New Travelling Wave Solutions of Burgers Equation with Finite Transport Memory

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The nonlinear evolution equations with finite memory have a wide range of applications in science and engineering. The Burgers equation with finite memory transport (time-delayed) describes convection-diffusion processes. In this paper, we establish the new solitary wave solutions for the time-delayed Burgers equation. The extended tanh method and the exp-function method have been employed to reveal these new solutions. Further, we have calculated the numerical solutions of the time-delayed Burgers equation with initial conditions by using the homotopy perturbation method (HPM). Our results show that the extended tanh and exp-function methods are very effective in finding exact solutions of the considered problem and HPM is very powerful in finding numerical solutions with good accuracy for nonlinear partial differential equations without any need of transformation or perturbation.

Key words: Travelling Wave Solutions; Time-Delayed Burgers Equation; Tanh-Function Method; Exp-Function Method; Burgers-Fisher Equation; Homotopy Perturbation Method.
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1. Introduction

In the mathematical description of a physical process, one generally assumes that the behaviour of the process considered depends only on the present state, an assumption which is verified for a large class of dynamical systems. However, there exist situations where this assumption is not satisfied and the use of a 'classical' model in system analysis and its design may lead to poor performance. In such cases, it is better to consider that the system behaviour includes information on the former state. These systems are called time-delay systems. Many of the processes, both natural and artificial, in biology, medicine, chemistry, physics, engineering, economics, etc. involve time delays.

A number of nonlinear phenomena such as physical, biochemical, and biological processes are described by the interplay of reaction and diffusion, or by the interaction between convection and diffusion [1 – 3]. The well-known partial differential equation (PDE) which governs a wide variety of them is the Burgers equation [4]. The diffusion processes, however, get significantly modified when the memory effects are taken into account, i. e., when the dispersal of the particles is not mutually independent. This implies that the correlation between the successive movements of the diffusing particles may be understood as a delay in the flux for a given concentration gradient. Thus existence of time-delay is an important feature in convection diffusion systems. It is interesting to point out that a well-known generalization of the Burgers-Fisher equation, namely the generalized time-delayed Burgers-Fisher equation, is given by [5]

$$\begin{aligned} &\tau u_{tt} + [1-\tau f_u]u_t = u_{xx} - pu^s u_x + f(u), \\ &f(u) = qu(1-u^s), \end{aligned} \tag{1}$$

where p, q, s are constants and τ is a time delayed constant. (1) reduces to the classical Burgers equation when $q=\tau=0$ and $p=s=1$.

On the other hand, directly seeking exact solutions of nonlinear partial differential equations has become one of the central themes of perpetual interest in mathematical physics. In order to understand better the nonlinear phenomena as well as further applications in the practical life, it is important to seek their more exact travelling solutions. Many methods are used to obtain travelling solitary wave solutions to nonlinear PDEs, such as the variational iteration method [6 – 10], the Adomian decomposition method [11, 12], the homotopy perturbation method [13, 14], the tanh-function method [15 – 18], the exp-function method [19 – 21], the sine-cosine method [22] and so on. In the present work, the extended tanh and exp-function methods are used to obtain travelling wave solutions of the time-delayed Burgers equation of the form

$$\tau u_{tt} + u_t = -pu^s u_x + u_{xx} \tag{2}$$

for different values of τ, p, and s. (2) has been applied to forest fire [23], population growth models, Neolithic transitions [24, 25] and many other areas of applied sciences. A search in the literature have revealed that the extended tanh method technique has successfully been used to find exact solutions of many nonlinear PDEs [26, 27]. Zhang [28] proposed a generalized transformation method to obtain more general exact solutions of the (2+1)-dimensional Konopelchenko-Dubrovsky equations. The exp-function method was first proposed in [19] and has been succesfully applied to many nonlinear problems [29, 30]. Xu [31] obtained generalized soliton solutions of Konopelchenko-Dubrovsky equations using the exp-function method. More recently, Wu and He [32 – 34] have illustrated some attractive merits of the exp-function method. However, practically there is no unified method that can be used to handle all types of nonlinear partial differential equations. On the other hand, the homotopy perturbation method, first proposed by He [35], is an elegant method which has proved its effectiveness and efficiency in obtaining numerical solutions of many types of nonlinear equations [36 – 39]. The stiff systems of ordinary differential equations have been solved by using the homotopy perturbation method in [40].

The organization of this paper is as follows: In Section 2, we apply the extended tanh-function method to obtain new soliton solutions of the time-delayed Burgers equation. The exp-function method is applied in Section 3 to obtain more generalized soliton solutions of the time-delayed Burgers equation. The numerical solutions of the time-delayed Burgers equation are obtained by using the homotopy perturbation method in Section 4. Also, a comparison between the exact solutions and the approximate solutions derived by the homotopy perturbation method is given in Section 4. A brief conclusion is given in Section 5.

2. Soliton Solutions via the Extended Tanh-Function Method

In this section we will use the extended tanh-function method in its systematized form as presented by [17]. The tanh technique is based on the a priori assumption that the travelling wave solutions can be expressed in terms of the tanh-function. A partial differential equation $P(u, u_t, u_x, u_{xx}, u_{xxx}, \cdots) = 0$ can be converted to an ordinary differential equation (ODE)

$$Q(u, u', u'', u''', \cdots) = 0 \tag{3}$$

upon using a wave variable $\eta = k(x - \omega t)$. The ordinary differential equation (3) is then integrated as long as all terms contain derivatives, where the integration constants are considered as zeros.

The tanh method introduces a new independent variable

$$Y = \tanh(\eta), \quad \eta = k(x - \omega t), \tag{4}$$

that leads to the change of derivatives:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\eta} &= (1 - Y^2)\frac{\mathrm{d}}{\mathrm{d}Y}, \\
\frac{\mathrm{d}^2}{\mathrm{d}\eta^2} &= (1 - Y^2)\left[-2Y\frac{\mathrm{d}}{\mathrm{d}Y} + (1 - Y^2)\frac{\mathrm{d}^2}{\mathrm{d}Y^2}\right], \\
\frac{\mathrm{d}^3}{\mathrm{d}\eta^3} &= (1 - Y^2)\left[(6Y^2 - 2)\frac{\mathrm{d}}{\mathrm{d}Y}\right. \\
&\qquad \left. -6Y(1 - Y^2)\frac{\mathrm{d}^2}{\mathrm{d}Y^2} + (1 - Y^2)^2\frac{\mathrm{d}^3}{\mathrm{d}Y^3}\right].
\end{aligned}$$

The extended tanh method [27] admits the use of the finite expansion

$$u(\eta) = S(Y) = \sum_{n=0}^{M} a_n Y^n + \sum_{n=1}^{M} b_n Y^{-n}, \tag{5}$$

where M is a positive integer, in most cases, that will be determined.

The expansion (5) reduces to the standard tanh method for $b_n = 0$, $1 \le n \le M$. The parameter M is usually obtained by balancing the linear terms of highest order in the resulting equation with the highest-order nonlinear terms. Substituting (5) in the ODE (3) results in an algebraic system of equations in powers of Y that will lead to the determination of the parameters a_n, $(n = 0, \ldots, M)$, k, and ω.

To obtain the solution of (2), we consider the transformation $u = U(\eta)$, $\eta = k(x - \omega t)$, where k and ω are constants to be determined later. Then we can rewrite the time-delayed Burgers equation (2) in the following nonlinear ordinary differential equation of the form:

$$(\tau\omega^2 - 1)k^2 U'' - k\omega U' + pkU^s U' = 0. \tag{6}$$

Balancing U^sU' with U'' in (6) gives $M = 1/s$. To get a closed form solution we use the transformation $U = v^{1/s}$ to change (6) into

$$(\tau\omega^2-1)k^2\left[v''v+\left(\frac{1}{s}-1\right)v'^2\right]-kwvv' + pkv^2v' = 0. \quad (7)$$

To determine the parameter M, we balance the linear terms of highest order in (7) with the highest-order nonlinear terms. This in turn gives $2M+2 = 2M+M+1$ so that $M = 1$. As a result, the extended tanh method (5) admits the use of the finite expansion

$$v(\eta) = a_0 + a_1Y + \frac{b_1}{Y}, \quad Y = \tanh(k(x-\omega t)).$$

Substituting this transformation formula in the reduced ODE (7), collecting the coefficients of Y, and solving the resulting system we find the following solutions:

$$a_0 = \frac{\omega(s+1)}{2p}, \quad a_1 = \pm\frac{\omega(s+1)}{2p}, \quad b_1 = 0, \quad k = \pm\frac{\omega s}{2(\tau\omega^2-1)}, \quad \omega = \omega, \quad (8)$$

$$a_0 = \frac{\omega(s+1)}{2p}, \quad a_1 = 0, \quad b_1 = \pm\frac{\omega(s+1)}{2p}, \quad k = \pm\frac{\omega s}{2(\tau\omega^2-1)}, \quad \omega = \omega, \quad (9)$$

$$a_0 = \frac{\omega(s+1)}{2p}, \quad a_1 = \pm\frac{\omega(s+1)}{4p}, \quad b_1 = \pm\frac{\omega(s+1)}{4p}, \quad k = \pm\frac{\omega s}{4(\tau\omega^2-1)}, \quad \omega = \omega. \quad (10)$$

This in turn gives the following soliton solutions of (2):

$$u_1(x,t) = \left[\frac{\omega(s+1)}{2p} \cdot\left\{1\pm\tanh\left(\pm\frac{\omega s}{2(\tau\omega^2-1)}(x-\omega t)\right)\right\}\right]^{1/s}, \quad (11)$$

$$u_2(x,t) = \left[\frac{\omega(s+1)}{2p} \cdot\left\{1\pm\coth\left(\pm\frac{\omega s}{2(\tau\omega^2-1)}(x-\omega t)\right)\right\}\right]^{1/s}, \quad (12)$$

$$u_3(x,t) = \left[\frac{\omega(s+1)}{4p} \cdot\left\{2\pm\tanh\left(\pm\frac{\omega s}{2(\tau\omega^2-1)}(x-\omega t)\right) \pm\coth\left(\pm\frac{\omega s}{4(\tau\omega^2-1)}(x-\omega t)\right)\right\}\right]^{1/s}. \quad (13)$$

Note 2.1 *It is observed that the solution of the ansatz (5) goes back to the solutions of standard tanh method once $b_n = 0$, $1 \le n \le M$. On the other hand in case of $b_n \ne 0$, the corresponding solutions are quite new and cannot be obtained from standard tanh method.*

Next, we obtain new soliton solutions of the time-delayed Burgers-Fisher equation. When $p = s = q = 1$, (1) reduces to the form

$$\tau u_{tt} + (1-\tau)u_t + 2\tau uu_t - u_{xx} + uu_x - u + u^2 = 0, \quad (14)$$

where τ is a time-delayed constant. This equation may be called as time-delayed Burgers-Fisher equation. This equation shows a prototypical model for describing the interaction between the reaction mechanism, convection effect, and diffusion transport [41]. It is clear that when $\tau = 0$, (14) reduces to the classical Burgers-Fisher equation discussed in [41]. After the customary transformation $u = v(\eta)$, $\eta = k(x-\omega t)$, we obtain

$$k^2(\tau\omega^2-1)v'' - k\omega(1-\tau)v' - (2k\omega\tau - k)vv' - v + v^2 = 0. \quad (15)$$

Balancing v'' and v^2 in (15) gives $M = 2$. The extended tanh method gives the finite expansion

$$v(\eta) = a_0 + a_1Y + a_2Y^2 + \frac{b_1}{Y} + \frac{b_2}{Y^2}.$$

Proceeding as before, we find

$$a_0 = \frac{1}{2}, \quad a_1 = \pm\frac{1}{2}, \quad a_2 = 0, \quad b_1 = 0, b_2 = 0, \quad k = \mp\frac{\sqrt{\tau}}{2}, \quad \omega = \pm\frac{1}{\sqrt{\tau}}, \quad (16)$$

$$a_0 = \frac{1}{2}, \quad a_1 = 0, \quad a_2 = 0, \quad b_1 = \pm\frac{1}{2}, \quad b_2 = 0, \quad k = \mp\frac{\sqrt{\tau}}{2}, \quad \omega = \pm\frac{1}{\sqrt{\tau}}, \quad (17)$$

$$a_0 = \frac{1}{2}, \quad a_1 = \pm\frac{1}{4}, \quad a_2 = 0, \quad b_1 = \pm\frac{1}{4}, \quad b_2 = 0, \quad k = \mp\frac{\sqrt{\tau}}{4}, \quad \omega = \pm\frac{1}{\sqrt{\tau}}. \quad (18)$$

This in turn gives the following soliton solutions of (14):

$$u_1(x,t) = \frac{1}{2}\left(1\pm\tanh\left[\mp\frac{\sqrt{\tau}}{2}\left(x\mp\frac{1}{\sqrt{\tau}}t\right)\right]\right), \quad (19)$$

$$u_2(x,t)=\frac{1}{2}\left(1\pm\coth\left[\mp\frac{\sqrt{\tau}}{2}\left(x\mp\frac{1}{\sqrt{\tau}}t\right)\right]\right), \quad (20)$$

$$u_3(x,t)=\frac{1}{4}\left(2\pm\tanh\left[\mp\frac{\sqrt{\tau}}{4}\left(x\mp\frac{1}{\sqrt{\tau}}t\right)\right]\right.$$
$$\left.\pm\coth\left[\mp\frac{\sqrt{\tau}}{4}\left(x\mp\frac{1}{\sqrt{\tau}}t\right)\right]\right). \quad (21)$$

3. New Generalized Wave Solutions via Exp-Function Method

In this section, we obtain the new more generalized soliton solutions of the time-delayed Burgers equation. According to the exp-function method, we assume that the solution of (2) can be expressed in the following form:

$$v(\eta)=\frac{\sum_{n=-c}^{\mathrm{d}}a_n\exp(n\eta)}{\sum_{m=-p}^{q}b_m\exp(m\eta)}, \quad (22)$$

where c, d, p, and q are unknown positive integers that will be determined by using the balance method, and a_n and b_m are unknown constants. To determine c and p we balance the linear term of the highest order in (7) with the highest-order nonlinear term. Similarly, we can determine d and q by balancing the linear term of the lowest order in (7) with the lowest-order nonlinear term.

In order to determine the values of c and p, we balance the linear term of highest order in (7) with the highest-order nonlinear term. By simple calculation, we have

$$v'^2=\frac{c_1\exp[2(p+c)\eta]+\cdots}{c_2\exp(4p\eta)+\cdots} \quad (23)$$

and

$$v^2v'=\frac{c_3\exp[(p+3c)\eta]+\cdots}{c_4\exp(4p\eta)+\cdots}, \quad (24)$$

where c_i are determined coefficients only for simplicity. Balancing highest order of exp-function in (23) and (24), we obtain

$$2p+2c=p+2c, \quad (25)$$

which gives

$$p=c. \quad (26)$$

Similarly, to determine the values of d and q, we balance the linear term of lowest order in (7) with the lowest-order nonlinear term

$$v'^2=\frac{\cdots+d_1\exp[-2(q+d)\eta]}{\cdots+d_2\exp[-4q\eta]} \quad (27)$$

and

$$v^2v'=\frac{\cdots+d_3\exp[-(q+3d)\eta]}{\cdots+d_4\exp[-4q\eta]}, \quad (28)$$

where d_i are determined coefficients only for simplicity. Balancing lowest order of exp-function in (27) and (28), we obtain

$$-2(q+d)=-(q+3d), \quad (29)$$

which gives

$$q=d. \quad (30)$$

We can freely choose the values of c and d, but the final solution does not strongly depend upon the choice of values of c and d. For simplicity, we set $p=c=1$ and $d=q=1$, then (22) reduces to

$$v(\eta)=\frac{a_1\exp(\eta)+a_0+a_{-1}\exp(-\eta)}{b_1\exp(\eta)+b_0+b_{-1}\exp(-\eta)}. \quad (31)$$

Substituting (31) in (7) and using the Maple program, equating to zero the coefficients of all powers of $\exp(n\eta)$ gives a set of algebraic equations for a_1, a_0, a_{-1}, b_0, b_1, b_{-1}, k, and ω. Solving the systems of algebraic equations using Maple gives the following sets of nontrivial solutions:

$$a_1=\frac{(s+1)\omega b_1}{p},\quad a_0=0,\quad a_{-1}=0,$$
$$b_1=b_1,\quad b_0=0,\quad b_{-1}=b_{-1}, \quad (32)$$
$$k=\frac{\omega s}{2(\tau\omega^2-1)},\quad \omega=\omega,$$

$$a_1=0,\quad a_0=a_0,\quad a_{-1}=\frac{(s+1)\omega b_{-1}}{p},$$
$$b_1=b_1,\quad b_0=\frac{D_1}{D_2},\quad b_{-1}=b_{-1}, \quad (33)$$
$$k=-\frac{\omega s}{\tau\omega^2-1},\quad \omega=\omega,$$

$$a_1=\frac{\omega(s+1)b_1}{p},\quad a_0=a_0,\quad a_{-1}=0,$$
$$b_1=b_1,\quad b_0=\frac{D_1}{D_2},\quad b_{-1}=b_{-1}, \quad (34)$$
$$k=\frac{\omega s}{\tau\omega^2-1},\quad \omega=\omega,$$

where

$$D_1 = 2b_{-1}b_1\omega^2 s + s^2\omega^2 b_1 b_{-1} + b_{-1}b_1\omega^2 + a_0^2 p^2, \quad (35)$$

$$D_2 = (s+1)pa_0\omega. \quad (36)$$

Substituting (32), (33), and (34) in (31), we obtain the following soliton solutions of (2):

$$u_1(x,t) = \left(\frac{(s+1)\omega b_1 \exp(\eta)}{pb_1\exp(\eta) + pb_{-1}\exp(-\eta)}\right)^{1/s}, \quad (37)$$

where $\eta = \frac{\omega s}{2(\tau\omega^2-1)}(x-\omega t)$.

$$u_2(x,t) = \left(\frac{pa_0D_2 + (s+1)\omega b_{-1}D_2\exp(-\eta)}{pb_1D_2\exp(\eta) + pD_1 + pb_{-1}D_2\exp(-\eta)}\right)^{1/s}, \quad (38)$$

where $\eta = -\frac{\omega s}{\tau\omega^2-1}(x-\omega t)$ and D_1, D_2 are defined as in (35) and (36).

$$u_3(x,t) = \left(\frac{\omega(s+1)b_{-1}D_2\exp(\eta) + pa_0D_2}{pb_1D_2\exp(\eta) + pD_1 + pb_{-1}D_2\exp(-\eta)}\right)^{1/s}, \quad (39)$$

where $\eta = \frac{\omega s}{\tau\omega^2-1}(x-\omega t)$ and D_1, D_2 are defined as in (35) and (36).

It should be noted that it is possible to obtain many soliton solutions, since a_0, b_1, b_{-1} are free parameters. If we take $b_1 = b_{-1} = 1$ then our solution of (37) turns out to the solution as expressed in [11]:

$$u(x,t) = \left[\frac{(s+1)\omega}{2p}\left\{1+\tanh\left(\frac{\omega s}{2(\tau\omega^2-1)}(x-\omega t)\right)\right\}\right]^{1/s}. \quad (40)$$

If we choose $b_1 = 1$, $b_{-1} = -1$ then our solution of (37) turns out to the solution as obtained in [11]:

$$u(x,t) = \left[\frac{(s+1)\omega}{2p}\left\{1+\coth\left(\frac{\omega s}{2(\tau\omega^2-1)}(x-\omega t)\right)\right\}\right]^{1/s}. \quad (41)$$

Also, it is worth to mention that a wide variety of distinct soliton solutions can be obtained by selecting arbitrary values for the constants p, c, q, and d provided that we fix the relation $p = c$, $d = q$.

4. Solutions by Homotopy Perturbation Method (HPM)

However, the previous methods can not be used directly to solve (2), it requires a transformation to convert the given nonlinear PDE into nonlinear ODE. Also another transformation to convert the nonlinear ODE into a new one, in which the balancing procedure becomes applicable. In this section, we show that the HPM method can be used directly to obtain the solution in the form of a convergent series without using any such transformations. The homotopy perturbation method has some flexibility in its use and is a very effective tool for solving nonlinear problems.

We obtain the numerical solutions of the time-delayed Burgers equation with initial conditions by using the homotopy perturbation method. In order to obtain the solutions, He's polynomials are introduced based on the homotopy perturbation method.

Consider the time-delayed Burgers equation of the form

$$\tau u_{tt} + u_t = -pu^s u_x + u_{xx}, \quad (42)$$

subject to the initial conditions

$$u(x,0) = g_1(x), \quad u_t(x,0) = g_2(x), \quad (43)$$

where $g_1(x)$ and $g_2(x)$ are known functions.

To solve (42) with the initial condition (43), we construct the following homotopy:

$$\tau u_{tt} + q[u_t + pH(u) - u_{xx}] = 0, \quad (44)$$

where $H(u) = u^s u_x$ and $q \in [0,1]$ is the embedding parameter. We assume that the solution of (42) and (43) is

$$u = u_0 + qu_1 + q^2u_2 + \dots. \quad (45)$$

The nonlinear term $H(u)$ in (44) can be expressed in the form

$$H(u) = H_0(u_0) + qH_1(u_0,u_1) + q^2H_2(u_0,u_1,u_2) + \dots, \quad (46)$$

where $H_n(u_0,\cdots,u_n)$ is called He's polynomial [42, 43] defined by

$$H_n(u_0,\cdots,u_n) = \frac{1}{n!}\frac{\partial^n}{\partial q^n}H\left(\sum_{k=0}^{n} q^k u_k\right)_{q=0}, \quad (47)$$
$$n = 0,1,2,3\dots.$$

Substituting (45) and (46) in (44), and equating the coefficients of like powers of q, we get following set of differential equations:

$$q^0: \ \tau u_{0tt} = 0, \quad u_0(x,0) = g_1(x), \\ u_{0t}(x,0) = g_2(x), \quad (48)$$

$$q^1: \ \tau u_{1tt} + u_{0t} + pH_0(u_0) - u_{0xx} = 0, \\ u_1(x,0) = 0, \quad u_{1t}(x,0) = 0, \quad (49)$$

$$q^2: \ \tau u_{2tt} + u_{1t} + pH_1(u_0,u_1) - u_{1xx} = 0, \\ u_2(x,0) = 0, \quad u_{2t}(x,0) = 0, \quad (50)$$

$$\vdots$$

$$q^{n+1}: \tau u_{(n+1)_{tt}} + u_{nt} + pH_n(u_0,u_1,\ldots u_n) - u_{nxx} = 0, \\ u_{n+1}(x,0) = 0, \quad u_{(n+1)_t}(x,0) = 0, \quad (51)$$

$$\vdots$$

The approximate solution of (42) can be obtained by setting $q = 1$:

$$u = u_0 + u_1 + u_2 + \ldots. \quad (52)$$

For concrete problems, where the exact solution is not easily obtainable, the n-term approximation to u

$$\Phi_n = \sum_{i=o}^{n} u_i(x,t) \quad (53)$$

can be used for numerical purposes. It is interesting to point out that He's polynomial plays an important role in handling the nonlinear term and making the solution procedure in a systematic way.

4.1. Numerical Example

Now we investigate how well the numerical scheme determines the solutions. Take $s = 1$, $p = 0.1$, $\tau = 0.5$, $k = -0.05$, $\omega = 0.1$, $t = 0.1$. The initial conditions can be obtained from the initial conditions (43). Now from (48)–(51), using Mathematica, we obtain $u_0(x,t)$, $u_1(x,t)$, $u_2(x,t)$,... as follows:

$$u_0(x,t) = 1 - \tanh[0.0502513x],$$

$$u_1(x,t) = 2(0.00251256t^2\operatorname{sech}^2[0.0502513x] \\ + 0.0000126259t^2\operatorname{sech}^2[0.0502513x] \\ \cdot \tanh[0.0502513x]),$$

$$u_2(x,t) = (2t^3\operatorname{sech}^2[0.0502513x](-2.8482\times10^{-8} \\ \cdot (1+\cosh[0.100503x]+\sinh[0.100503x])^2 \\ \cdot (58514.9+59106\cosh[0.100503x] \\ +59106\sinh[0.100503x]) \\ +2.13615\times10^{-8}t(-195.045 \\ -198.945\cosh[0.100503x] \\ +187.065\cosh[0.201005x] \\ +199.005\cosh[0.301508x] \\ -198.945\sinh[0.100503x] \\ +187.065\sinh[0.201005x] \\ +199.005\sinh[0.301508x]))) \\ /(1+\cosh[0.100503x]+\sinh[0.100503x])^3,$$

$$\vdots$$

In this manner, the other components of the series can be easily obtained.

When $s = 1$, $p = 0.5$, $\tau = 1$, $k = -0.01$, $\omega = 0.1$, $t = 0.1$, proceeding as before we obtain $u_0(x,t)$, $u_1(x,t)$, $u_2(x,t)$, ... as follows:

$$u_0(x,t) = 0.2(1-\tanh[0.0505051x]),$$

$$u_1(x,t) = 0.00010101t^2\operatorname{sech}^2[0.0505051x] \\ + 0.000409142t^2\operatorname{sech}^2[0.0505051x] \\ \cdot \tanh[0.0505051x],$$

$$u_2(x,t) = (t^3\operatorname{sech}^2[0.0505051x](-8.3582610^{-7} \\ \cdot (1.+\cosh[0.10101x]+\sinh[0.10101x])^2 \\ \cdot (-122.885+203.452\cosh[0.10101x] \\ +203.452\sinh[0.10101x]) \\ +6.2686910^{-7}t(-0.252432 \\ +4.64879\cosh[0.10101x] \\ -5.96732\cosh[0.20202x] \\ +0.691944\cosh[0.30303x] \\ +4.64879\sinh[0.10101x] \\ -5.96732\sinh[0.20202x] \\ +0.691944\sinh[0.30303x]))),$$

$$\vdots$$

Table 1 represents the values of the exact solution $u_E(x,t)$ and the numerical solution $\Phi_2(x,t)$ when $s = 1$, $p = 0.1$, $\tau = 0.5$, $k = -0.05$, $\omega = 0.1$, $t = 0.1$. Table 2 represents the values of the exact solution $u_E(x,t)$ and the numerical solution $\Phi_2(x,t)$ when $s = 1$, $p = 0.5$, $\tau = 1$, $k = -0.01$, $\omega = 0.1$, $t = 0.1$. It is clear from Table 1 and Table 2 that the absolute error $|\Phi_2(x,t) - u_E(x,t)|$ is very small. It is important to note that the exact solutions in Table 1 and Table 2 were obtained with the method presented in Section 1. Table 1 and Table 2 show that a good approximation is achieved

Table 1. Values of $\Phi_2(x,t)$, $u_\mathrm{E}(x,t)$, and $|\Phi_2(x,t)-u_\mathrm{E}(x,t)|$, when $s=1$, $p=0.1$, $\tau=0.5$, $k=-0.05$, $\omega=0.1$, $t=0.1$.

x_i	$\Phi_2(x,t)$	$u_\mathrm{E}(x,t)$	$\lvert\Phi_2(x,t)-u_\mathrm{E}(x,t)\rvert$
−120	1.99999	1.99999	1.05417×10^{-8}
−100	1.99991	1.99991	7.86744×10^{-8}
−80	1.99936	1.99936	5.86874×10^{-7}
−60	1.9952	1.99521	4.36206×10^{-6}
−40	1.96473	1.99477	0.0000315703
−20	1.76372	1.76391	0.000189879
0	1.00005	1.0005	0.000455611
20	0.236323	0.236513	0.000189889
40	0.0352725	0.035304	0.0000315724
60	0.00479917	0.00480353	4.36237×10^{-6}
80	0.000644341	0.000644928	5.86915×10^{-7}
100	0.0000863542	0.0000864329	7.86799×10^{-8}
120	0.0000115703	0.0000115809	1.05425×10^{-8}

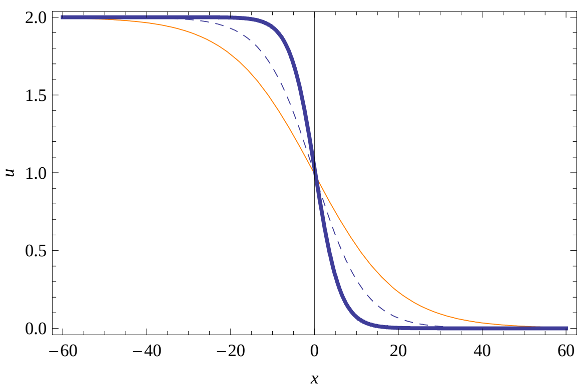

Fig. 1. The thin curve represents the numerical solution $\Phi_2(x,t)$ for $s=1$, $p=0.1$, $\tau=0.5$, $\omega=0.1$, $k=-0.05$, and $t=0.1$. The dashed and thick curves represent the numerical solution $\Phi_2(x,t)$ when $s=1$, $p=0.1$, $\tau=40$, $\omega=0.1$, $k=-0.05$, $t=2$ and $s=1$, $p=0.1$, $\tau=70$, $\omega=0.1$, $k=-0.05$, $t=3$, respectively.

Table 2. Values of $\Phi_2(x,t)$, $u_\mathrm{E}(x,t)$, and $|\Phi_2(x,t)-u_\mathrm{E}(x,t)|$ when $s=1$, $p=0.1$, $\tau=0.5$, $k=-0.05$, $\omega=0.1$, $t=0.1$.

x_i	$\Phi_2(x,t)$	$u_\mathrm{E}(x,t)$	$\lvert\Phi_2(x,t)-u_\mathrm{E}(x,t)\rvert$
−120	0.399998	0.399998	2.26283×10^{-9}
−100	0.399984	0.399984	1.70602×10^{-8}
−80	0.399876	0.399876	1.28559×10^{-7}
−60	0.399069	0.39907	9.65263×10^{-7}
−40	0.393085	0.393092	7.05339×10^{-6}
−20	0.35316	0.353203	0.0000426048
0	0.200001	0.200101	0.000100034
20	0.0468409	0.046881	0.000040132
40	0.00691484	0.00692138	6.54117×10^{-6}
60	0.000931068	0.000931961	8.93074×10^{-7}
80	0.000123736	0.000123855	1.18907×10^{-7}
100	0.0000164154	0.0000164311	1.57787×10^{-8}
120	2.17723×10^{-6}	2.17932×10^{-6}	2.09285×10^{-9}

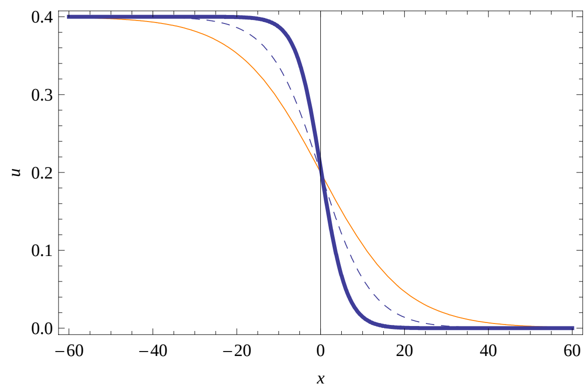

Fig. 2. The thin curve represents the numerical solution $\Phi_2(x,t)$ for $s=1$, $p=0.5$, $\tau=1$, $k=-0.01$, $\omega=0.1$, and $t=0.1$. The dashed and thick graphs represent the numerical solution $\Phi_2(x,t)$ when $s=1$, $p=0.5$, $\tau=40$, $\omega=0.1$, $k=-0.01$, $t=2$ and $s=1$, $p=0.5$, $\tau=70$, $\omega=0.1$, $k=-0.01$, $t=3$, respectively.

using a few terms of the series solution. It is also clear that the overall errors can be made smaller by adding more terms of the series. The effect of the time-delay is shown in Figure 1 and Figure 2. We found that the time delay is effective in smoothing out the shock-wave nature of the travelling wave.

5. Conclusion

We obtained many new travelling wave solutions of the time-delayed Burgers equation using exp-function and extended tanh method with the help of a suitable transformation. The computer symbolic systems such as Maple and Mathematica allow us to perform complicated and tedious calculations. The extended tanh and exp-function methods are reliable algorithms to obtain new solutions if compared with the existing methods. The solutions have different physical structures and depend on the real parameters. It is shown in this paper that the approximate solutions can be calculated by using the HPM without any need of transformation or perturbation of the equation. It is concluded that the exp-function and extended tanh methods are very efficient in finding exact solutions for the nonlinear evolution equations while the HPM method is very powerful in finding numerical solutions.

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