Adomian Decomposition Method for Approximating the Solutions of the Bidirectional Sawada-Kotera Equation

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In this paper, the decomposition method is implemented for solving the bidirectional Sawada-Kotera (bSK) equation with two kinds of initial conditions. As a result, the Adomian polynomials have been calculated and the approximate and exact solutions of the bSK equation are obtained by means of Maple, such as solitary wave solutions, doubly-periodic solutions, two-soliton solutions. Moreover, we compare the approximate solution with the exact solution in a table and analyze the absolute error and the relative error. The results reported in this article provide further evidence of the usefulness of the Adomian decomposition method for obtaining solutions of nonlinear problems.

Key words: Adomian Decomposition Method; Bidirectional Sawada-Kotera Equation; Adomian Polynomials.

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1. Introduction

Nonlinear phenomena play a crucial role in applied mathematics and physics. Explicit solutions to the nonlinear equations are of fundamental importance. Various methods for obtaining explicit solutions to nonlinear evolution equations have been proposed [1 – 11]. Among them are Hirota's bilinear transformation method, the inverse scattering transform, and the Bäcklund transformation. A feature common to all these methods is that they use transformations to reduce the equation to a more simple one and then solve it. Unlike classical techniques, the nonlinear equations are solved easily and elegantly without transforming the equation by using the Adomian decomposition method (ADM) [12 – 16].

Adomian polynomials decompose a function $\Phi(x,t)$ into a sum of components

$$\Phi(x,t) = \sum_{n=0}^{\infty} \Phi_n(x,t), \tag{1}$$

for a nonlinear operator F as

$$F[\Phi(x,t)] = \sum_{n=0}^{\infty} A_n, \tag{2}$$

where the components Φ_n will be determined recurrently, and A_n are the so-called Adomian polynomials of $\Phi_0, \Phi_1, \dots, \Phi_n$ defined by

$$A_n = \frac{1}{n!}\frac{\mathrm{d}^n}{\mathrm{d}\lambda^n}\left[F\left(\sum_{i=0}^{\infty}\lambda^i \Phi_i\right)\right]_{i=0}, \quad n = 0,1,2,\dots. \tag{3}$$

These polynomials have the property that A_n depends only on $\Phi_0, \Phi_1, \dots, \Phi_n$ and that the sum of subscripts for the component Φ_n is equal to n.

This technique has many advantages, mainly, it avoids linearization and perturbation in order to find solutions of a given nonlinear equation. On the other hand, the ADM provides efficiently explicit and numerical solutions with high accuracy.

Here we consider the bidirectional Sawada-Kotera (bSK) equation [17]

$$\begin{aligned} &5\partial_x^{-1}u_{tt} + 5u_{xxt} - 15uu_t - 15u_x\partial_x^{-1}u_t - 45u^2u_x \\ &\quad + 15u_xu_{xx} + 15uu_{xxx} - u_{xxxxx} = 0. \end{aligned} \tag{4}$$

It was formulated there as a bidirectional generalization of the Sawada-Kotera (SK) equation [18]

$$u_t + 45u^2u_x - 15u_xu_{xx} - 15uu_{xxx} + u_{xxxxx} = 0. \tag{5}$$

By constructing its bilinear form, we are able to identify the bSK equation with the well-known Ramani equation [19]. The bSK equation has likewise received considerable attention over the last two decades. Authors of [20] pointed out its bidirectional nature and relation to the SK equation (5) through the normal form

(4). The mathematical properties that assure the complete integrability of the bSK equation have largely been established. In view of its connection with the SK equation (5), the bSK equation (4) also belong to the Kadomtsev-Petviashvili equations of B type (BKP) hierarchy [17]. The equation appears in [21] neither in its normal form (4) nor in the form of Ramani's (7), but rather as the coupled bilinear form with $u(x,t) = -\partial_x^2 \ln f(x,t)$,

$$\begin{aligned}&(80\mathrm{D}_t^2 + 20\mathrm{D}_x^3\mathrm{D}_t - \mathrm{D}_x^6) f\cdot f + 30(4\mathrm{D}_x\mathrm{D}_t - \mathrm{D}_x^4) f\cdot g = 0,\\ &\mathrm{D}_x^2 f\cdot f - 2f\cdot g = 0,\end{aligned} \tag{6}$$

in which $g(x,t)$ is some auxiliary function, which can be found in [20]. On the other hand, when $u = -2\partial_x^2 \ln f(x,t)$ one obtains the single bilinear equation

$$(5\mathrm{D}_t^2 + 5\mathrm{D}_x^3\mathrm{D}_t - \mathrm{D}_x^6) f\cdot f = 0. \tag{7}$$

To our knowledge, the Adomian decomposition method was not extended to investigate approximate solutions of the bSK equation. In this paper, we extend the decomposition method to the bSK equation and seek numerical solutions, such as solitary wave solutions, two-soliton solutions, and doubly-periodic solutions.

2. Description of the ADM

In this section, our attention will focus on the bSK equation with the initial condition based on the ADM. Firstly, we perform the transformation $w = \partial_x^{-1} u_t$ to (4), then we have a couple of equations

$$\begin{aligned}&5w_t + 5w_{xxx} - 15uw_x - 15u_x w - 45u^2 u_x\\ &\quad + 15u_x u_{xx} + 15uu_{xxx} - u_{xxxxx} = 0,\end{aligned} \tag{8}$$

$$u_t - w_x = 0, \tag{9}$$

with the initial conditions $w(x,0) = g_1(x), u(x,0) = g_2(x)$.

Next, we describe the algorithm of the ADM as it applies to the bSK equation. For that we rewrite (8) and (9) in the following operator form:

$$\begin{aligned}L_t w = &-L_{3x} w + 3uw_x + 3u_x w + 9u^2 u_x\\ &- 3u_x u_{xx} - 3uu_{xxx} + \frac{1}{5} L_{5x} u,\end{aligned} \tag{10}$$

$$L_t u = L_x w. \tag{11}$$

Following [12, 13] we define for (10) and (11) the linear operators

$$L_t \equiv \frac{\partial}{\partial t}, \quad L_x \equiv \frac{\partial}{\partial x}, \quad L_{3x} \equiv \frac{\partial^3}{\partial x^3}, \quad L_{5x} \equiv \frac{\partial^5}{\partial x^5}. \tag{12}$$

By defining the onefold right-inverse operator

$$L_t^{-1} \equiv \int_0^t (.)\mathrm{d}t, \tag{13}$$

without loss of generality, we choose L_t^{-1} to operate on both sides of (10) and (11) to obtain

$$\begin{aligned}w(x,t) = &\, w(x,0) + L_t^{-1}\Big[- L_{3x} w + 3uw_x\\ &+ 3u_x w + 9u^2 u_x - 3u_x u_{xx} - 3uu_{xxx} + \frac{1}{5} L_{5x} u\Big],\end{aligned} \tag{14}$$

$$u(x,t) = u(x,0) + L_t^{-1}[L_x w]. \tag{15}$$

Therefore,

$$\begin{aligned}w(x,t) = g_1(x) + L_t^{-1}\Big[&- L_{3x} w + 3N_1(u,w)\\ &+ 3N_2(u,w) + 9N_3(u,w) - 3N_4(u,w)\\ &- 3N_5(u,w) + \frac{1}{5} L_{5x} u\Big],\end{aligned} \tag{16}$$

$$u(x,t) = g_2(x) + L_t^{-1}[L_x w], \tag{17}$$

where $N_i(u,w), i = 1,\ldots,4$, are the nonlinear operators. The decomposition method [12–16] suggests that the terms $u(x,t)$ and $w(x,t)$ can be decomposed by an infinite series of components

$$w(x,t) = \sum_{n=0}^{\infty} w_n(x,t), \quad u(x,t) = \sum_{n=0}^{\infty} u_n(x,t) \tag{18}$$

and the nonlinear operators $N_1(u,w)$, $N_2(u,w)$, $N_3(u,w)$, $N_4(u,w)$, and $N_5(u,w)$ by the infinite series

$$N_i(u,w) = \sum_{n=0}^{\infty} A_{i,n}, \quad i = 1,2,\ldots,5. \tag{19}$$

That means that the nonlinear terms uw_x, $u_x w$, $u^2 u_x$, $u_x u_{xx}$, and uu_{xxx} are represented by series of $A_{i,n}$, $i = 1,2,\ldots,5$, which are called Adomian polynomials.

Next we determine $w(x,t)$ and $u(x,t)$. Hence, we obtain the components series solutions by the following recursive relationship:

$$\begin{aligned}&w_0 = g_1(x), \quad u_0 = g_2(x),\\ &w_{n+1} = L_t^{-1}\Big[- L_{3x} w_n + 3A_{1,n} + 3A_{2,n} + 9A_{3,n}\\ &\qquad - 3A_{4,n} - 3A_{5,n} + \frac{1}{5} L_{5x} u_n\Big],\\ &u_{n+1} = L_t^{-1}[L_x w_n],\end{aligned} \tag{20}$$

where $n \geq 0$.

The Adomian polynomials $A_{i,n}$, $i=1,2,\ldots,5$ can be generated for all forms of nonlinearity which are generated according to the following algorithm:

$$A_{i,n}=\frac{1}{n!}\left[\frac{\mathrm{d}^n}{\mathrm{d}\lambda^n}N_i\left(\sum_{k=0}^{n}\lambda^k w_k,\sum_{k=0}^{n}\lambda^k u_k\right)\right]_{\lambda=0},\quad n\ge 0. \tag{21}$$

This formulae is easy to set as computer code to get as many polynomial as we need in the calculation.

3. Numerical Doubly-Periodic Solutions

In the following we would like to consider the numerical doubly-periodic solutions of (8) and (9) with the following initial conditions:

$$\begin{aligned}u(x,0)&=\frac{1}{3k}(\omega-4k^3m^2-4k^3)+4k^2m^2\mathrm{sn}(\Theta)^2,\\ w(x,0)&=\frac{1}{15k^2}\big(16k^6m^2-16k^6m^4+5\omega^2\\ &\quad+20k^3\omega-20k^3\omega m^2-16k^6\big)\\ &\quad-4\omega km^2\mathrm{sn}(\Theta)^2,\end{aligned}\tag{22}$$

where $\mathrm{sn}(\Theta)\equiv\mathrm{sn}(kx,m)$ is the Jacobian elliptic sine function, m is the modulus of Jacobi elliptic functions. Applying the inverse operator L_t^{-1} to the bSK equation and using the decomposition series (18) and (19) yields

$$\begin{aligned}\sum_{n=0}^{\infty}w_n(x,t)&=-\frac{1}{15k^2}\big(-16k^6m^2+16k^6m^4-5\omega^2\\ &-20k^3\omega-20k^3\omega m^2+16k^6\big)-4\omega km^2\mathrm{sn}(\Theta)^2\\ &+L_t^{-1}\Big[-L_{3x}\sum_{n=0}^{\infty}w_n+3\sum_{n=0}^{\infty}A_{1,n}+3\sum_{n=0}^{\infty}A_{2,n}+9\sum_{n=0}^{\infty}A_{3,n}\\ &-3\sum_{n=0}^{\infty}A_{4,n}-3\sum_{n=0}^{\infty}A_{5,n}+\frac{1}{5}L_{5x}\sum_{n=0}^{\infty}u_n\Big],\end{aligned}\tag{23}$$

$$\begin{aligned}\sum_{n=0}^{\infty}u_n(x,t)&=-\frac{1}{3k}(-\omega+4k^3m^2+4k^3)\\ &\quad+4k^2m^2\mathrm{sn}(\Theta)^2+L_t^{-1}\left[L_x\textstyle\sum_{n=0}^{\infty}w_n\right],\end{aligned}\tag{24}$$

where $A_{i,n}$, $i=1,2,\ldots,5$ is the Adomian polynomial that represent the nonlinear operator $uw_x, u_xw, u^2u_x, u_xu_{xx}$, and uu_{xxx}, respectively. Setting $w_0=w(x,0), u_0=u(x,0)$ and making the symbolic computation (Maple), we have

$$\begin{aligned}w_1&=L_t^{-1}\Big[-L_{3x}w_0+3A_{1,0}+3A_{2,0}+9A_{3,0}-3A_{4,0}\\ &\quad-3A_{5,0}+\frac{1}{5}L_{5x}u_0\Big]=8km^2\mathrm{cn}(\Theta)\mathrm{dn}(\Theta)\mathrm{sn}(\Theta)t\omega^2,\end{aligned}$$

$$u_1=L_t^{-1}[L_xw_0]=-8\omega k^2m^2\mathrm{sn}(\Theta)\mathrm{cn}(\Theta)\mathrm{dn}(\Theta)t,\tag{25}$$

$$\begin{aligned}w_2&=L_t^{-1}\Big[-L_{3x}w_1+3A_{1,1}+3A_{2,1}+9A_{3,1}\\ &-3A_{4,1}-3A_{5,1}+\frac{1}{5}L_{5x}u_1\Big]=-12k\omega^3m^4\mathrm{sn}(\Theta)^4t^2\\ &+8k\omega^3t^2m^2(1+m^2)\mathrm{sn}(\Theta)^2-4k\omega^3m^2t^2,\\ u_2&=L_t^{-1}[L_xw_1]=12k^2m^4\mathrm{sn}(\Theta)^4t^2\omega^2\\ &-8k^2t^2\omega^2m^2(1+m^2)\mathrm{sn}(\Theta)^2+4k^2m^2t^2\omega^2,\end{aligned}\tag{26}$$

$$\begin{aligned}w_3&=L_t^{-1}\Big[-L_{3x}w_2+3A_{1,2}+3A_{2,2}+9A_{3,2}-3A_{4,2}\\ &-3A_{5,2}+\frac{1}{5}L_{5x}u_2\Big]=16km^4\omega^4t^3\mathrm{sn}(\Theta)^3\mathrm{cn}(\Theta)\mathrm{dn}(\Theta)\\ &-\frac{16}{3}km^2\omega^4t^3(1+m^2)\mathrm{cn}(\Theta)\mathrm{dn}(\Theta)\mathrm{sn}(\Theta),\\ u_3&=L_t^{-1}[L_xw_2]=-16k^2\omega^3m^4t^3\mathrm{sn}(\Theta)^3\mathrm{cn}(\Theta)\mathrm{dn}(\Theta)\\ &+\frac{16}{3}k^2\omega^3m^2t^3(1+m^2)\mathrm{cn}(\Theta)\mathrm{dn}(\Theta)\mathrm{sn}(\Theta),\end{aligned}\tag{27}$$

$$\begin{aligned}w_4&=L_t^{-1}\Big[-L_{3x}w_3+3A_{1,3}+3A_{2,3}+9A_{3,3}\\ &-3A_{4,3}-3A_{5,3}+\frac{1}{5}L_{5x}u_3\Big]=-20k\omega^5m^6t^4\mathrm{sn}(\Theta)^6\\ &+20k\omega^5t^4m^4(1+m^2)\mathrm{sn}(\Theta)^4-k\omega^5t^4\Big(\frac{8}{3}m^2+\frac{8}{3}m^6\\ &+\frac{52}{3}m^4\Big)\mathrm{sn}(\Theta)^2+\frac{4}{3}k\omega^5t^4m^2(1+m^2),\\ u_4&=L_t^{-1}[L_xw_3]=20k^2\omega^4m^6t^4\mathrm{sn}(\Theta)^6\\ &-20k^2\omega^4t^4m^4(1+m^2)\mathrm{sn}(\Theta)^4+k^2\omega^4t^4\Big(\frac{52}{3}m^4\\ &+\frac{8}{3}m^6+\frac{8}{3}m^2\Big)\mathrm{sn}(\Theta)^2-\frac{4}{3}k^2\omega^4m^2t^4(1+m^2),\end{aligned}\tag{28}$$

$$\begin{aligned}w_5&=L_t^{-1}\Big[-L_{3x}w_4+3A_{1,4}+3A_{2,4}+9A_{3,4}-3A_{4,4}\\ &-3A_{5,4}+\frac{1}{5}L_{5x}u_4\Big]=24k\omega^6m^6t^5\mathrm{sn}(\Theta)^5\mathrm{cn}(\Theta)\mathrm{dn}(\Theta)\\ &-16k\omega^6m^4t^5(1+m^2)\mathrm{dn}(\Theta)\mathrm{cn}(\Theta)\mathrm{sn}(\Theta)^3+k\omega^6t^5\\ &\cdot\Big(\frac{16}{15}m^2+\frac{16}{15}m^6+\frac{104}{15}m^4\Big)\mathrm{dn}(\Theta)\mathrm{cn}(\Theta)\mathrm{sn}(\Theta),\\ u_5&=L_t^{-1}[L_xw_4]=-24k^2\omega^5m^6\mathrm{sn}(\Theta)^5\mathrm{cn}(\Theta)\mathrm{dn}(\Theta)t^5\\ &+16k^2\omega^5t^5m^4(1+m^2)\mathrm{dn}(\Theta)\mathrm{cn}(\Theta)\mathrm{sn}(\Theta)^3\\ &-k^2\omega^5t^5\Big(\frac{16}{15}m^6+\frac{16}{15}m^2+\frac{104}{15}m^4\Big)\mathrm{dn}(\Theta)\mathrm{cn}(\Theta)\mathrm{sn}(\Theta),\end{aligned}\tag{29}$$

$$\begin{aligned}w_6&=L_t^{-1}\Big[-L_{3x}w_5+3A_{1,5}+3A_{2,5}+9A_{3,5}\\ &-3A_{4,5}-3A_{5,5}+\frac{1}{5}L_{5x}u_5\Big]=-28k\omega^7m^8t^6\mathrm{sn}(\Theta)^8\end{aligned}$$

$$+\frac{112}{3}k\omega^7t^6m^6(1+m^2)\mathrm{sn}(\Theta)^6-k\omega^7t^6\frac{56}{5}(m^8+m^4+4m^6)\mathrm{sn}(\Theta)^4+k\omega^7t^6\Big(\frac{32}{3}m^6+\frac{32}{3}m^4+\frac{16}{45}m^2+\frac{16}{45}k\omega^7m^8t^6\Big)\mathrm{sn}(\Theta)^2-k\omega^7t^6\Big(\frac{52}{45}m^4+\frac{8}{45}m^2+\frac{8}{45}m^6\Big),$$

$$u_6=L_t^{-1}[L_xw_5]=28k^2\omega^6m^8\mathrm{sn}(\Theta)^8t^6-\frac{112}{3}k^2\omega^6t^6m^6\cdot(1+m^2)\mathrm{sn}(\Theta)^6+k^2\omega^6t^6\frac{56}{5}(m^8+m^4+4m^6)\mathrm{sn}(\Theta)^4-k^2\omega^6t^6\Big(\frac{16}{45}m^8+\frac{32}{3}m^6+\frac{32}{3}m^4+\frac{16}{45}m^2\Big)\mathrm{sn}(\Theta)^2+k^2\omega^6t^6\Big(\frac{52}{45}m^4+\frac{8}{45}m^2+\frac{8}{45}m^6\Big),$$

$$\ldots. \quad (30)$$

Therefore, we have the approximate solution of (8) and (9) as

$$\begin{aligned}w(x,t)&=w_0(x,t)+w_1(x,t)+w_2(x,t)+w_3(x,t)\\&+w_4(x,t)+w_5(x,t)+w_6(x,t)+\ldots,\\u(x,t)&=u_0(x,t)+u_1(x,t)+u_2(x,t)+u_3(x,t)\\&+u_4(x,t)+u_5(x,t)+u_6(x,t)+\ldots,\end{aligned}\quad (31)$$

where $w_i(x,t),u_i(x,t),i=0,1,\ldots$ are given by (22), (25)–(30).

In order to proof numerically whether the new scheme obtained from the Adomian decomposition method for numerical doubly-periodic solutions of (8) and (9) leads to accuracy, we evaluate the approximate solution using the five-term approximation

$$\phi_{\mathrm{appr}}=w_0(x,t)+w_1(x,t)+w_2(x,t)+w_3(x,t)+w_4(x,t), \quad (32)$$

$$\varphi_{\mathrm{appr}}=u_0(x,t)+u_1(x,t)+u_2(x,t)+u_3(x,t)+u_4(x,t), \quad (33)$$

where $w_n,u_n,\ n=0,1,\ldots,4$ is given by (22), (25)–(28).

4. Numerical Two-Soliton Solutions

In this section, we consider the application of the decomposition method to (8) and (9) with the following initial valuable problem:

$$\begin{aligned}w(x,0)&=-22\big[-2\exp(2x+2)-\exp(3x+3)-11\exp(x+1)\big]\big[11+22\exp(x+1)+\exp(2x+2)\big]^{-2},\\u(x,0)&=-22\big[22\exp(x+1)+4\exp(2x+2)+2\exp(3x+3)\big]\big[11+22\exp(x+1)+\exp(2x+2)\big]^{-2}.\end{aligned}\quad (34)$$

Setting $w_0=w(x,0),u_0=u(x,0)$, the corresponding recursive relationship is

$$\begin{aligned}w_0&=[44\exp(2x+2)+22\exp(3x+3)+242\exp(x+1)]g^{-2},\\u_0&=-22[22\exp(x+1)+4\exp(2x+2)+2\exp(3x+3)]g^{-2},\\w_{n+1}&=L_t^{-1}\Big[-L_{3x}w_n+3A_{1,n}+3A_{2,n}+9A_{3,n}-3A_{4,n}-3A_{5,n}+\frac{1}{5}L_{5x}u_n\Big],\\u_{n+1}&=L_t^{-1}[L_xw_n],\quad n\ge 0.\end{aligned}\quad (35)$$

Using the Adomian polynomials (21) the recursive relationship (35) gives

$$w_1=\frac{22}{5}t\exp(x+1)[7\exp(4x+4)+54\exp(3x+3)-594\exp(x+1)-847]g^{-3},$$

$$u_1=-22t\exp(x+1)[198\exp(x+1)-18\exp(3x+3)+\exp(4x+4)-121]g^{-3},$$

$$w_2=-\frac{44}{5}t^2\exp(x+1)[-2\exp(6x+6)+25\exp(5x+5)+803\exp(4x+4)+2772\exp(3x+3)+8833\exp(2x+2)+3025\exp(x+1)-2662]g^{-4},$$

$$u_2=\frac{11}{5}t^2\exp(x+1)[1573\exp(4x+4)+200\exp(5x+5)-7\exp(6x+6)+4752\exp(3x+3)+24200\exp(x+1)+17303\exp(2x+2)-9317]g^{-4}, \quad (36)$$

$$w_3=\frac{11}{75}t^3\exp(x+1)[47\exp(8x+8)-3602\exp(7x+7)-87098\exp(6x+6)-394702\exp(5x+5)+4341722\exp(3x+3)+10538858\exp(2x+2)+4794262\exp(x+1)-688127]g^{-5},$$

$$u_3=\frac{44}{15}t^3\exp(x+1)[29282+182\exp(7x+7)-8228\exp(5x+5)-2\exp(8x+8)+1463\exp(6x+6)+90508\exp(3x+3)-177023\exp(2x+2)-242242\exp(x+1)]g^{-5},$$

$$\ldots,$$

where

$$g=11+22\exp(x+1)+\exp(2x+2).$$

Table 1. Numerical results (in x-direction) for absolute and relative error, where $w(x,t) = \frac{17}{15} - \mathrm{sn}(-x+0.1,0.5)^2, u(x,t) = -\frac{4}{3} + \mathrm{sn}(-x+0.1,0.5)^2$ for (8) and (9).

spatial variable	temporal variable	absolute error		relative error	
x_i	t_i	$\lvert w(x,t) - \phi_{\mathrm{appr}}(x,t)\rvert$	$\lvert u(x,t) - \varphi_{\mathrm{appr}}(x,t)\rvert$	$\frac{\lvert w(x,t)-\phi_{\mathrm{appr}}(x,t)\rvert}{\lvert w(x,t)\rvert}$	$\frac{\lvert u(x,t)-\varphi_{\mathrm{appr}}(x,t)\rvert}{\lvert u(x,t)\rvert}$
2	0.1	0.9917586342e-2	0.9917586342e-2	0.6834193216e-1	0.2873686841e-1
2	0.2	0.2870741159e-1	0.2870741158e-1	0.2101237591	0.8528099447e-1
2	0.3	0.5719222723e-1	0.5719222723e-1	0.4287784375	0.1715505531
5	0.1	0.3950771439e-2	0.3950771439e-2	0.2829972079e-1	0.1163344481e-1
5	0.2	0.1738277575e-1	0.1738277575e-1	0.1154856874	0.4959155419e-1
5	0.3	0.3952410864e-1	0.3952410864e-1	0.2357809014	0.1075103703
10	0.1	0.2605698235e-1	0.2605698235e-1	0.2514664468e-1	0.2107827045e-1
10	0.2	0.6457769070e-1	0.6457769080e-1	0.6880017152e-1	0.5671541003e-1
10	0.3	0.1042967862	0.1042967862	0.1264904872	0.1017983962

The other components of the decomposition series (18) can be determined in a similar way. Substituting (36) into (18), we obtain the closed form solutions

$$w(x,t) = w_0(x,t) + w_1(x,t) + w_2(x,t) + w_3(x,t) + \ldots, \tag{37}$$

$$u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + u_3(x,t) + \ldots. \tag{38}$$

Similar to the doubly-periodic solution of the bSK equation, we evaluate the approximate solution using the four-term approximation to proof whether the obtained numerical solution leads to accuracy

$$\phi_{\mathrm{appr}} = w_0(x,t) + w_1(x,t) + w_2(x,t) + w_3(x,t), \tag{39}$$

$$\varphi_{\mathrm{appr}} = u_0(x,t) + u_1(x,t) + u_2(x,t) + u_3(x,t), \tag{40}$$

where w_n, u_n, $n = 0, 1, \ldots, 3$ is given by (36).

5. Comparison of the Approximate and the Exact Solution

In the following, we compare the approximate solution $\{\phi_{\mathrm{appr}}, \varphi_{\mathrm{appr}}\}$ with the exact solution $\{w, u\}$ using tables and graphs. It is worth noting that the advantage of the decomposition methodology is the fast convergence of the solutions in physical problems. The theoretical treatment of convergence of the decomposition method has been considered in the literature. Authors of [13] have proposed a new approach of convergence of the decomposition series and have given a new condition for obtaining convergence of the decomposition series to the classical presentation of the ADM.

Moreover, as the decomposition method does not require discretization of the variables, i. e., time and space, it is not effected by computation round off errors and one is not faced with necessity of large computer memory and time. The accuracy of the decomposition method for (8) and (9) are controllable. The absolute errors $|w - \phi_{\mathrm{appr}}|$(or $|u - \varphi_{\mathrm{appr}}|$) and the relative errors are very small with the present choice of x and t which are given in Tables 1 and 2. In most cases ϕ_{appr}(or φ_{appr}), the $(n+1)$-term approximations for w(or u), is accurate for quite low values of n in expression (18). As an example, we achieve a very good approximation to the partial exact solution by using only five terms of the decomposition series with the initial condition (22) and four terms of the decomposition series with the initial condition (34), which shows that the speed of convergence of this method is very fast. It is evident that the overall errors can be made smaller by adding new terms of the decomposition series.

6. Conclusions

In this paper, we investigated the approximate solutions of the bSK equation with two kinds of initial conditions based on ADM. As a result, the approximate solutions, such as solitary wave solutions, two-soliton solutions, and doubly-periodic solutions, are obtained. Moreover, we compared the approximate solution with the exact solution in a table, and analyzed the absolute and relative error. The obtained numerical results compared with the analytical solution show that the method provides remarkable accuracy especially for small values of time t. The accuracy can be further improved by considering more terms in the series expansion. A clear conclusion can be draw from the numerical results that the ADM algorithm provides highly accurate numerical solutions without spatial discretizations for the nonlinear partial differential equation. It is

Table 2. Values (in x-direction) for numerical result, exact result, absolute error, and relative error, where $w(x,t) = -2\frac{\partial^2}{\partial x \partial t}\ln f$, $u(x,t) = -2\frac{\partial^2}{\partial x^2}\ln f$, with $f = 1+\exp\left[x+(-\frac{1}{2}+\frac{3}{10}\sqrt{5})t+1\right]+\exp\left[x+(-\frac{1}{2}-\frac{3}{10}\sqrt{5})t+1\right]+\frac{1}{11}\exp(2x-t+2)$ for (8) and (9).

spatial variable x_i	temporal variable t_i	absolute error $\lvert w(x,t)-\phi_{\text{appr}}(x,t)\rvert$	$\lvert u(x,t)-\varphi_{\text{appr}}(x,t)\rvert$	relative error $\frac{\lvert w(x,t)-\phi_{\text{appr}}(x,t)\rvert}{\lvert w(x,t)\rvert}$	$\frac{\lvert u(x,t)-\varphi_{\text{appr}}(x,t)\rvert}{\lvert u(x,t)\rvert}$
2	0.1	0.9406011069e-6	0.2492275680e-6	0.3182192565e-5	0.4538633807e-6
2	0.2	0.1307103460e-4	0.2883382551e-5	0.4136216493e-4	0.5252769532e-5
2	0.3	0.5551209076e-4	0.8692030688e-5	0.1657285055e-3	0.1586981016e-4
5	0.1	0.1460467047e-6	0.1214196039e-6	0.2592317884e-5	0.1175273920e-5
5	0.2	0.2275224870e-5	0.1892152592e-5	0.3555100150e-4	0.1741246313e-4
5	0.3	0.1116616023e-4	0.9289348856e-5	0.1539274027e-3	0.8098542122e-4
10	0.1	0.3427729007e-8	0.2928719131e-8	0.8129075977e-5	0.3783851674e-5
10	0.2	0.5616050275e-7	0.4798032888e-7	0.1163511575e-3	0.5859490865e-4
10	0.3	0.2912722955e-6	0.2488447040e-6	0.5284741139e-3	0.2858875004e-3

also worth noting that the advantage of the decomposition methodology is the fast convergence of the series solutions. The method can also easy be extended to other similar physical equations, with the routine algebraic computations carried out using a package such as Maple (or Matlab, Mathematica, etc.).

Finally, we point out that, although the decomposition series (18) obtained by using ADM is infinite, we often replace the exact solutions with two N-term finite series

$$w \simeq \phi_N = \sum_{n=0}^{N-1} w_n(x,t), \quad u \simeq \varphi_N = \sum_{n=0}^{N-1} u_n(x,t),$$

which is quickly convergent towards the accurate solution for quite low values of N. On this account, the time variable is taken small scales (see Table 1 and 2). Since the Taylor series method provides the same answer obtained by the ADM, we can proceed from the nature of Taylor series [22, 23] to study this phenomenon. The Taylor series expansion of the function $Q(x,t)$ about $t = t_0$ is given by

$$Q(x,t) = \sum_{n=0}^{\infty} \frac{Q^{(n)}(x,t_0)}{n!}(t-t_0)^n, \tag{41}$$

or, equivalently,

$$Q(x,t) = \sum_{n=0}^{N-1} \frac{Q^{(n)}(x,t_0)}{n!}(t-t_0)^n + R_{N-1}. \tag{42}$$

Here, R_{N-1} is a remainder term known as the Lagrange remainder, which is given by

$$R_{N-1} = \underbrace{\int \dots \int_{t_0}^{t}}_{N} Q^{(N)}(x,t)(\mathrm{d}t)^N$$
$$= \frac{(t-t_0)^N}{(N)!} Q^{(N)}(t^*), \quad t^* \in [t_0,t] \tag{43}$$

As we know, the decomposition series (18) are exactly two Taylor series of exact solutions w and u about a point $t = 0$, that is

$$\sum_{n=0}^{N-1} w_n(x,t) = \sum_{n=0}^{N-1} \frac{w^{(n)}(x,0)}{n!} t^n,$$
$$\sum_{n=0}^{N-1} u_n(x,t) = \sum_{n=0}^{N-1} \frac{u^{(n)}(x,0)}{n!} t^n.$$

Then the remainder terms $R_{w,N-1}$ and $R_{u,N-1}$, i. e. the errors between analytical and approximate solutions, are

$$\begin{aligned} R_{w,N-1} &= w - \sum_{n=0}^{N-1} \frac{w^{(n)}(x,0)}{n!} t^n = w - \phi_N \\ &= \underbrace{\int \dots \int_0^t}_{N} w^{(N)}(x,t)(\mathrm{d}t)^N \\ &= \frac{(t)^N}{N!} w^{(N)}(t^*), \\ R_{u,N-1} &= u - \sum_{n=0}^{N-1} \frac{u^{(n)}(x,t_0)}{n!}(t)^n = u - \varphi_N \\ &= \underbrace{\int \dots \int_0^t}_{N} u^{(N)}(x,t)(\mathrm{d}t)^N \\ &= \frac{(t)^N}{N!} u^{(N)}(t^*), \\ & t^* \in [0,t] \end{aligned} \tag{44}$$

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