Long-Wave Instabilities of Viscoelastic Fluid Film Flowing Down an Inclined Plane with Linear Temperature Variation

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The two dimensional flow of a viscoelastic fluid (weakly elastic) represented by Walters' B″ model running down an inclined heated plane with linear temperature variation has been investigated in the finite amplitude regime. A long-wave expansion method obtains a nonlinear evolution equation of the film interface. A normal mode approach and the method of multiple scales are used to obtain the linear and nonlinear stability solutions for the film flow. The study reveals that both supercritical stability and subcritical instability are possible for this type of film flow. The influence of the viscoelastic parameter and the Marangoni number on the different zones, the amplitude of the disturbances on sub/supercritical region, and the nonlinear phase speed in the supercritical region are also investigated. It is interesting to note that the influence of the viscoelastic parameter on the above aspects are very significant at low Marangoni number, while it has very feeble impact at high Marangoni number. Finally, the results thus determined are interpreted physically.

Key words: Thin Film; Viscoelastic Fluid; Finite Amplitude Stability Analysis; Marangoni Instability.
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1. Introduction

Fluids which can not be completely characterised by the linear law of viscous fluids are usually called non-Newtonian fluids. There are many avenues in which the deviation from Newtonian behaviour exhibits itself. Some of these fluids exhibit shear thinning or thickening while some others possesses elastic properties or something else. There are many examples of fluids which possess the characteristic of elastic properties and the so-called cross-viscosity. When such a fluid is in motion, the flow energy is partially reserved as strain energy and partially assimilated by the internal viscous forces and dissipated as heat to the environment; these type of fluids are known as viscoelastic fluids. The elastic properties of these type of fluids can create the shear on an element of the fluid to be reverted and the elastic stresses can not be relaxed at a certain frequency as the flow moves down a vertical wall. There are numerous models suggested in the literature to describe viscoelastic fluids, such as the second-order model, the Oldroyd-B model, the Walters B″ model etc. The rheological behaviour of the fluids described by each and every model is completely different and almost every one has some limitations but almost each one is very successful in describing a particular type of fluid. As for example, the second-order fluid model describes very successfully the fluids with gradually fading memory while Walters' B″ fluid model is valid for short or rapidly fading memory liquids (weakly elastic). To get some insight into the flow behaviour of the viscoelastic fluids, it is preferable to restrict oneself to a model with a minimum number of parameters in the constitutive equations. We choose Walters' B″ viscoelastic fluid model for our study as it involves only one non-Newtonian parameter.

Linear stability of a falling second-order viscoelastic fluid down an incline plane was first investigated by Gupta [1]. He found that flow becomes unstable for two-dimensional disturbances of large wave length and the viscosity of the fluid tends to reduce the critical Reynolds number for the onset of instability. Lai [2], Gupta and Rai [3] have studied the long wave disturbances for the flow of an Oldroyd-B fluid and recalculated the critical conditions and found identical results to the second-order fluid at sufficiently small Deborah number and also yield qualitatively reasonable predictions at large Deborah numbers. Later, Dandapat [4] investigated the stability of a falling film of an incompressible second-order fluid with respect to two-dimensional disturbances of small but finite amplitude. He obtained that in the presence of surface tension the falling film is supercritically stable and an initially growing monochromatic wave reaches an equilibrium state of finite amplitude. Further, he found that the equilibrium amplitude first increases with the elastic parameter of the fluid, reaches a maximum and then decreases with the increase of the elastic parameter. Shaqfeh et al. [5] investigated the stability of the gravity driven viscoelastic film flow at low moderate Reynolds number using the Oldroyd-B fluid model. They found that at small Reynolds numbers, the viscoelastic effects are destabilizing while at moderate Reynolds number they produce a stabilizing effect. On the other hand the growth rate of the resulting purely elastic waves remains very small. Later, Kang and Chen [6] studied the nonlinear elastic instability of thin viscoelastic Oldroyd-B fluids without any shear thinning behaviours in the long-wave limit. Dandapat and Gupta [7] studied the existence and role of the stationary wave in the finite amplitude instability of a layer of second-order fluid flowing down an inclined plane. They observed that the number of solitary waves decreases with the increase in the viscoelastic parameter. Stability analysis of thin viscoelastic Walters' B'' fluid flowing down a vertical wall was investigated by Cheng et al. [8] by using the long wave perturbation method. Recently, Uma and Usha [9] investigated the linearised instability for the permanent waves on a thin viscoelastic Walters' B'' fluid layer flowing down an inclined plane. They also classified the eigen value properties of the fixed points in various parametric regimes.

Viscoelastic models have been found to be successful in describing the behaviour of colloids and suspensions and some manmade fluids, such as polymeric fluids, animal blood, fluid with additives, and liquids crystals etc. The problem of the stability of the flow of a viscoelastic fluid film has fundamental importance for the technology of the production of polymer coating products. For example, a thin polymer film is applied to a flat substrate with some localised defects or topography. A number of mechanisms leading to flow instability are known; such surface tension can play a significant role in the neighbourhood of the surface features. On the other hand, surface tension is a function of the thermodynamic quantities such as the temperature. Therefore, it is very important to study what are the impact of variation of the surface tension due to the temperature gradient for a viscoelastic film.

An enormous number of studies on isothermal viscoelastic thin film flow problem has been investigated during the last few years but the flow of non-isothermal viscoelastic fluids attracted less attention in the past. On the other hand, much has been learned about the non-isothermal Newtonian film flows.

The interfacial instability generated by the surface tension gradient due to a temperature gradient is known as Marangoni instability. The study of thermocapillary flows is pertinent in various technological applications where free surface flows are encounterd, such as in lubricating and coating flows where temperature control is indispensable to maintain uniformity in the film thickness; any slight variation on the temperature could enhance the instabilities that could disrupt the entire coating layer. Thermocapillary instability in a liquid film falling down an inclined uniformly heated plane was first studied by Goussis and Kelly [10]. They performed linear stability analysis based on Orr-Somerfield and linearized energy equations. They found that a heated wall has a destabilizing effect on the free surface but a cooled wall stabilizes the flow. Joo et al. [11] investigated the long wave instability of a uniform film of volatile viscous liquid falling down a uniformly heated inclined plane for two-dimensional disturbances, taking into account the surface tension, thermocapillary, and evaporation effects. Smith and Davice [12] studied the thermocapillary instability in the case of temperature gradient along the liquid layer. They discussed both thermal convective and surface wave instabilities in two consecutive papers. Kalliadasis et al. [13] have studied the thermocapillary instability of a viscous fluid film flowing down a uniformly heated plane by using the integral boundary layer approach and they predict the existence of solitary waves for all Reynolds number. Ruyer-Quil et al. [14] studied the same problem as Kalliadasis et al. [13] by using a higher-order weighted residual approach with polynomial expansions for both velocity and temperature field, which introduces the second-order dissipative effects that are known to determine the amplitude of the front-running capillary waves. In the above study they have shown that the transport of heat by the flow plays a double role; for small amplitude it stabilizes the flow but for large amplitude waves it promotes the instability. Kalitzova-Kurteva et al. [15] studied the linear stability analysis of a thin liquid layer on a non-uniformly heated plate in the case of infinitesimal wave number and demonstrate the role of the Marangoni effect on the critical Reynolds number

Symbol	Meaning
Nomenclature:	
θ	inclination angle
∇	Laplacian operator
$\boldsymbol{v}$	velocity vector
$\boldsymbol{g}$	gravitational acceleration
h	film thickness
t	time
x, z	coordinates along and transverse to the inclined plane
u, v	components of velocity in x- and z-directions
ρ	density of the fluid
ν	kinematic viscosity of the liquid
κ	thermal diffusivity
κ_T	thermal conductivity
C_P	specific heat at constant pressure
T	absolute temperature
T_g	temperature of the gas phase
T_H	temperature of the hotter part of the plane
T_C	temperature of the colder part of the plane
$\boldsymbol{n}$	outward pointing unit vector normal to the interface
$\boldsymbol{\tau}$	stress tensor
$\boldsymbol{t}$	unit vector tangential to the interface
σ	surface tension of the fluid
p	fluid pressure
μ	dynamical viscosity
$\mathbf{e}$	rate of strain tensor
p_a	pressure of the ambient gas phase
κ_g	heat transfer coefficient between the liquid and air
σ_0	surface tension at T_g
Γ_0	viscoelastic coefficient of the fluid
γ	coefficient of surface tension
l_0	longitudinal length scale
h_0	film thickness in equilibrium state
u_0	velocity on free surface in equilibrium state
ε	aspect ratio
Fr	Froude number
Re	Reynolds number
Pr	Prandtl number
Pe	Peclect number
Mn	Marangoni number
We	Weber number
Ca	Capillary number
Bi	Biot number
Cr	Crispation number
Re_c	critical Reynolds number
k	dimensionless wave number
Γ	viscoelastic parameter
η	dimensionless perturbation
Λ	amplitude of the disturbance
ω	complex frequency = $\omega_r + i\omega_i$
λ	wave length
ζ	small perturbation parameter
a	amplitude
Nc_r	nonlinear wave speed
c_g	group velocity

Symbol	Meaning
Superscripts:	
*	dimensionless quantity
′	differentiation with respect to h
Subscripts:	
,	partialderivatives with respect to the following variables
$0, 1, 2 \ldots$	expansion order of the long wave
$\sim$	transformed co-ordinates

for the onset of instability. Later, Miladinova et al. [16] extended their study for the long wave instabilities of two-dimensional films in the finite amplitude regime. Recently, Demekhin et al. [17] examined the linear stability of a falling film down an inclined wall heated by a downstream linearly increasing temperature distribution. They found that the thermocapillary forces play a double role depending on the Prandtl number. For liquids with sufficiently large Prandtl numbers increasing the temperature gradient they first destabilizes the flow and then stabilizes it. On the other hand, for small Prandtl numbers, increasing the temperature gradient leads to a monotonic stabilization of all instability modes.

In this paper we shall present the analysis of finite amplitude of long wave instability of a thin viscoelastic Walters' B″ fluid film down an inclined heated plane with linear temperature variation. We have assumed that the liquid is non-volatile, so that evaporation effects can be neglected while the film is sufficiently thin so that buoyancy effects can also be neglected, but it is not so thin that intermolecular forces come into play. However, the heating on the inclined plane produces surface tension gradients, which induce thermocapillary stresses on the free surface, that effects considerably on the flow dynamics.

2. Formulation of the Problem

Consider a two-dimensional flow of a incompressible viscoelastic fluid film on an inclined heated plane of inclination θ with the horizon under the action of gravity. The co-ordinate system is chosen such that the x-axis is along the flow and the z-axis normal to the inclined plane.

2.1. Governing Equations

The governing equations consist of the conservation of mass and momentum equations for the flow of the liquid layer and the energy equation for the temperature field. In dimensional form they can be written as

$$\nabla \cdot \boldsymbol{v} = 0, \tag{1}$$

$$\rho(\boldsymbol{v}_t + (\boldsymbol{v} \cdot \nabla)\boldsymbol{v}) = \rho \boldsymbol{g} + \nabla \cdot \boldsymbol{\tau}, \tag{2}$$

$$\partial_t T + (\boldsymbol{v}\cdot\nabla)T = \kappa\nabla^2 T, \tag{3}$$

where $\nabla = (\partial/\partial x, 0, \partial/\partial z)$, $\boldsymbol{v} = (u, 0, v)$ is the velocity vector, $\boldsymbol{g} = (g\sin\theta, 0, -g\cos\theta)$ is the acceleration due to the gravity vector, ρ, T denote the density and absolute temperature, respectively, and $\boldsymbol{\tau}$ is the Cauchy stress tensor given by Beard and Walters [18] as

$$\boldsymbol{\tau} = -p\mathbf{I} + 2\mu\mathbf{e} - 2\Gamma_0\frac{\delta\mathbf{e}}{\delta t}, \tag{4a}$$

where p is the isotropic pressure, $\mathbf{I}$ is the identity tensor, μ is the dynamic viscosity of the fluid at small rates of shear, Γ_0 is the short memory coefficient or viscoelastic coefficient of the fluid, and the rate of strain tensor $\mathbf{e}$ is defined by

$$\mathbf{e} = (\nabla\boldsymbol{v} + \nabla\boldsymbol{v}^T)/2. \tag{4b}$$

$\delta/\delta t$ denoted the convected differentiation of a tensor quantity in relation to the material in motion. The convected differentiation of the rate of strain tensor is given by

$$\frac{\delta\mathbf{e}}{\delta t} = \frac{\partial\mathbf{e}}{\partial t} + \boldsymbol{v}\cdot\nabla\,\mathbf{e} - \mathbf{e}\cdot\nabla\boldsymbol{v} - (\nabla\boldsymbol{v})^T\cdot\mathbf{e}. \tag{4c}$$

Also $\kappa = k_\mathrm{T}/(\rho C_\mathrm{p})$ denote the thermal diffusivity, k_T the thermal conductivity, C_p the specific heat at constant pressure of the fluid. The dynamic influence of the air above the liquid film is ignored.

The pertinent boundary conditions are the usual noslip condition and a linear temperature distribution as the thermal boundary condition on the inclined plane at $z = 0$:

$$\boldsymbol{v} = 0, \quad T = T_\mathrm{g} + \mathcal{A}x, \tag{5}$$

where T_g denotes the temperature in the gas phase and $\mathcal{A} = (T_\mathrm{H} - T_\mathrm{C})/l_0$ (T_H and T_C denote the temperatures at hotter part and the colder part, respectively, along the inclined plane and l_0 is the characteristic longitudinal length scale whose order may be considered same as the wave length λ). The temperature T increases (decreases) in the stream-wise direction with positive (negative) $\mathcal{A}$. In our present study $\mathcal{A}$ is taken as positive.

At the free surface $z = h(x,t)$ dynamic and kinematic conditions along with Newton's law of cooling as the thermal boundary condition are

$$\boldsymbol{n}\cdot\boldsymbol{\tau}\cdot\boldsymbol{t} = \nabla\,\sigma\cdot\boldsymbol{t}, \tag{6}$$

$$p_\mathrm{a} + \boldsymbol{n}\cdot\boldsymbol{\tau}\cdot\boldsymbol{n} = -\sigma(T)\,\nabla\cdot\boldsymbol{n}, \tag{7}$$

$$h_\mathrm{t} + \boldsymbol{v}\cdot\nabla\,(h - \boldsymbol{z}) = 0, \tag{8}$$

$$k_\mathrm{T}\nabla T\cdot\boldsymbol{n} + k_\mathrm{g}(T - T_\mathrm{g}) = 0, \tag{9}$$

where $\boldsymbol{n}$ and $\boldsymbol{t}$ are the normal and tangent vectors pointing outward to the interface, respectively, p_a is the pressure of the ambient gas phase, and k_g is the heat transfer coefficient between the liquid and the air.

The above equations are quite general regarding various coefficients (k_T, κ, μ, σ etc). It is well known that a temperature variation in the fluid can cause dramatic changes in the above coefficients, but approximations can be made depending on the type of the problem being examined. In the foregoing analysis we have the variation of surface tension assumed as

$$\sigma(T) = \sigma_0 - \gamma(T - T_\mathrm{g}), \tag{10}$$

where σ_0 is the surface tension at T_g, which is taken as the reference temperature, and $\gamma = -\partial\sigma/\partial T|_{T=T_\mathrm{g}}$ is a positive constant for most common fluids. The assumption of the linear variation of the surface tension with temperature is very much compatible with the experimental data for many fluids [19, 20]. It is also assumed that all other physical properties of the fluid are constant.

2.2. Scalings and Non-Dimensionalization

Before solving the problem we want to rewrite the problem precisely in dimensionless form. We define the dimensionless quantities as

$$\begin{aligned}
&x = l_0x^*, \quad t = (l_0/u_0)t^*, \quad (h,z) = h_0(h^*, z^*),\\
&u = u_0u^*, \quad v = (h_0u_0/l_0)v^*, \quad p = \rho u_0{}^2p^*,\\
&T = T_\mathrm{g} + T^*(T_\mathrm{H} - T_\mathrm{C}),\\
&(\tau_{xx}, \tau_{zz}) = \mu(u_0/l_0)(\tau_{xx}{}^*, \tau_{zz}{}^*),\\
&(\tau_{xz}, \tau_{zx}) = \mu(u_0/h_0)\tau_{xz}{}^*, \tau_{zx}{}^*),
\end{aligned} \tag{11}$$

where l_0 as mentioned before, h_0 is the mean film thickness, and $u_0 = gh_0{}^2\sin\theta/3\nu$ the Nusselt velocity, which are taken as the length scale in transverse direction and characteristic velocity, respectively.

Using the above dimensionless quantities, the governing equations (1) – (3) and the boundary conditions (5) – (9) are reduced to the form, after dropping the asterisk, as:

I. Governing Equations:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial z} = 0, \tag{12}$$

$$\varepsilon\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial z}\right) = \frac{3}{\mathrm{Re}}+\frac{1}{\mathrm{Re}}\left(\varepsilon^2\frac{\partial \tau_{xx}}{\partial x}+\frac{\partial \tau_{xz}}{\partial z}\right), \tag{13}$$

$$\varepsilon^2\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial z}\right) = -\frac{3\cot\theta}{\mathrm{Re}}+\frac{\varepsilon}{\mathrm{Re}}\left(\frac{\partial \tau_{zx}}{\partial x}+\frac{\partial \tau_{zz}}{\partial z}\right), \tag{14}$$

$$\varepsilon\mathrm{RePr}\left(\frac{\partial T}{\partial t}+u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial z}\right)=\varepsilon^2\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial z^2}. \tag{15}$$

II. Boundary conditions at the wall $(z=0)$ are

$$u=0=v, \tag{16}$$

$$T=x. \tag{17}$$

III. Boundary conditions at the free surface $(z=h)$ are

$$\left[\left(1-\varepsilon^2\left(\frac{\partial h}{\partial x}\right)^2\right)\tau_{zx}+\varepsilon^2(\tau_{zz}-\tau_{xx})\frac{\partial h}{\partial x}\right] \cdot\left(1+\varepsilon^2\left(\frac{\partial h}{\partial x}\right)^2\right)^{-1/2}=-\mathrm{Mn}\left(\frac{\partial T}{\partial x}+\frac{\partial h}{\partial x}\frac{\partial T}{\partial z}\right), \tag{18}$$

$$p_{\mathrm{a}}+\frac{\varepsilon}{\mathrm{Re}}\left[\varepsilon^2\tau_{xx}\left(\frac{\partial h}{\partial x}\right)^2-2\tau_{zx}\frac{\partial h}{\partial x}+\tau_{zz}\right] \cdot\left(1+\varepsilon^2\left(\frac{\partial h}{\partial x}\right)^2\right)^{-1} = \varepsilon^2\mathrm{We}(1-\mathrm{Ca}T)\frac{\partial^2 h}{\partial x^2}\left(1+\varepsilon^2\left(\frac{\partial h}{\partial x}\right)^2\right)^{-3/2}, \tag{19}$$

$$v=\frac{\partial h}{\partial t}+u\frac{\partial h}{\partial x}, \tag{20}$$

$$\left(\frac{\partial T}{\partial z}-\varepsilon^2\frac{\partial h}{\partial x}\frac{\partial T}{\partial x}\right)\left(1+\varepsilon^2\left(\frac{\partial h}{\partial x}\right)^2\right)^{-1/2}+\mathrm{Bi}T=0. \tag{21}$$

In dimensionless form the component of stress tensor given in (4a) can be written as

$$\tau_{xx}=-\frac{\mathrm{Re}}{\varepsilon}p+2\frac{\partial u}{\partial x}-2\varepsilon\mathrm{Re}\Gamma\left[\frac{\partial^2 u}{\partial x\partial t}+u\frac{\partial^2 u}{\partial x^2} +v\frac{\partial^2 u}{\partial x\partial z}-2\left(\frac{\partial u}{\partial x}\right)^2-\frac{\partial u}{\partial z}\left(\frac{1}{\varepsilon^2}\frac{\partial u}{\partial z}+\frac{\partial v}{\partial x}\right)\right], \tag{22}$$

$$\tau_{zz}=-\frac{\mathrm{Re}}{\varepsilon}p+2\frac{\partial v}{\partial z}-2\varepsilon\mathrm{Re}\Gamma\left[\frac{\partial^2 v}{\partial z\partial t}+v\frac{\partial^2 v}{\partial z^2} +u\frac{\partial^2 v}{\partial x\partial z}-2\left(\frac{\partial v}{\partial z}\right)^2-\frac{\partial v}{\partial x}\left(\frac{\partial u}{\partial z}+\varepsilon^2\frac{\partial v}{\partial x}\right)\right], \tag{23}$$

$$\tau_{xz}=\left(\frac{\partial u}{\partial z}+\varepsilon^2\frac{\partial v}{\partial x}\right)-\varepsilon\mathrm{Re}\Gamma\left[\frac{\partial^2 u}{\partial z\partial t}+\varepsilon^2\frac{\partial^2 v}{\partial x\partial t} +u\left(\varepsilon^2\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 u}{\partial x\partial z}\right)+v\left(\varepsilon^2\frac{\partial^2 v}{\partial x\partial z}+\frac{\partial^2 u}{\partial z^2}\right) -2\frac{\partial u}{\partial z}\frac{\partial v}{\partial z}-2\varepsilon^2\frac{\partial u}{\partial x}\frac{\partial v}{\partial x}\right], \tag{24}$$

where $\mathrm{Fr}(\equiv {u_0}^2/gh_0)$ is the Froude number, $\mathrm{Re}(\equiv u_0h_0/\nu)$ is the Reynolds number, $\mathrm{Pr}(\equiv \nu/\kappa)$ is the Prandtl number, $\mathrm{Mn}(\equiv 3\gamma(T_{\mathrm{H}}-T_{\mathrm{C}})/\rho g l_0 h_0\sin\theta)$ is the Marangoni number[1], $\mathrm{We}(\equiv \sigma_0/\rho {u_0}^2h_0)$ is the Weber number, $\mathrm{Ca}(\equiv \gamma(T_{\mathrm{H}}-T_{\mathrm{C}})/\sigma_0)$ is the capillary number, $\mathrm{Bi}(\equiv k_{\mathrm{g}}h_0/k_{\mathrm{T}})$ is the Biot number, $\Gamma(\equiv \Gamma_0/\rho {h_0}^2)$ is the viscoelastic parameter, and $\varepsilon(\equiv h_0/l_0)$ is the aspect ratio for long wave expansion. We have used the relation $\sin\theta/\mathrm{Fr}=3/\mathrm{Re}$, which can be shown by using the definition of the Nusselt velocity.

2.3. Long-Wave Approximation

The complexity of the surface problem posed by (12)–(21) can be overcome by invoking long wave (lubrication) approximation for $\varepsilon \ll 1$. This gives us an opportunity for asymptotic reduction of the governing equations and boundary conditions to a single nonlinear partial differential of Benney [21] type in term of local film thickness $h(x,t)$. A comprehensive review on the long-wave approximation method can be found in Chang [22], Oron and Gottlieb [23], and in many other papers. Experimental observations confirm that the interfacial disturbances exhibits long-wave characteristic within the regime of small Reynolds number. Therefore, we have to assume the Reynolds number is $O(1)$ in the forgoing study. Introducing the stream function ψ defined by

$$u=\psi_{,z},\quad v=-\psi_{,x}, \tag{25}$$

where (,) denotes the partial derivatives with respect to the following variables, we can write the above non-

[1] $\mathrm{Mn}=\varepsilon\mathrm{Ma}/\mathrm{RePr}$, $\mathrm{Ca}=\mathrm{CrMa}$, where $\mathrm{Ma}=\gamma(T_{\mathrm{H}}-T_{\mathrm{C}})h_0/\mu\kappa$ and $\mathrm{Cr}=\mu\kappa/\sigma_0h_0$ are, respectively, the Marangoni numbers and Crispation/capillary in usual definition.

dimensional governing equations and boundary conditions as

$$\begin{aligned}\psi_{,zzz} = -3 + \varepsilon\mathrm{Re}[(p_{,x} + \psi_{,tz} + \psi_{,z}\psi_{,xz} - \psi_{,x}\psi_{,zz}) \\ + \Gamma(\psi_{,tzzz} - \psi_{,xzz}\psi_{,zz} + \psi_{,xz}\psi_{,zzz} \\ + \psi_{,z}\psi_{,xzzz} - \psi_{,x}\psi_{,zzzz})] + O(\varepsilon^2),\end{aligned} \tag{26}$$

$$p_{,z} = -\varepsilon\mathrm{Re}^{-1}\psi_{,xzz} - (3\cot\theta/\mathrm{Re}) + O(\varepsilon^2), \tag{27}$$

$$T_{,zz} = \varepsilon\mathrm{Pe}(T_{,t} + \psi_{,z}T_{,x} - \psi_{,x}T_{,z}) + O(\varepsilon^2). \tag{28}$$

At $z = 0$:

$$\psi_{,x} = 0 = \psi_{,z}, \quad T = x. \tag{29}$$

At $z = h$:

$$\begin{aligned}\psi_{,zz} = -\mathrm{Mn}(T_{,x} + h_{,x}T_{,z}) + \varepsilon\mathrm{Re}\Gamma(2\psi_{,zz}^2 h_{,x} \\ + \psi_{,tzz} + \psi_{,z}\psi_{,xzz} - \psi_{,x}\psi_{,zzz} + 2\psi_{,zz}\psi_{,xz}) \\ + O(\varepsilon^2),\end{aligned} \tag{30}$$

$$\begin{aligned}p = p_\mathrm{a} - \varepsilon^2\mathrm{We}h_{,xx} - \varepsilon(2\mathrm{Re}^{-1}(\psi_{,zz}h_{,x} + \psi_{,xz}) \\ - \varepsilon\mathrm{WeCa}T h_{,xx}) + O(\varepsilon^2),\end{aligned} \tag{31}$$

$$h_{,t} + \psi_{,z}h_x + \psi_{,x} = 0, \tag{32}$$

$$T_{,z} = 0. \tag{33}$$

In writing (26) to (33) we have assumed that the Weber number is $O(1/\varepsilon^2)$, i.e. large enough which is a fact for a thin film. We have also assumed that the Marangoni number is of order $O(1)$, the Capillary number is $O(\varepsilon)$, the Prandtl number is $O(1)$, the Biot number is $O(\varepsilon^2)$, and the viscoelastic parameter is $O(1)$. It is customary to consider the Newton's law of cooling as a guiding law to describe the heat transfer on the interface but when the heat transfer on the interface is very small, the interface is almost adiabatic. As a consequence (21) reduced to (33) in the limit of small Biot numbers. The estimation of the physical parameters is discussed in the results and discussion section.

At the end of this subsection we like to mention here that the success of the lubrication approximation is its firmness [24] and a tendency for it to produce good agreement, with the experiment, in parameter regions on the optimal limits of the expected range of validity or beyond.

2.4. Evolution Equation

We are now interested in yielding a nonlinear evolution equation in terms of film thickness $h(x,t)$. Expanding the stream function, the pressure, and the temperature in powers of ε as

$$\psi = \psi_0 + \varepsilon\psi_1 + \ldots, \quad p = p_0 + \varepsilon p_1 + \ldots$$
$$\text{and } T = T_0 + \varepsilon T_1 + \ldots$$

and substituting the above into the governing equations (26) to (28) and the boundary conditions (29) to (33), we obtain a set of partial differential equations for different order of ε. The zeroth-order and the first-order problem and their solutions are obtained as equations at $O(\varepsilon^0)$:

$$\psi_{0,zzz} = -3, \quad p_{0,z} = -(3\cot\theta/\mathrm{Re}), \quad T_{0,zz} = 0 \tag{34}$$

with the boundary conditions

$$\psi_0 = \psi_{0,z} = 0, \quad T_0 = x \text{ on } z = 0, \tag{35}$$

$$\begin{aligned}\psi_{0,zz} = -\mathrm{Mn}(T_{0,x} + h_{,x}T_{0,z}), \\ p_0 = p_\mathrm{a} - \varepsilon^2\mathrm{We}h_{,xx}, \quad T_{0,z} = 0 \text{ on } z = h.\end{aligned} \tag{36}$$

Solution at $O(\varepsilon^0)$:

$$\begin{aligned}p_0 = p_\mathrm{a} + (3\cot\theta/\mathrm{Re})(h - z) - \varepsilon^2\mathrm{We}h_{,xx}, \\ T_0 = x,\end{aligned} \tag{37}$$

$$\psi_0 = -\frac{z^3}{2} + (3h - \mathrm{Mn})\frac{z^2}{2}. \tag{38}$$

Equations at $O(\varepsilon)$:

$$\begin{aligned}\psi_{1,zzz} = \mathrm{Re}[(p_{0,x} + \psi_{0,tz} + \psi_{0,z}\psi_{0,xz} - \psi_{0,x}\psi_{0,zz}) \\ + \Gamma(\psi_{0,tzzz} - \psi_{0,xzz}\psi_{0,zz} + \psi_{0,zzz}\psi_{0,xz} \\ + \psi_{0,z}\psi_{0,xzzz} - \psi_{0,x}\psi_{0,zzzz})],\end{aligned} \tag{39}$$

$$p_{1,z} = -\mathrm{Re}^{-1}\psi_{0,xzz}, \tag{40}$$

$$T_{1,zz} = \mathrm{Pe}(T_{0,t} + \psi_{0,z}T_{0,x} - \psi_{0,x}T_{0,z}) \tag{41}$$

with boundary conditions

$$\psi_1 = \psi_{1,z} = 0, \quad T_1 = 0 \text{ on } z = 0, \tag{42}$$

$$\begin{aligned}\psi_{1,zz} = -\mathrm{Mn}(T_{1,x} + h_{,x}T_{1,z}) \\ + \mathrm{Re}\Gamma(2\psi_{0,zz}^2 h_x + \psi_{0,tzz} + \psi_{0,z}\psi_{0,xzz} \\ - \psi_{0,x}\psi_{0,zzz} + 2\psi_{0,zz}\psi_{0,xz}) \text{ on } z = h,\end{aligned} \tag{43}$$

$$\begin{aligned}p_1 = -[2\mathrm{Re}^{-1}(\psi_{0,zz}h_{,x} + \psi_{0,xz}) - \varepsilon\mathrm{WeCa}T_0 h_{,xx}], \\ T_{1,z} = 0 \text{ on } z = h.\end{aligned} \tag{44}$$

Solution at $O(\varepsilon)$:

$$p_1 = \mathrm{Re}^{-1}h_{,x}\{2\mathrm{Mn} - 3(z + h)\} + \varepsilon\mathrm{WeCa}xh_{,xx}, \tag{45}$$

$$T_1 = \mathrm{Pe}\left[-\frac{z^4}{8} + \frac{1}{6}(3h-\mathrm{Mn})z^3 - \frac{h^2}{2}(2h-\mathrm{Mn})z\right], \quad (46)$$

$$\begin{aligned}\psi_1 = \mathrm{Re}\bigg[& -\frac{1}{6}\varepsilon^2 \mathrm{We}(z^3-3hz^2)h_{,xxx} \\ & + \bigg\{\frac{1}{40}(3h-\mathrm{Mn})(z^5-5hz^4+20h^3z^2) \\ & + \frac{\cot\theta}{2\mathrm{Re}}(z^3-3hz^2) \\ & + \frac{1}{4}\mathrm{PrMn}(5h-2\mathrm{Mn})h^2z^2\bigg\}h_{,x} \\ & + \frac{\Gamma}{2}(3h-\mathrm{Mn})\{(3h-\mathrm{Mn})z^2-z^3\}h_{,x}\bigg]. \end{aligned} \quad (47)$$

Invoking the solutions (38) and (47) into the expanded kinematic boundary condition (32), we get the generalized nonlinear kinematic equation characterising the evolution of the free surface as

$$h_{,t} + A(h)h_{,x} + \varepsilon(B(h)h_{,x} + \varepsilon^2 C(h)h_{,xxx})_{,x} = 0, \quad (48)$$

where

$$A(h) = 3h^2 - \mathrm{Mn}h,$$

$$B(h) = -\cot\theta h^3 + \mathrm{Re}\bigg[(3h-\mathrm{Mn}) \cdot\left\{\tfrac{2}{5}h^5 + \Gamma(h-\mathrm{Mn})h^2\right\} + \tfrac{1}{4}\mathrm{PrMn}(5h-2\mathrm{Mn})h^4\bigg],$$

$$C(h) = \frac{1}{3}\mathrm{ReWe}h^3.$$

3. Stability Analysis

The variation of the film thickness in the basic flow is found to be very small, so it is reasonable to assume that the local dimensionless film thickness is equal to one. Thus, the dimensionless film thickness in the perturbed state may be written as

$$h(x,t) = 1 + \eta, \quad (49)$$

where $\eta \ll 1$ is the dimensionless perturbation of the film thickness.

Setting the transformation

$$t = \varepsilon\tilde{t} \;\text{ and }\; x = \varepsilon\tilde{x} \quad (50)$$

and using (49) and (50) in (48) and retaining the terms up to the second-order fluctuations after dropping the tilde sign it can be written as

$$\begin{aligned} & \eta_{,t} + A\eta_{,x} + B\eta_{,xx} + C\eta_{,xxxx} + A'\eta\eta_{,x} \\ & + B'(\eta\eta_{,xx} + \eta_{,x}^2) + C'(\eta\eta_{,xxxx} + \eta_{,x}\eta_{,xxx}) \\ & + \frac{1}{2}A''\eta^2\eta_{,x} + B''\left(\frac{1}{2}\eta^2\eta_{,xx} + \eta\eta_{,x}^2\right) \\ & + C''\left(\frac{1}{2}\eta^2\eta_{,xxxx} + \eta\eta_{,x}\eta_{,xxx}\right) + O(\eta^4) = 0, \end{aligned} \quad (51)$$

where A, B, C and their corresponding derivatives ($'$) are evaluated at $h = 1$. (51) characterise the behaviour of finite amplitude perturbations on the thin viscous film and it is used to signify the time sapient behaviour of an initially sinusoidal perturbation on the viscous film.

Let us note that the assumption of constant film thickness with long wave perturbation for the basic flow, as used in (49), is a judicious one. Otherwise, if we take $h(x,t) = h_B(x) + \eta(x,t)$, instead of (49), where $h_B(x)$ is the time-averaged part of $h(x,t)$, then to obtain (51) we have (i) to neglect the derivatives of the averaged film thickness $h_B(x)$ with respect to x in the unsteady equation and (ii) to allow $h_B \to 1$ in the unsteady equation. The first assumption corresponds to parallel flow approximation, while the second one leads to the restriction that the unsteady equation (51) is only locally valid.

3.1. Linear Stability Analysis

In this section we are interested to study the linear response for a sinusoidal perturbation of the film by assuming the perturbation of the form

$$\eta = \Lambda[\exp\{\mathrm{i}(kx-\omega t)\}] + \mathrm{c.c}, \quad (52)$$

where Λ is the amplitude of the disturbance and c.c represents the complex conjugate. Here the wave number k is real and $\omega = \omega_r + \mathrm{i}\omega_i$ is the complex frequency. Using (52) in the linearized part of (51), we get the dispersion relation as

$$D(\omega,k) \equiv -\mathrm{i}\omega + \mathrm{i}Ak - Bk^2 + Ck^4 = 0. \quad (53)$$

Equating the real and imaginary parts of (53), we get

$$\omega_r = Ak \;\text{ and }\; \omega_i = Bk^2 - Ck^4. \quad (54)$$

Therefore, the phase speed is

$$c_r = \omega_r/k = 3 - \mathrm{Mn}. \quad (55)$$

It is to be noted here that the phase speed is independent of k, implying the wave is non-dispersive in nature. Let us point out here that the viscoelastic parameter Γ has no influence on the linear phase speed, but the phase speed decreases with the increase of Mn.

In the neutral state $\omega_\mathrm{i} = 0$ gives two relations:

$$k = 0, \tag{56a}$$

$$k_c = \sqrt{B/C}, \tag{56b}$$

which correspond to two branches of the neutral curves and the flow instability takes place in between them. Further, the neutral curves intersect at the bifurcation point $\mathrm{Re} = \mathrm{Re}_\mathrm{c}$, $k = 0$. The wave number of the waves with a maximum growth is obtained from the relation $\mathrm{d}\omega_\mathrm{i}/\mathrm{d}k = 0$, which gives $k_\mathrm{m} = k_\mathrm{c}/\sqrt{2}$.

The onset of instability is obtained by considering the zero critical wave number as given in (56b), which gives the critical condition

$$\cot\theta = \mathrm{Re}\left[(3-\mathrm{Mn})\left\{\frac{2}{5} + \Gamma(1-\mathrm{Mn})\right\} + \frac{1}{4}\mathrm{Pr}\,\mathrm{Mn}(5-2\mathrm{Mn})\right]. \tag{57}$$

The above relation defines the critical Reynolds number

$$\mathrm{Re}_\mathrm{c} = 20\cot\theta\left[8(3-\mathrm{Mn}) + 20\Gamma(3-\mathrm{Mn})(1-\mathrm{Mn}) + 5\mathrm{Pr}\mathrm{Mn}(5-2\mathrm{Mn})\right]^{-1} \tag{58}$$

above which the flow loses stability.

For $\mathrm{Mn} = 0$, $\mathrm{Re}_\mathrm{c} = \cot\theta/(1.2 + 3\Gamma)$, which has the same functional form to the one obtained by Uma and Usha [9] except the term 1.2 instead 1 (eqn. (53) in that work) in the denominator of the right hand side of (58), which is due to the well-known fact that they [9] have calculated the film evolution equation by the integral method of Karman. Moreover, for $\mathrm{Mn} = 0 = \Gamma$, the relation (58) leads to $\mathrm{Re}_\mathrm{c} = (5/6)\cot\theta$ which is exactly the same to that obtained by Benjamin [25] and Yih [26].

In the next section we will study the growth of weakly nonlinear waves and therefore we need to consider two or more characteristic time scales of different order of magnitude. As a rule, problems of this kind can not be solved by means of the classical perturbation method or by the method of matched asymptotic expansions. The problem can be overcomes by introducing the method of multiple scales.

3.2. Weakly Nonlinear Stability Analysis

Now we are interested to study those small amplitude waves which develop immediately just after the break down of the flat film solution($\eta = 0$).

From the linear stability analysis we see that in the neutral state all modes are neither stable nor unstable and therefore $\omega_\mathrm{i} = 0$, which corresponds to two branches of the neutral curve that intersect at the bifurcation point $(\mathrm{Re}_\mathrm{c}, 0)$. Thus, one expect in the vicinity of the upper branch of the neutral curve a thin band of width $\varsigma \ll 1$(say) of unstable modes around a central one appearing over a time $O(\varsigma^{-2})$ such that $\omega_\mathrm{i} \sim O(\varsigma^2)$. Therefore the main features of the system behaviour can be obtained from an analysis of these modes and so we investigate the nonlinear evolution of the unstable linear waves in the region where $\omega_\mathrm{i} \sim O(\varsigma^2)$.

The above expectation together with the anticipation that the wave packet may travel at a group velocity of order one, suggest a scale

$$x_1 = \varsigma x, \quad t_1 = \varsigma t, \quad t_2 = \varsigma^2 t, \quad \ldots \tag{59}$$

for multiple scale analysis. Here t and x are fast scales, whereas t_1, x_1, and so on are slow scales. It is assumed that these variables are mutually independent, so the temporal and spatial derivatives become

$$\partial_t \to \partial_t + \varsigma\partial_{t_1} + \varsigma^2\partial_{t_2}, \quad \partial_x \to \partial_x + \varsigma\partial_{x_1}. \tag{60}$$

Now the surface deformation will be expressed as $\eta = \sum \eta_\mathrm{i}\varsigma^i$.

Using (59) and (60) in (51) we get

$$(L_0 + \varsigma L_1 + \varsigma^2 L_2 + \ldots)(\varsigma\eta_1 + \varsigma^2\eta_2 + \varsigma^3\eta_3 + \ldots) = -\varsigma^2 N_2 - \varsigma^3 N_3 - \ldots, \tag{61}$$

where L_0, L_1, L_2 etc. are the operators and N_2, N_3 are the nonlinear terms of (61) which are given in the appendix.

In the lowest-order of ς, we have

$$L_0\eta_1 = 0, \tag{62}$$

which has a solution of the form

$$\eta_1 = \Lambda(x_1, t_1, t_2)[\exp \mathrm{i}\Theta] + \mathrm{c.c}, \tag{63}$$

where $\Theta = kx - \omega_\mathrm{r} t$ and c.c denotes the complex conjugates. It is to be noted here that the above solution given in (63) is already obtained in connection with

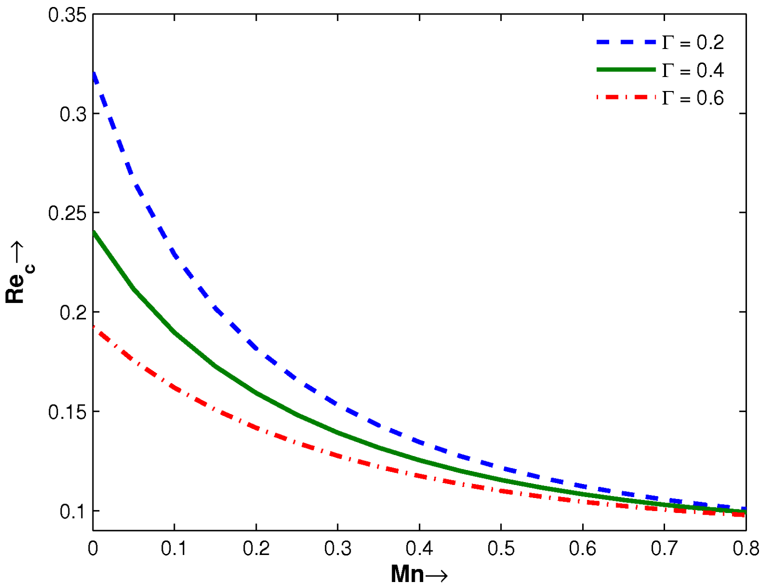

Fig. 1 (colour online). Variation of $\mathrm{Re_c}$ with Mn for different Γ and for fixed $\theta = \pi/3$, $\mathrm{Pr} = 7$.

the linear stability analysis except ω is replaced by ω_r, since in the vicinity of the neutral curve $\omega_\mathrm{i} = O(\varsigma^2)$, so that the function $\exp(\omega_\mathrm{i} t)$ is slowly varying and may be absorbed in $\Lambda(x_1, t_1, t_2)$.

In the second order, the perturbation system yields

$$L_0\eta_2 = -L_1\eta_1 - N_2. \tag{64}$$

Invoking (63) in (64), we have

$$L_0\eta_2 = -\mathrm{i}\left[\frac{\partial D(\omega_\mathrm{r},k)}{\partial \omega_\mathrm{r}}\frac{\partial \Lambda}{\partial t_1} - \frac{\partial D(\omega_\mathrm{r},k)}{\partial k}\frac{\partial \Lambda}{\partial x_1}\right]\mathrm{e}^{\mathrm{i}\Theta} - \Omega\Lambda^2\mathrm{e}^{2\mathrm{i}\Theta} + \mathrm{c.c}, \tag{65}$$

where $D(\omega_\mathrm{r},k)$ is given by (53) and

$$\Omega = \mathrm{i}A'k - 2B'k^2 + 2C'k^4.$$

The uniform valid solution for η_2 is obtained from (65) as

$$\eta_2 = -\frac{\Omega\Lambda^2\mathrm{e}^{2\mathrm{i}\Theta}}{D(2\omega_\mathrm{r},2k)} + \mathrm{c.c.} \tag{66}$$

Introducing the co-ordinate transformation $\xi = (x_1 - c_\mathrm{g}t_1)$, where c_g $(= -D_k/D_{\omega_\mathrm{r}})$ is the group velocity, and using the solvability condition on the third-order equation, we get

$$\frac{\partial \Lambda}{\partial t_2} + J_1\frac{\partial^2\Lambda}{\partial \xi^2} - \varsigma^{-2}\omega_\mathrm{i}\Lambda + (J_2 + \mathrm{i}J_4)|\Lambda|^2\Lambda = 0, \tag{67}$$

where

$$J_1 = B - 6Ck^2,$$

$$J_2 = \frac{1}{2}(-B''k^2 + C''k^4) + \left[\frac{(A')^2k^2 - 2(B'k^2 - 7C'k^4)(B'k^2 - C'k^4)}{16Ck^4 - 4Bk^2}\right],$$

and

$$J_4 = \frac{1}{2}A''k + \left[\frac{A'k(B'k^2 - 7C'k^4) + 2A'k(B'k^2 - C'k^4)}{16Ck^4 - 4Bk^2}\right].$$

For filtered waves there is no spatial modulation and the diffusion term vanishes; we get

$$\frac{\partial \Lambda}{\partial t_2} - \varsigma^{-2}\omega_\mathrm{i}\Lambda + (J_2 + \mathrm{i}J_4)|\Lambda|^2\Lambda = 0. \tag{68}$$

The solution of this equation may be written as

$$\Lambda = a\mathrm{e}^{-\mathrm{i}b(t_2)t_2}, \tag{69}$$

which gives

$$\frac{\partial a}{\partial t_2} = [\varsigma^{-2}\omega_\mathrm{i} - J_2a^2]a \tag{70}$$

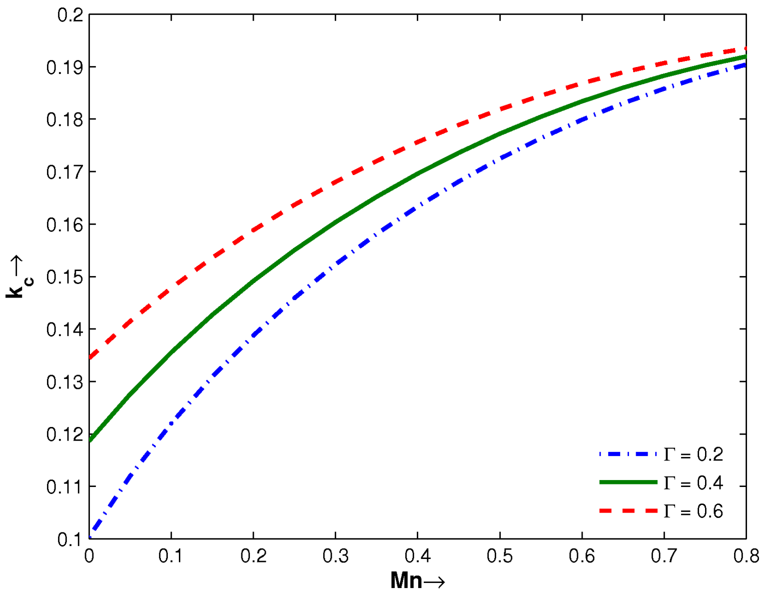

Fig. 2 (colour online). Variation of k_c with Mn for different Γ and for fixed We $= 450$, $\mathrm{Re}_c = 2$, $\theta = \pi/3$, Pr $= 7$.

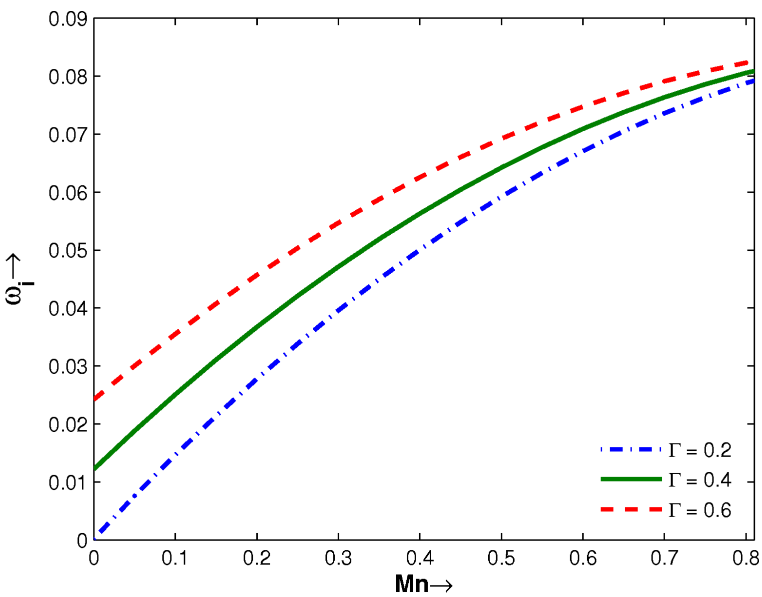

Fig. 3 (colour online). Variation of ω_i with Mn for different Γ and for fixed We $= 450$, $\mathrm{Re}_c = 2$, $\theta = \pi/3$, Pr $= 7$, $k = 0.1$.

and

$$\frac{\partial(b(t_2)t_2)}{\partial t_2} = J_4 a^2. \tag{71}$$

(70) is nothing but the Landau equation. The second term on the right hand side of (70) is due to nonlinearity and may moderate or accelerate the exponential growth of the linear disturbance. The perturbed wave speed caused by the infinitesimal disturbances appearing in the nonlinear system can be modified using (71).

For the existence of a supercritical stable zone in the linearly unstable region ($\omega_i < 0$), the second Landau constant J_2 should be positive and the threshold amplitude will be

$$\varsigma a = [\omega_i / J_2]^{1/2}. \tag{72}$$

On the other hand, in the linear unstable zone ($\omega_i < 0$) if J_2 is negative, the flow will be subcritically unstable and ςa is the threshold amplitude. The nonlinear wave

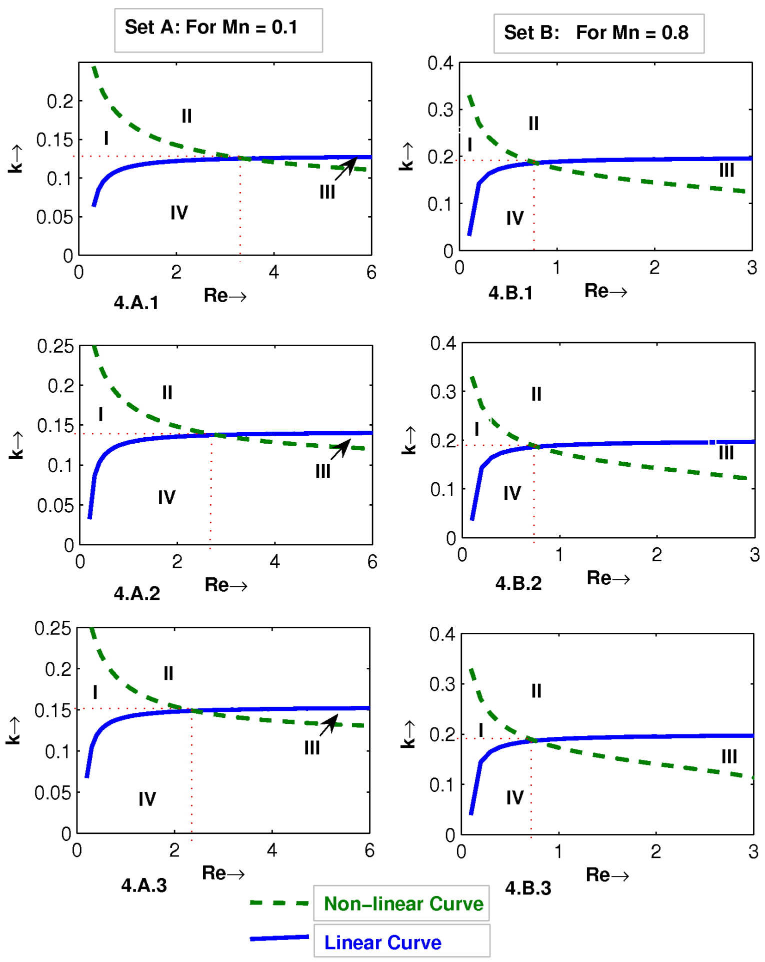

Fig. 4 (colour online). **Set A**: Stability curve k versus Re for fixed values of the parameter We $= 450$, $\theta = \pi/3$, Pr $= 7$, Mn $= 0.1$ varying Γ as (1): $\Gamma = 0.2$, (2): $\Gamma = 0.4$, (3): $\Gamma = 0.6$. **Set B**: Stability curve k versus Re for fixed values of the parameter We $= 450$, $\theta = \pi/3$, Pr $=$ 7, Mn $= 0.8$ varying Γ as (1): $\Gamma = 0.2$, (2): $\Gamma = 0.4$, (3): $\Gamma = 0.6$. Zone I, II, III, and IV denote unconditional, subcritical, explosive, and supercritical zone, respectively.

speed in the supercritical region will be

$$Nc_r = c_r + c_i(J_4/J_2), \quad \text{where} \quad c_i = \omega_i/k. \qquad (73)$$

4. Results and Discussion

The objective of the present study is to quantify the effect of thermocapillarity on a viscoelastic film flowing down an inclined plane with linear temperature variation in the finite amplitude regime. Physical parameters chosen for this investigation are (1) Reynolds number ranging from 0 to 6, (2) Marangoni number ranging from 0 to 0.8, (3) Prandtl number 7, (4) viscoelastic parameter ranging from 0 to 0.6.

The ongoing analysis is a completely theoretical one. Since a complete set of data regarding the necessary physical properties for any particular viscoelastic fluid that can perfectly match with this problem are not available, therefore it is very difficult to estimate the values of the parameter involved in this problem for a particular fluid. To estimate the value of the viscoelastic parameter, we have considered [9], a mixture of polymethyle methacrylate in pyridine with density $\rho = 0.98 \times 10^3$ Kg m^{-3}, limiting viscosity $\mu = 0.79$ Ns m^{-2}, and viscoelastic coefficient $\Gamma_0 = 0.4$ Ns^2 M^{-2}. The viscoelastic parameter Γ for a film of this mixture with $h_0 = 10^{-2}$ m becomes 0.4. To estimate the value of the Marangoni number, we have considered [13] water at 25 ℃ with $\gamma = 5 \times 10^{-5}$ Kg S^{-2} K^{-1}, $\mu = 10^{-3}$ Kg m^{-1}S^{-1}, $\rho = 10^3$ Kg m^{-3} gives Pr $\simeq 7$ and with temperature difference 11 K, we obtain Mn $= 0.5$, where $\varepsilon = 0.01$ for which our long wave assumption is satisfied.

Therefore, to capture the instability mechanism, the estimated value of the Marangoni number and the viscoelastic parameter should be wide ranging. For this

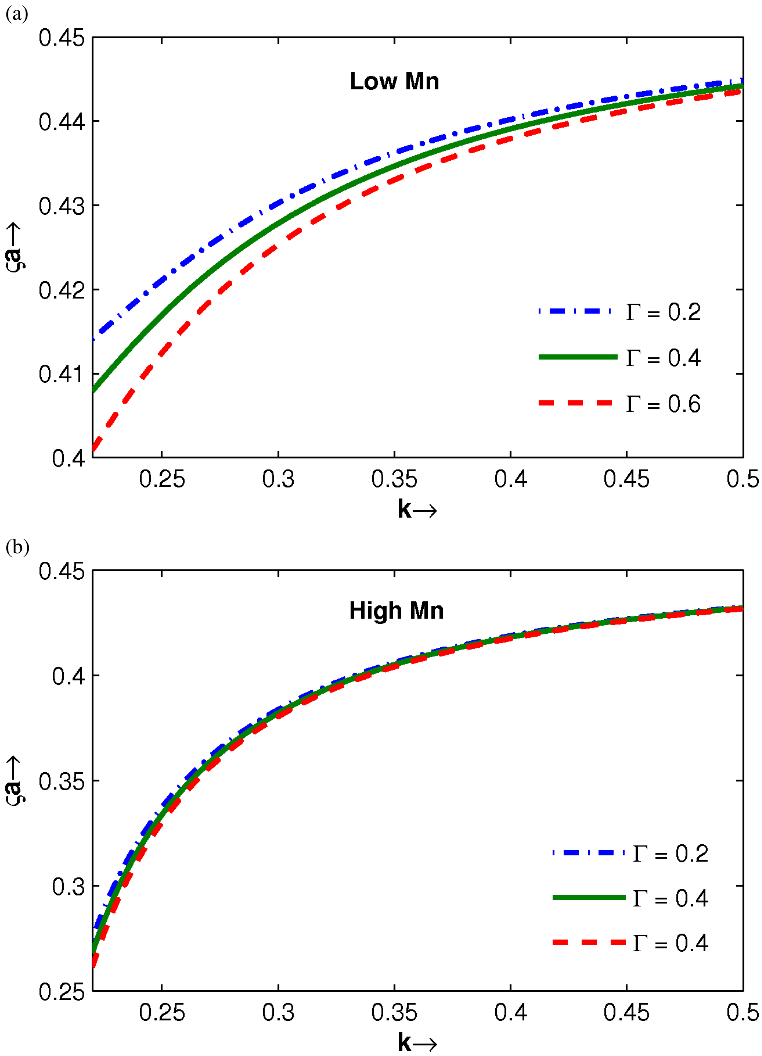

Fig. 5 (colour online). Amplitude of the disturbances in the subcritical region for We = 450, $\theta = \pi/3$, Pr = 7, Re = 2; (a) Mn = 0.1 and (b) Mn = 0.8 with different Γ.

reason we have considered the value of Mn from 0 to 0.8 and that of Γ from 0 to 0.6.

Figure 1 depicts the variation of the critical Reynolds number $\mathrm{Re_c}$ with the Marangoni number Mn for different values of the viscoelastic parameter Γ. It reveals from the figure that $\mathrm{Re_c}$ decreases with the increase of both Mn and Γ exhibiting the destabilising role of Marangoni number and the viscoelastic parameter. Again it is clear from the figure that with the increase of the viscoelastic parameter Γ, $\mathrm{Re_c}$ decreases for small Marangoni number, but for large Marangoni number the influence of viscoelastic parameter Γ is very small on the criticality.

Figure 2 displays the variation of the critical wave number k_c with the Marangoni number Mn. It is clear from the figure that k_c increases with Mn and also increases with Γ for the small value of Mn while at large value of Mn the variation is very feeble.

Figure 3 shows that the viscoelastic parameter enhances the growth of the instability for small values of Mn while it has very small influence at large values of Mn.

By observing the span of the zones from Figures 4.A.1 – A.3, it is clear that at low Mn the supercritical zone enlarges with the increase of Γ whereas the unconditional stable zone shrinks with the increase of Γ and the other zones almost remain unchanged. On the other hand, Figures 4.B.1 – B.3 reveals that at high Mn the zones are almost same, i. e., the effect of Γ on the zones are insignificant.

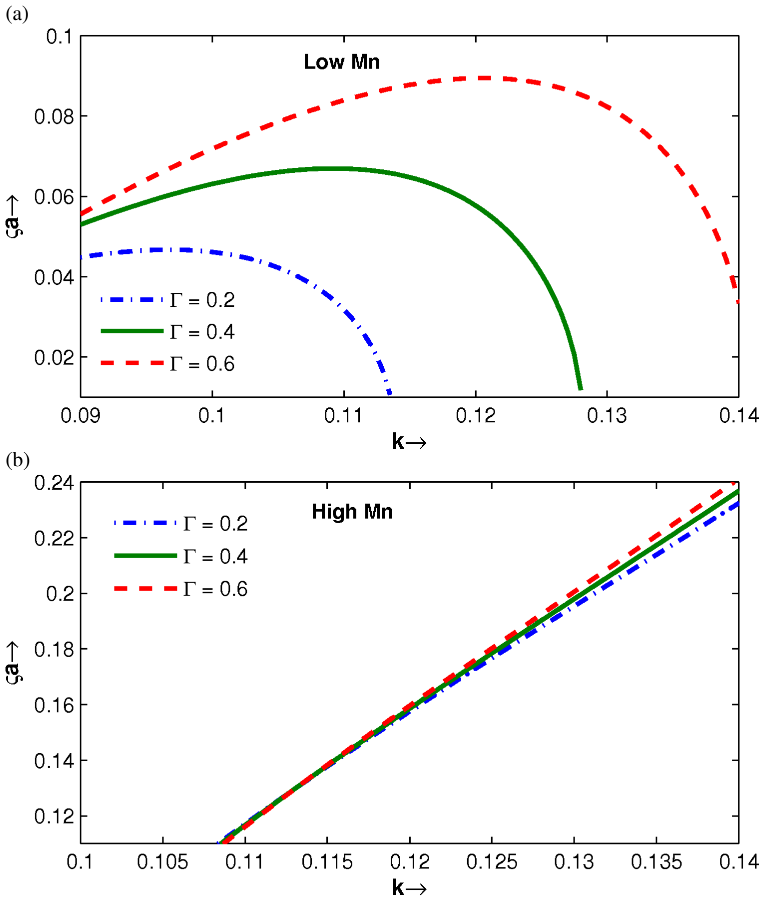

Fig. 6 (colour online). Amplitude of the disturbances in the supercritical region for We $= 450$, $\theta = \pi/3$, Pr $= 7$, Re $= 1$; (a) Mn $= 0.1$ and (b) Mn $= 0.8$ with different Γ.

Figures 5a and b display the amplitude in the subcritical region for different viscoelastic parameters Γ for low and high Marangoni number, respectively. It is clear from the figures that the amplitude of the nonlinear waves decreases with the increase of both Γ and Mn, but the variation of the wave amplitude is very small for high Marangoni number compared with that of low Marangoni number.

Figures 6a and b show the amplitude of the supercritical waves for different viscoelastic parameters Γ for low and high Marangoni number, respectively. It reveals from the figures that the amplitude of the nonlinear waves increases with the increases of both Γ and Mn, but the variation of the wave amplitude with Γ is very low for high Marangoni number compared with that of low Marangoni number.

For the supercritical region, the variation of the nonlinear phase speed Nc_r with respect to the wave number for different Γ with low and high values of the Marangoni number Mn are shown in Figures 7a and b, respectively. It reveals that the nonlinear phase speed Nc_r in the supercritical region increases with both Mn and Γ and confirms the destabilizing role of the parameter Mn and Γ. As a consequence at high Mn the nonlinear phase speed increases more with the increase of Γ than that of the low Mn, which is also confirmed by the Figures 7a and b.

The destabilizing role of the Marangoni number Mn can be physically interpreted as follows: the instabilities of the thin falling films are convective, the disturbances are transported down stream by the mean flow. Again due to thermocapillarity, the liquid moves from warmer region to the neighbouring colder region. In the present case the increase of Mn implies that the temperature of the inclined plane is increasing with down stream direction. Thus, the fluid becomes warmer in the downward and the thermocapillary force appearing at the interface acts in the opposite direction of the gravitational acceleration and then enhances the growth of the surface instabilities.

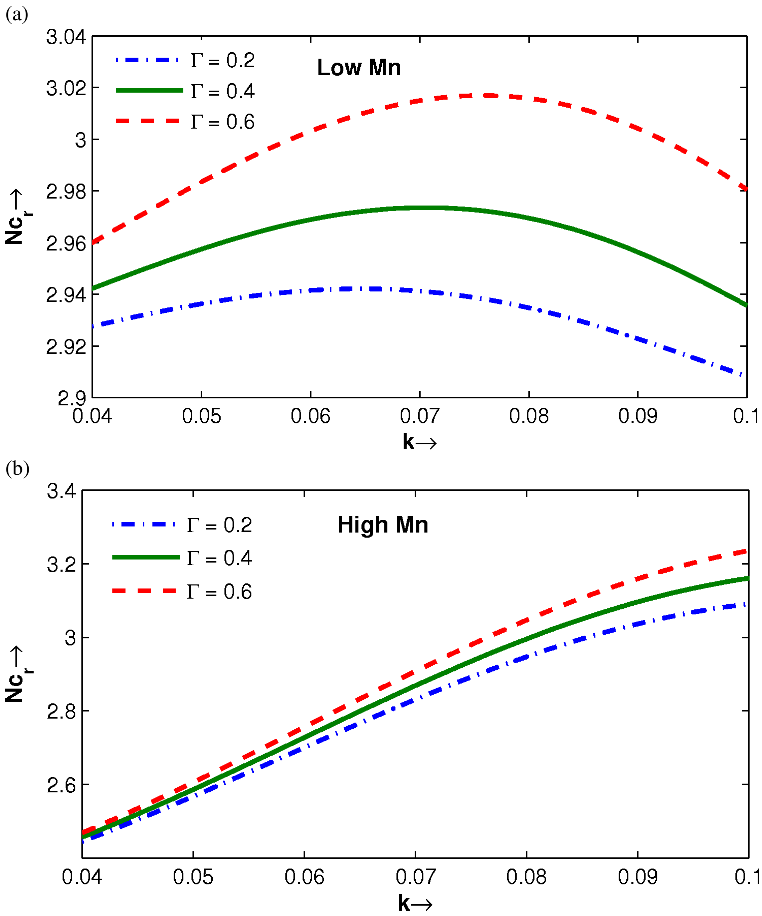

Fig. 7 (colour online). Nonlinear wave speed in the supercritical region for We $= 450$, $\theta = \pi/3$, Pr $= 7$, Re $= 1.4$; (a) Mn $= 0.1$ and (b) Mn $= 0.8$ with different Γ.

On the other hand, the less prominance of the viscoelastic behaviour (feeble impact of Γ) with increasing temperature (increasing Mn) can be interpreted as follows: the increase of the temperature leads to smaller coherent groups of molecules, owing to the increased agitation of individual molecules, and as a consequence, there is less tendency in the fluid to relax back to its previous shape, though these tendency (i. e. memory) is very short or rapidly fading for Walters' B$''$ fluids.

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Appendix

$$L_0 \equiv \frac{\partial}{\partial t} + A\frac{\partial}{\partial x} + B\frac{\partial^2}{\partial x^2} + C\frac{\partial^4}{\partial x^4}, \quad L_1 \equiv \frac{\partial}{\partial t_1} + A\frac{\partial}{\partial x_1} + 2B\frac{\partial^2}{\partial x \partial x_1} + 4C\frac{\partial^4}{\partial x^3 \partial x_1}, \quad L_2 \equiv \frac{\partial}{\partial t_2} + B\frac{\partial^2}{\partial x_1^2} + 6C\frac{\partial^4}{\partial x^2 \partial x_1^2},$$

$$N_2 = A'\eta_1\frac{\partial \eta_1}{\partial x} + B'\left[\eta_1\frac{\partial^2\eta_1}{\partial x^2} + \left(\frac{\partial \eta_1}{\partial x}\right)^2\right] + C'\left[\eta_1\frac{\partial^4\eta_1}{\partial x^4} + \frac{\partial \eta_1}{\partial x}\frac{\partial^3\eta_1}{\partial x^3}\right],$$

$$N_3 = A'\left[\eta_1\left(\frac{\partial \eta_2}{\partial x} + \frac{\partial \eta_1}{\partial x_1}\right) + \eta_2\frac{\partial \eta_1}{\partial x}\right] + B'\left[\eta_1\left(\frac{\partial^2\eta_2}{\partial x^2} + 2\frac{\partial^2\eta_1}{\partial x \partial x_1}\right) + \eta_2\frac{\partial^2\eta_1}{\partial x^2} + 2\frac{\partial \eta_1}{\partial x}\left(\frac{\partial \eta_2}{\partial x} + \frac{\partial \eta_1}{\partial x_1}\right)\right]$$

$$+ C' \left[\eta_1 \left(\frac{\partial^4 \eta_2}{\partial x^4} + 4 \frac{\partial^4 \eta_1}{\partial x^3 \partial x_1} \right) + \eta_2 \frac{\partial^4 \eta_1}{\partial x^4} + \frac{\partial \eta_1}{\partial x} \left(\frac{\partial^3 \eta_2}{\partial x^3} + 3 \frac{\partial^3 \eta_1}{\partial x^2 \partial x_1} \right) + \frac{\partial^3 \eta_1}{\partial x^3} \left(\frac{\partial \eta_2}{\partial x} + \frac{\partial \eta_1}{\partial x_1} \right) \right]$$

$$+ \frac{1}{2} A'' {\eta_1}^2 \frac{\partial \eta_1}{\partial x} + B'' \left(\frac{1}{2} {\eta_1}^2 \frac{\partial^2 \eta_1}{\partial x^2} + \eta_1 \left(\frac{\partial \eta_1}{\partial x} \right)^2 \right) + C'' \left(\frac{1}{2} {\eta_1}^2 \frac{\partial^4 \eta_1}{\partial x^4} + \eta_1 \frac{\partial \eta_1}{\partial x} \frac{\partial^3 \eta_1}{\partial x^3} \right).$$

[1] A. S. Gupta, J. Fluid Mech. **28**, 17 (1967).
[2] W. Lai, Phys. Fluids **10**, 844 (1967).
[3] A. S. Gupta and L. Rai, J. Fluid Mech. **33**, 87 (1968).
[4] B. S. Dandapat, Rheol. Acta **17**, 492 (1978).
[5] E. S. G. Shaqfeh, R. G. Larson, and G. H. Fredrickson, J. Non-Newtonian Fluid Mech. **31**, 87 (1989).
[6] F. Kang and K. P. Chen, J. Non-Newtonian Fluid Mech. **57**, 243 (1995).
[7] B. S. Dandapat and A. S. Gupta, Rheol. Acta. **36**, 135 (1997).
[8] P. J. Cheng, H. Y. Lai, and C. K. Chen, J. Phys. D: Appl. Phys. **33**, 1674 (2000).
[9] B. Uma and R. Usha, Int. J. Eng. Sci. **44**, 1449 (2006).
[10] D. A. Goussis and R. E. Kelly, J. Fluid Mech. **223**, 25 (1991).
[11] S. W. Joo, S. H. Davis, and S. G. Bankoff, J. Fluid Mech. **230**, 117 (1991).
[12] M. K. Smith and S. H. Davis, J. Fluid Mech. **132**, 119 (1983).
[13] S. Kalliadasis, E. A. Demekhin, C. Ruyer-Quil, and M. G. Velarde, J. Fluid Mech. **492**, 303 (2003).
[14] C. Ruyer-Quil, B. Scheid, S. Kalliadasis, M. G. Valarde, and R. Kh. Zeytounian, J. Fluid Mech. **538**, 199/223 (2005).
[15] P. Kalitzova-Kurteva, S. Slavtchev, and I. Kurtev, J. Theor. Appl. Mech. **30**, 12 (2000).
[16] S. Miladinova, S. Slavtchev, G. Lebon, and J. C. Legros, J. Fluid Mech. **453**, 153 (2002).
[17] E. A. Demekhin, S. K. Kalliadasis, and M. G. Velarde, Phys. Fluid **18**, 042111.1 (2006).
[18] D. W. Beard and K. Walters, Proc. Comb. Philos. Soc. **60**, 667 (1964).
[19] F. M. White, Viscous Fluid Flow 3rd edn, McGraw Hill, Singapore (2006).
[20] W. K. Rhim, K. Ohsaka, P. F. Paradis and R. E. Spjut, Rev. Sci. Instrum. **70**, 2796 (1999).
[21] D. J. Benney, J. Math. Phy. **45**, 150 (1966).
[22] H. C. Chang, Annu. Rev. Fluid Mech. **36**, 103 (1994).
[23] A. Oron and O. Gottlieb, J. Eng. Math. **50**, 121 (2004).
[24] A. F. M. Leenaars and S. B. G. O'Brien, J. Res. **44**, 183 (1989).
[25] T. B. Benjamin, Phys. Fluids **2**, 554 (1957).
[26] C. S. Yih, Phys. Fluids **6**, 321 (1963).