^3M^2(54+3y^2M^4We^2\varepsilon-2M^2(3y^2+We^2\varepsilon))3F \cdot\{4+2h^5M^5+2y^2M^2-6Mh(2+y^2M^2) -3h^4M^4(2+We^2M^2\varepsilon)+2h^3(M^3-M^5(y^2-4We^2\varepsilon)) +h^2M^2(10+3y^2We^2M^4\varepsilon+M^2(6y^2-2We^2\varepsilon))\},$

$B_{12}(y) = h^3M^3(-4+3M^2We^2\varepsilon)+h^2M^2 \cdot(-12+We^2M^2\varepsilon+6FM^3We^2\varepsilon)-4+3F^2We^2M^4\varepsilon +hM(-12+2FWe^2M^3\varepsilon+3F^2We^2M^4\varepsilon),$

$B_{13}(y) = 1+M^6h^4We^2\varepsilon+M^4F^2We^2\varepsilon(-1+My) +h(M-2FM^4We^2\varepsilon-FM^5We^2\varepsilon(F-2y) +M^6F^2yWe^2\varepsilon)+h^3M^3(-1-M^2We^2\varepsilon +M^3We^2\varepsilon(2F+y))+h^2M^2\{-1-M^2We^2\varepsilon +M^3We^2\varepsilon(-2F+y)+FM^4We^2\varepsilon(F+2y)\},$

$B_{14}(y) = 1+h^4M^6We^2\varepsilon+F^2M^4We^2\varepsilon(-1+My) +h^3(M^3+M^5We^2\varepsilon+M^6We^2\varepsilon(2F-y)) +h^2M^2(-1-M^2We^2\varepsilon+FM^4We^2\varepsilon(F-2y) +M^3We^2\varepsilon(F+2y))-h(M+2FM^4We^2\varepsilon +F^2M^6yWe^2\varepsilon-FM^5We^2\varepsilon(F+2y)),$

$B_{15}(y) = 4-F^2M^4We^2\varepsilon+h^3M^3(-4+3M^2We^2\varepsilon) +h^2M^2(12-M^2We^2\varepsilon+6FM^3We^2\varepsilon) \cdot hM(-12-2FM^3We^2\varepsilon+3F^2M^4We^2\varepsilon).$

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