Variational Iteration Method for the Hirota-Satsuma Model Using He's Polynomials

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This paper out lines the implementation of the variational iteration method using He's polynomials (VMHP) for solving the Hirota-Satsuma model which occurs quite often in applied sciences. Numerical results show that the proposed VIMHP is quite efficient.

Key words: Variational Iteration Method; He's Polynomials; Hirota-Satsuma Coupled KdV Systems.

1. Introduction

Recently, Ji-Huan He [1 – 15] developed two very efficient techniques which are named as variational iteration (VIM) and homotopy perturbation (HPM) methods. These schemes completely changed the research scenario in nonlinear sciences due to their simplicity coupled with tangible accuracy. VIM and HPM have been applied on a wide range of physical problems, see [1 – 39] and the references therein. In a subsequent work, Ghorbani and Nadjfi [26, 27] introduced He's polynomials which are calculated from the homotopy perturbation method (HPM). Most recently, Noor and Mohyud-Din [33 – 35] made the elegant coupling of He's polynomials and the correction functional of VIM. This very reliable modified version (VIMHP) has been proved to be useful in coping with the physical nature of the nonlinear problems and, hence, absorbs all the positive features of the coupled techniques, see [39 – 42]. Inspired and motivated by the ongoing research in this area, we applied the variational iteration method using He's polynomials (VIMHP) for solving the Hirota-Satsuma model which arises quite often in applied sciences, see [18, 29, 39 – 47]. The obtained results are very encouraging.

2. Variational Iteration Method using He's Polynomials (VIMHP)

To illustrate the basic concept of VIMHP, we consider the following general differential equation:

$$Lu + Nu = g(x), \tag{1}$$

where L is a linear operator, N a nonlinear operator, and $g(x)$ is the forcing term. According to VIM [5, 10 – 17, 19 – 25, 30, 32 – 38], we can construct a correction functional as follows:

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda(\xi)(Lu_n(\xi) + N\tilde{u}_n(\xi) - g(\xi))\mathrm{d}\xi, \tag{2}$$

where λ is a Lagrange multiplier [5, 10 – 15], $\tilde{u}_n$ is a restricted variation; (2) is called a 'correction functional'. Now, we apply He's polynomials [13, 14]

$$\sum_{n=0}^{\infty} p^{(n)} u_n = u_0(x) + p\int_0^x \lambda(\xi)\left(\sum_{n=0}^{\infty} p^{(n)} L(u_n) + \sum_{n=0}^{\infty} p^{(n)} N(\tilde{u}_n)\right)\mathrm{d}\xi - \int_0^x \lambda(\xi) g(\xi)\mathrm{d}\xi, \tag{3}$$

which is the VIMHP [33 – 35] and is formulated by the coupling of VIM and He's polynomials. The comparison of like powers of p gives solutions of various orders.

3. Solution Procedure

Consider the following Hirota-Satsuma coupled Korteweg-de Vries (KdV) system:

$$u_t - \frac{1}{2}u_{xxx} + 3uu_x - 3(vw)x = 0,$$

$$v_t - v_{xxx} - 3uv_x = 0,$$
$$w_t + w_{xxx} - 3uw_x = 0$$

with the initial conditions

$$u(x,0) = \frac{1}{3}(\beta - 2k^2) + 2k^2 \tanh^2(kx),$$
$$v(x,0) = \frac{-4k^2 c_0(\beta + k^2)}{3c_1^2} + \frac{4k^2(\beta + k^2)}{3c_1} \tanh(kx),$$
$$w(x,0) = c_0 + c_1 \tanh(kx),$$

where c_0, c_1, and β are constants. The exact solution of the problem is given by

$$u(x,t) = \frac{1}{3}(\beta - 2k^2) + 2k^2 \tanh^2(k(x+\beta t)),$$
$$v(x,t) = \frac{-4k^2 c_0(\beta + k^2)}{3c_1^2} + \frac{4k^2(\beta + k^2)}{3c_1} \tanh(k(x+\beta t)),$$
$$w(x,t) = c_0 + c_1 \tanh(k(x+\beta t)).$$

The correction functionals for the above system are given as

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda_1(s)\Big[u_{nt} - \frac{1}{2}\tilde{u}_{n_{xxx}} + 3\tilde{u}_n\tilde{u}_{n_x} - 3\tilde{v}_n\tilde{w}_n x\Big]\mathrm{d}s,$$
$$v_{n+1}(x,t) = v_n(x,t) + \int_0^t \lambda_2(s)(v_{nt} - \tilde{v}_{n_{xxx}} - 3\tilde{u}_n\tilde{v}_{n_x})\mathrm{d}s,$$
$$w_{n+1}(x,t) = w_n(x,t) + \int_0^t \lambda_3(s)(w_{nt} + \tilde{w}_{n_{xxx}} - 3\tilde{u}_n\tilde{w}_{n_x})\mathrm{d}s.$$

Making the correction functional stationary, the Lagrange multipliers can easily be identified as $\lambda_1 = \lambda_2 = \lambda_3 = -1$, consequently,

$$u_{n+1}(x,t) = u_n(x,t) - \int_0^t \Big(u_{nt} - \frac{1}{2}u_{n_{xxx}} + 3u_n u_{n_x} - 3v_n w_n x\Big)\mathrm{d}s,$$
$$v_{n+1}(x,t) = v_n(x,t) - \int_0^t (v_{nt} - v_{n_{xxx}} - 3u_n v_{n_x})\mathrm{d}s,$$
$$w_{n+1}(x,t) = w_n(x,t) - \int_0^t (w_{nt} + w_{n_{xxx}} - 3u_n w_{n_x}))\mathrm{d}s.$$

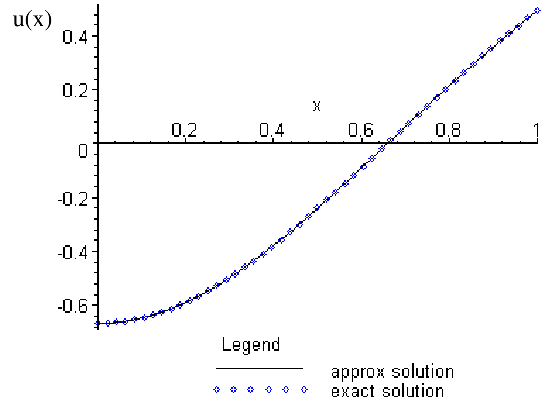

Fig. 1. Solution u with $b = 0$, $k = t = 1$.

Applying the variational iteration method using He's polynomials (VIMHP), we get

$$u_0 + pu_1 + \cdots = u_0 - p\int_0^t \Big[\Big(\frac{\partial u_0}{\partial s} + p\frac{\partial u_1}{\partial s} + \cdots\Big) - \frac{1}{2}\Big(\frac{\partial^3 u_0}{\partial x^3} + p\frac{\partial^3 u_1}{\partial x^3} + \cdots\Big) + 3(u_0 + pu_1 + \cdots)\Big(\frac{\partial u_0}{\partial x} + p\frac{\partial u_1}{\partial x^3} + \cdots\Big)\Big]\mathrm{d}s - 3p\int_0^t (v_0 + pv_1 + \cdots)(w_0 + pw_1 + \cdots)x\mathrm{d}s,$$
$$v_0 + pv_1 + \cdots = v_0 - p\int_0^t \Big[\Big(\frac{\partial v_0}{\partial s} + p\frac{\partial v_1}{\partial s} + \cdots\Big) - \Big(\frac{\partial^3 v_0}{\partial x^3} + p\frac{\partial^3 v_1}{\partial x^3} + \cdots\Big) + 3(u_0 + pu_1 + \cdots)\Big(\frac{\partial v_0}{\partial x} + p\frac{\partial v_1}{\partial x^3} + \cdots\Big)\Big]\mathrm{d}s,$$
$$w_0 + pw_1 + \cdots = w_0 - p\int_0^t \Big[\Big(\frac{\partial w_0}{\partial s} + p\frac{\partial w_1}{\partial s} + \cdots\Big) + \Big(\frac{\partial^3 w_0}{\partial x^3} + p\frac{\partial^3 w_1}{\partial x^3} + \cdots\Big) + 3(u_0 + pu_1 + \cdots)\Big(\frac{\partial w_0}{\partial x} + p\frac{\partial w_1}{\partial x^3} + \cdots\Big)\Big]\mathrm{d}s.$$

Comparing the coefficient of like powers of p, following approximants are obtained:

$$p^{(0)}: \begin{cases} u_0(x,t) = \frac{1}{3}(\beta - 2k^2) + 2k^2\tanh^2(kx), \\ v_0(x,t) = \frac{-4k^2 c_0(\beta + k^2)}{3c_1^2} + \frac{4k^2(\beta + k^2)}{3c_1}\tanh(kx), \\ w_0(x,y,t) = c_0 + c_1\tanh(kx), \end{cases}$$

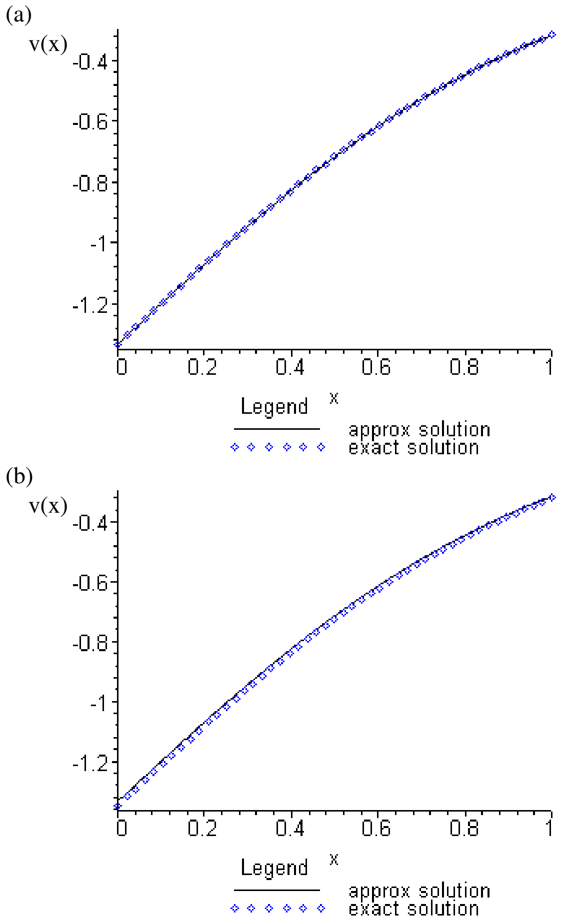

Fig. 2. Solution v with $b = 0$, $k = t = c_0 = c_1 = 1$ (a), and $b = 0.01$, $k = t = c_0 = c_1 = 1$ (b).

$$p^{(1)}: \begin{cases} u_1(x,t) = \frac{1}{3}(\beta - 2k^2) + 2k^2 \tanh^2(kx) \\ \qquad - \frac{2\cosh x = 2t \sinh x}{\cos^3 x}, \\ v_1(x,t) = \frac{-4k^2 c_0(\beta + k^2)}{3c_1^2} \\ \qquad + \frac{4k^2(\beta + k^2)}{3c_1} \tanh(kx) \\ \qquad + \frac{\cosh^2 x \cosh x + t \sinh x}{\cosh^2 x}, \\ w_1(x,y,t) = c_0 + c_1 \tanh(kx) \\ \qquad + 2\frac{-\cosh^2 + \cosh x + t \sinh x}{\cosh^2 x}, \end{cases}$$

$$\vdots$$

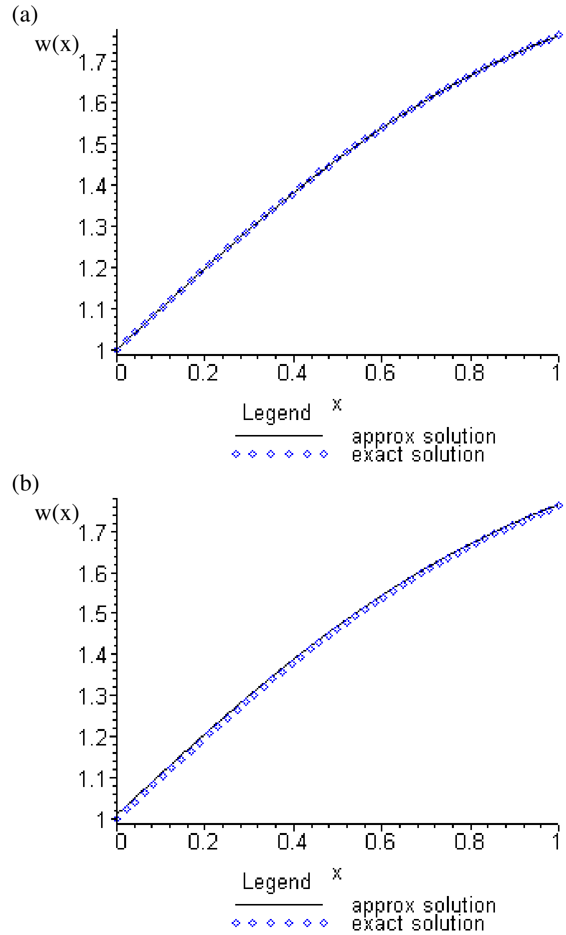

Fig. 3. Solution w with $b = 0$, $k = t = c_0 = c_1 = 1$ (a), and $b = 0.01$, $k = t = c_0 = c_1 = 1$ (b).

The closed form solution is given as

$$(u, v, w) = (\mathrm{e}^{x+y-t}, \mathrm{e}^{x-y+t}, \mathrm{e}^{-x+y+t}),$$

and is graphically depicted in Figures 1 – 3.

4. Conclusion

In this paper, we applied the variational iteration method using He's polynomials (VIMHP) for solving Hirota-Satsuma coupled KdV systems. The method is applied in a direct way without using linearization, transformation, discretization or restrictive assumptions. It may be concluded that the VIMHP is very powerful and efficient in finding the analytical solutions for a wide class of boundary value problems. The method gives more realistic series solutions that converge very rapidly in physical problems. It is worth mentioning that the method is capable of reducing the volume of the computational work as compare to the classical methods while still maintaining the high accuracy of the numerical result. The fact that the VIMHP solves nonlinear problems without using Adomian's polynomials is a clear advantage of this technique over the decomposition method.

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[1] A. Golbabai and M. Javidi, Comput. Math. Appl. **54**, 1003 (2007).
[2] J. H. He and M. A. Abdou, Chaos, Solitons, and Fractals, **34**, 1421 (2007).
[3] J. H. He, Int. J. Mod. Phys. B **22**, 3487 (2008).
[4] J. H. He, Top. Meth. Nonlinear Anal. **31**, 205 (2008).
[5] J. H. He, Int. J. Mod. Phys. (20)**10**, 1144 (2006).
[6] J. H. He, Appl. Math. Comput. **156**, 527 (2004).
[7] J. H. He, Int. J. Nonlinear Sci. Numer. Simul. **6**, 207 (2005).
[8] J. H. He, Appl. Math. Comput. **151**, 287 (2004).
[9] J. H. He, Int. J. Nonlinear Mech. **35**, 115 (2000).
[10] J. H. He, J. Comput. Appl. Math. **207**, 3 (2007).
[11] J. H. He and X. Wu, Comput. Math. Appl. **54**, 881 (2007).
[12] J. H. He, Int. J. Nonlinear Mech. **34**, 699 (1999).
[13] J. H. He, Appl. Math. Comput. 114 (2 – 3) (2000), 115 – 123.
[14] J. H. He and X. H. Wu, Chaos, Solitons, and Fractals, **29**, 108 (2006).
[15] J. H. He, Phys. Scr. **76**, 680 (2007).
[16] S. Abbasbandy, J. Comput. Appl. Math. **207**, 59 (2007).
[17] S. Abbasbandy, Int. J. Numer. Meth. Eng. **70**, 876 (2007).
[18] S. Abbasbandy, Phys. Lett. A **361**, 478 (2007).
[19] M. A. Abdou and A. A. Soliman, J. Comput. Appl. Math. **181**, 245 (2005).
[20] M. A. Abdou and A. A. Soliman, Phys. D **211**, 1 (2005).
[21] T. A. Abassy, M. A. El-Tawil and H. El-Zoheiry, J. Comput. Appl. Math. **207**, 73 (2007).
[22] B. Batiha, M. S. M. Noorani and I. Hashim, Comput. Math. Appl. **54**, 903 (2007).
[23] J. Biazar and H. Ghazvini, Comput. Math. Appl. **54**, 1047 (2007).
[24] S. A. El-Wakil, M. A. Madkour and M. A. Abdou, Phys. Lett. A **369**, 62 (2007).
[25] D. D. Ganji, H. Tari, M. B. Jooybari, Comput. Math. Appl. **54**, 1018 (2007).
[26] A. Ghorbani and J. S. Nadjfi, Int. J. Nonlinear Sci. Numer. Simul. **8**, 229 (2007).
[27] A. Ghorbani, Chaos, Solitons, and Fractals **39**, 1486 (2009).
[28] J. Lu, J. Comput. Appl. Math. **207**, 92 (2007).
[29] W. X. Ma and D. T. Zhou, Acta Math. Scita. **17**, 168 (1997).
[30] S. Momani and Z. Odibat, Chaos, Solitons, and Fractals, **27**, 1119 (2006).
[31] S. T. Mohyud-Din, M. A. Noor, and K. I. Noor, Int. J. Nonlinear Sci. Numer. Simul. **10**, 223 (2009).
[32] S. T. Mohyud-Din and M. A. Noor, Z. Naturforsch. **64a**, 157 (2009).
[33] M. A. Noor and S. T. Mohyud-Din, Int. J. Nonlinear Sci. Numer. Simul. **9**, 141 (2008).
[34] M. A. Noor and S. T. Mohyud-Din, Acta Appl. Math. (2008), DOI: 10.1007/s10440 – 008 – 9255-x.
[35] M. A. Noor and S. T. Mohyud-Din, Math. Prob. Eng. (2008), Article ID 696734, doi:10.1155/2008/696734.
[36] M. Rafei and H. Daniali, Comput. Math. Appl. **54**, 1079 (2007).
[37] N. H. Sweilam, J. Comput. Appl. Math. **207**, 64 (2007).
[38] M. Tatari and M. Dehghan, J. Comput. Appl. Math. **207**, 121 (2007).
[39] L. Xu, Comput. Math. Appl. **54**, 1067 (2007).
[40] X. H. Wu and J. H. He, Comput. Math. Appl. **54**, 966 (2007).
[41] E. Yusufoglu, Phys. Lett. A **372**, 442 (2008).
[42] S. Zhang, Chaos, Solitons, and Fractals, **365**, 448 (2007).
[43] D. Kaya, Intern. J. Comput. Math. **72**, 531 (1999).
[44] W. X. Ma and Y. Trans. American Math. Soc. **357**, 1753 (2004).
[45] S. D. Zhu, Int. J. Nonlinear Sci. Numer. Simul. **8**, 461 (2007).
[46] S. D. Zhu, Int. J. Nonlinear Sci. Numer. Simul. **8**, 465 (2007).
[47] X. W. Zhou, Y. X. Wen, and J. H. He, Int. J. Nonlinear Sci. Numer. Simul. **9**, 301 (2008).