Homotopy Perturbation Method for Solving Nonlinear Differential-Difference Equations

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In this paper, the homotopy perturbation method (HPM) is extended to obtain analytical solutions for some nonlinear differential-difference equations (NDDEs). The discretized modified Korteweg-de Vries (mKdV) lattice equation and the discretized nonlinear Schrödinger equation are taken as examples to illustrate the validity and the great potential of the HPM in solving such NDDEs. Comparisons between the results of the presented method and exact solutions are made. The results reveal that the HPM is very effective and convenient for solving such kind of equations.

Key words: Homotopy Perturbation Method; Nonlinear Differential-Difference Equation; Discretized mKdV Lattice Equation; Discretized Nonlinear Schrödinger Equation.

1. Introduction

Many interesting physical phenomena, such as ladder type electric circuit, vibration of particles, collapse of langmuir waves in plasma physics (see [1] and references therein), molecular crystals [2], biophysical systems [3], electrical lattices [4], and recently in arrays of coupled nonlinear optical wave guides [5, 6], to cite a few, can be modelled by nonlinear differential-difference equations (NDDEs). Recently, differential-difference equations became a very interesting topic due to the development of nanotechnology. Generally we can use differential equations to describe various physical problems, but when time or space become discontinuous, the differential model becomes invalid. According to El-Naschie E-infinity theory [7], space and time are discontinuous. In many applications, time can be approximately considered to be continuous. Moreover, in nano scales or smaller scales, many problems become discontinuous, where differential-difference model can be powerfully applied, see explanations in [8 – 10]. Unlike difference equations which are fully discretized, differential-difference equations are semi-discretized, with some (or all) of their spatial variables discretized, while time variable is usually kept continuous. In this paper we will be primarily concerned with outlining an effective procedure that allows us to implement the homotopy perturbation method (HPM) for solving some important NDDE initial value problems. For illustration, we apply it to the discretized mKdV lattice equation and the discretized nonlinear Schrödinger equation. Recently the HPM is applied to solve two kinds of NDDEs. The discrete KdV equation was studied by Yildirim [11] and the nonlinear relativistic Toda lattice equation was studied by Zhu [12].

In recent years a lot of attention has been drawn to solve differential-difference equations using new developed analytical methods such as the exp-function method [13 – 15], the variational iteration method [16, 17], and the parameterized perturbation method [18].

Recently, the homotopy perturbation method [19, 20] has drawn lot of attention to investigate various scientific models and solve various kinds of differential equations. For example, it is employed in [21] for determining the frequency-amplitude relation of a nonlinear oscillator with discontinuities. Application of this method to squeezing flow of a Newtonian fluid is investigated in [22]. He's homotopy perturbation method is used in [23] for solving linear and nonlinear Schrödinger equations and obtaining exact solutions. The authors in [24] applied the homotopy perturbation Padé technique for constructing approximate and exact solutions of Boussinesq equations that describe motions of long waves in shallow water

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under gravity and in a one-dimensional nonlinear lattice. Dehghan et al. successfully used the homotopy perturbation method to solve several problems which have nice applications in science and engineering [25 – 30]. Recently, He made some developments in the HPM [31, 32]. The homotopy perturbation method, based on series approximation, is one among the newly developed analytical methods for strongly nonlinear problems and has been proven successful in solving a wide class of differential equations. The method provides the solution in a rapidly convergent series with components that can be simply computed. The HPM is useful for obtaining both closed form explicit solutions and numerical approximations of linear or nonlinear differential equations, integral equations, and differential-difference equations as we will see in the present paper, and it is of great interest to applied science, engineering, physics, biology, etc.

2. Basic Idea of the Homotopy Perturbation Method

To illustrate the basic ideas of this method, we consider the following nonlinear differential equation [19]:

$$A(u) - f(r) = 0, \quad r \in \Omega, \tag{1}$$

with the boundary conditions

$$B(u, \partial u/\partial n) = 0, \quad r \in \Gamma, \tag{2}$$

where A is a general differential operator, B a boundary operator, $f(r)$ a known analytical function, and Γ is the boundary of the domain Ω.

Generally speaking, the operator A can be divided into two parts, which are L and N, where L is linear, but N is nonlinear. Therefore, (1) can be rewritten as follows:

$$L(u) + N(u) - f(r) = 0. \tag{3}$$

By the homotopy technique, we construct a homotopy $V(r; p) : \Omega \times [0,1] \to \mathbb{R}$ which satisfies

$$\begin{aligned} H(V,p) &= (1-p)[L(V) - L(u_0)] \\ &\quad + p[A(V) - f(r)] = 0, \\ p &\in [0,1], \quad r \in \Omega, \end{aligned} \tag{4}$$

where $p \in [0,1]$ is an embedding parameter, u_0 is an initial approximation of (1), which satisfies the boundary conditions. In most cases $L(u_0)$ is equal to zero.

Obviously, from (4) we will have

$$H(V,0) = L(V) - L(u_0) = 0, \tag{5}$$

$$H(V,1) = A(V) - f(r) = 0. \tag{6}$$

The changing process of p from zero to unity is just that of $V(r; p)$ from $u_0(r)$ to $u(r)$. In topology, this is called deformation, and $L(V) - L(u_0)$ and $A(V) - f(r)$ are called homotopic.

According to the HPM, we can first use the embedding parameter p as a ‘small parameter’, and assume that the solution of (4) can be written as a power series in p:

$$V = V_0 + pV_1 + p^2 V_2 + \cdots \tag{7}$$

Setting $p = 1$ results in the approximate solution of (1):

$$u = \lim_{p \to 1} V = V_0 + V_1 + V_2 + \cdots \tag{8}$$

The series in (8) is convergent for most cases, and also the rate of convergent depends on the nonlinear operator $A(V)$ [19].

Let’s denote the m-term(s) approximate solution φ_m by

$$u \simeq \varphi_m = \sum_{k=0}^{m} V_k(r). \tag{9}$$

3. Applications

3.1. The Discretized mKdV Lattice Equation

Consider the discretized mKdV lattice equation

$$\frac{\partial u_n}{\partial t} = (1 - u_n^2)(u_{n+1} - u_{n-1}) \tag{10}$$

with the initial condition

$$u_n(0) = A \tanh(kn), \tag{11}$$

where k is an arbitrary constant and $A = \tanh(k)$.

The exact solution of the problem was given by Wu et al. [33] as

$$u_n(t) = A \tanh(kn + 2At). \tag{12}$$

According to (4), a homotopy $V(n,t; p) : \Omega \times [0,1] \to \mathbb{R}$ can be constructed as follows:

$$\begin{aligned} &(1-p)[V_t(n,t) - u_t(n,0)] \\ &+ p\Big\{ V_t(n,t) - \big[1 - V^2(n,t)\big] \\ &\quad \cdot [V(n+1,t) - V(n-1,t)] \Big\} = 0, \end{aligned} \tag{13}$$

then (13) can be written as

$$V_t(n,t) - p\left[1 - V^2(n,t)\right] \cdot [V(n+1,t) - V(n-1,t)] = 0. \quad (14)$$

One can now try to obtain solutions of $V(n-1,t)$, $V(n,t)$, and $V(n+1,t)$ as:

$$V(n+i,t) = \sum_{k=0}^{\infty} p^k V_k(n+i,t), \quad i = 1,0 \text{ and } 1. \quad (15)$$

Substituting (15) into (14), and comparing coefficients of terms with identical powers of p, yield

$$\begin{aligned}
p^0 &: \frac{\partial}{\partial t} V_0(n,t) = 0,\\
p^1 &: \frac{\partial}{\partial t} V_1(n,t) - [1 - (V_0(n,t))^2] \\
&\quad \cdot [V_0(n+1,t) - V_0(n-1,t)] = 0,\\
p^2 &: \frac{\partial}{\partial t} V_2(n,t) - [1 - (V_0(n,t))^2][V_1(n+1,t) \\
&\quad - V_1(n-1,t)] + 2V_0(n,t)V_1(n,t) \\
&\quad \cdot [V_0(n+1,t) - V_0(n-1,t)] = 0,\\
p^3 &: \frac{\partial}{\partial t} V_3(n,t) - [1 - (V_0(n,t))^2][V_2(n+1,t) \\
&\quad - V_2(n-1,t)] + 2V_0(n,t)V_1(n,t)[V_1(n+1,t) \\
&\quad - V_1(n-1,t)] + [2V_0(n,t)V_2(n,t) + (V_1(n,t))^2)] \\
&\quad \cdot [V_0(n+1,t) - V_0(n-1,t)]) = 0,\\
&\vdots
\end{aligned} \quad (16)$$

with the following initial conditions:

$$V_i(n,0) = \begin{cases} A\tanh(kn)), & i = 0,\\ 0, & i = 1,2,3,\dots \end{cases} \quad (17)$$

Solving the system (16) with the conditions (17) yields

$$\begin{aligned}
V_0(n,t) &= A\tanh(kn),\\
V_1(n,t) &= A[\tanh(kn+k) - \tanh(kn-k)] \\
&\quad \cdot [1 - A^2\tanh^2(kn)]t,\\
V_2(n,t) &= \frac{A}{2}[A^2\tanh^2(kn) - 1] \\
&\cdot \Big[A^2\tanh(kn)\tanh^2(k(n+1)) - 4A^2\tanh(kn) \\
&\quad \cdot \tanh(k(n-1))\tanh(k(n+1)) + A^2\tanh(kn) \\
&\quad \cdot \tanh^2(k(n-1)) + A^2\tanh(k(n-2))\tanh^2(k(n-1)) \\
&\quad + A^2\tanh(k(n+2))\tanh^2(k(n+1)) + 2\tanh(kn) \\
&\quad - \tanh(k(n+2)) - \tanh(k(n-2))\Big]t^2,\\
&\vdots
\end{aligned} \quad (18)$$

In this manner, other components can be easily obtained using any symbolic computation program.

Substituting solutions (18) into (9) gives

$$\begin{aligned}
u_n(t) &\simeq \varphi_2 = A\tanh(kn) + A[\tanh(kn+k) \\
&- \tanh(kn-k)][1 - A^2\tanh^2(kn)]t \\
&+ \frac{A}{2}[A^2\tanh^2(kn) - 1]\Big[A^2\tanh(kn)\tanh^2(k(n+1)) \\
&- 4A^2\tanh(kn)\tanh(k(n-1))\tanh(k(n+1)) \\
&+ A^2\tanh(kn)\tanh^2(k(n-1)) + A^2\tanh(k(n-2)) \\
&\cdot \tanh^2(k(n-1)) + A^2\tanh(k(n+2))\tanh^2(k(n+1)) \\
&+ 2\tanh(kn) - \tanh(k(n+2)) - \tanh(k(n-2))\Big]t^2.
\end{aligned} \quad (19)$$

For this test problem, we continued solving (16) for V_n, $n = 0,1,\dots$ until V_6 and hence obtained the approximate solution $\varphi_6 = \sum_{j=0}^{6} V_j(n,t)$.

A comparison between approximate solutions φ_2, φ_6 and the exact solution (12) for $k = 0.1$ is illustrated in Figure 1. The behaviours of the approximate solutions φ_2 and φ_6 in comparison with the exact solution illustrated in Figures 1a and 1b, respectively, show that the more term-approximate solution is the more accurate one for relatively high values of t. This is due to the fact that the increase in the number of terms used for calculating φ_m will lead to increase the approximate solution radius of convergence.

Moreover, the radius of convergence of the approximate solution can be increased by applying Padé approximants to the truncate series solution φ_m [24].

3.2. The Discretized Nonlinear Schrödinger Equation

Consider the discretized nonlinear Schrödinger equation

$$\mathrm{i}\frac{\partial u_n}{\partial t} = (u_{n+1} + u_{n-1} - 2u_n) - |u_n|^2(u_{n+1} + u_{n-1}) \quad (20)$$

with the initial condition

$$u_n(0) = \tanh(k)\mathrm{e}^{\mathrm{i}pn}\tanh(kn), \quad (21)$$

where k and p are arbitrary constants and $\mathrm{i} = \sqrt{-1}$.

The exact solution of the problem was given in [34, 35] as

$$u_n(t) = \tanh(k)\mathrm{e}^{\mathrm{i}[pn + (2 - 2\cos(p)\operatorname{sech}(k))t]} \cdot \tanh(kn + 2\sin(p)\tanh(k)t). \quad (22)$$

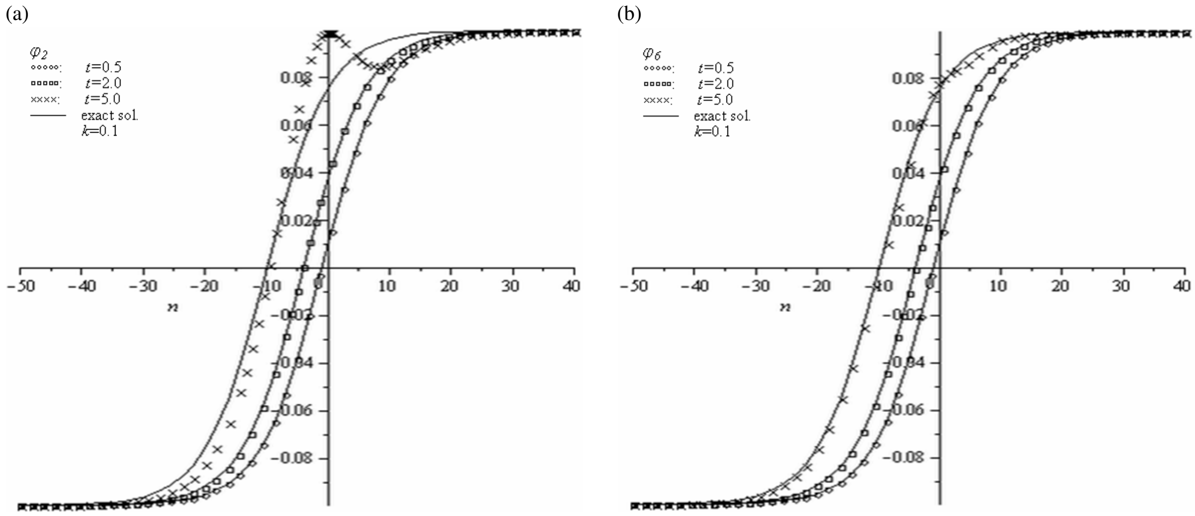

Fig. 1. Results for $t = 0.5$, 2, and 5 obtained form the exact solution (solid line) and the HPM solution for the discretized mKdV lattcie equation when $k = 0.1$; (a) in comparison with φ_2, (b) in comparison with φ_6.

As done in [23], we can construct the following homotopy $V(n,t,p) : \Omega \times [0,1] \to \mathbb{C}$ which satisfies

$$V_t(n,t) + \mathrm{i}p\{V(n+1,t) + V(n-1,t) - 2V(n,t) \\ -V(n,t)\bar{V}(n,t)[V(n+1,t) + V(n-1,t)]\} = 0, \tag{23}$$

where $\bar{V}$ is the conjugate of V.

Let's consider the conjugate series solution $\bar{V}$ as

$$\bar{V}(n,t) = \sum_{j=0}^{\infty} p^j \bar{V}_j(n,t). \tag{24}$$

Substituting (15) and (24) into (23), and arranging the coefficients of 'p' powers yield

$$\begin{aligned}
p^0 &: \frac{\partial}{\partial t}V_0(n,t) = 0, \\
p^1 &: \frac{\partial}{\partial t}V_1(n,t) - \mathrm{i}V_0(n,t)\bar{V}_0(n,t)[V_0(n+1,t) \\
&\quad + V_0(n-1,t)] + \mathrm{i}[V_0(n+1,t) - 2V_0(n,t) \\
&\quad + V_0(n-1,t)] = 0, \\
p^2 &: \frac{\partial}{\partial t}V_2(n,t) - \mathrm{i}V_0(n,t)[V_1(n+1,t) + V_1(n-1,t)] \\
&\quad - \mathrm{i}[V_0(n,t)\bar{V}_1(n,t) + V_1(n,t)\bar{V}_0(n,t)][V_0(n+1,t) \\
&\quad + V_0(n-1,t)] + \mathrm{i}[-2V_1(n,t) + V_1(n-1,t) \\
&\quad + V_1(n+1,t)] = 0, \\
&\vdots
\end{aligned} \tag{25}$$

with the following initial conditions:

$$V_j(n,0) = \begin{cases} \tanh(k)\mathrm{e}^{\mathrm{i}pn}\tanh(kn), & j = 0, \\ 0, & j = 1,2,3,\ldots \end{cases} \tag{26}$$

We continued solving the system (25) with the conditions (26) for V_n, $n = 0$, 1, 2, and hence obtained the approximate solution $\varphi_2 = \sum_{j=0}^{2} V_j(n,t)$, for simplicity we write down only φ_1

$$\begin{aligned}
\varphi_1 = {} & \tanh(k)\mathrm{e}^{\mathrm{i}pn}\tanh(kn) \\
& + \mathrm{i}\tanh(k)\Big[-\mathrm{e}^{\mathrm{i}p(n-1)}\tanh(kn-k) \\
& + \tanh^2(k)\tanh^2(kn)\mathrm{e}^{\mathrm{i}p(n+1)}\tanh(kn+k) \\
& + \tanh^2(k)\tanh^2(kn)\mathrm{e}^{\mathrm{i}p(n-1)}\tanh(kn-k) \\
& - \mathrm{e}^{\mathrm{i}p(n+1)}\tanh(kn+k) + 2\mathrm{e}^{\mathrm{i}pn}\tanh(kn)\Big]\,t.
\end{aligned} \tag{27}$$

Some graphical comparisons between the approximate solution φ_2 and the exact solution (22) for $k = 0.1$ and $p = 0.2$ are illustrated in Figures 2–4. From the graphical comparisons it is clear that the three-term approximate solution φ_2 is an acceptable solution even for relatively high values of t. We can obtain more accurate solutions by solving more equations in the system (25) and get an approximate solution with high radius of convergence with the aid of any symbolic computation program. Moreover, we can accelerate the convergence of the HPM approximate solution using Padé approximants [24].

(a) $\mathrm{Re}(\varphi_2)$ (b) $\mathrm{Im}(\varphi_2)$

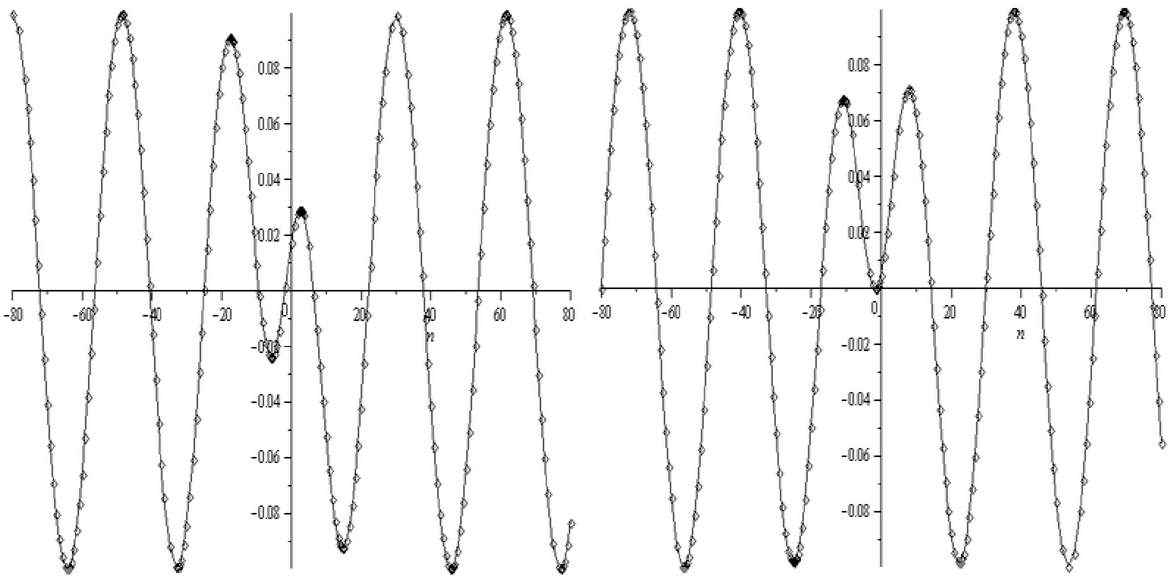

Fig. 2. Results for $t = 5$, obtained form the exact solution, real and imaginary parts (solid line), and the HPM solution $\mathrm{Re}(\varphi_2)$ shown in (a), and $\mathrm{Im}(\varphi_2)$ shown in (b), for equation (20) when $k = 0.1$ and $p = 0.2$.

(a) $\mathrm{Re}(\varphi_2)$ (b) $\mathrm{Im}(\varphi_2)$

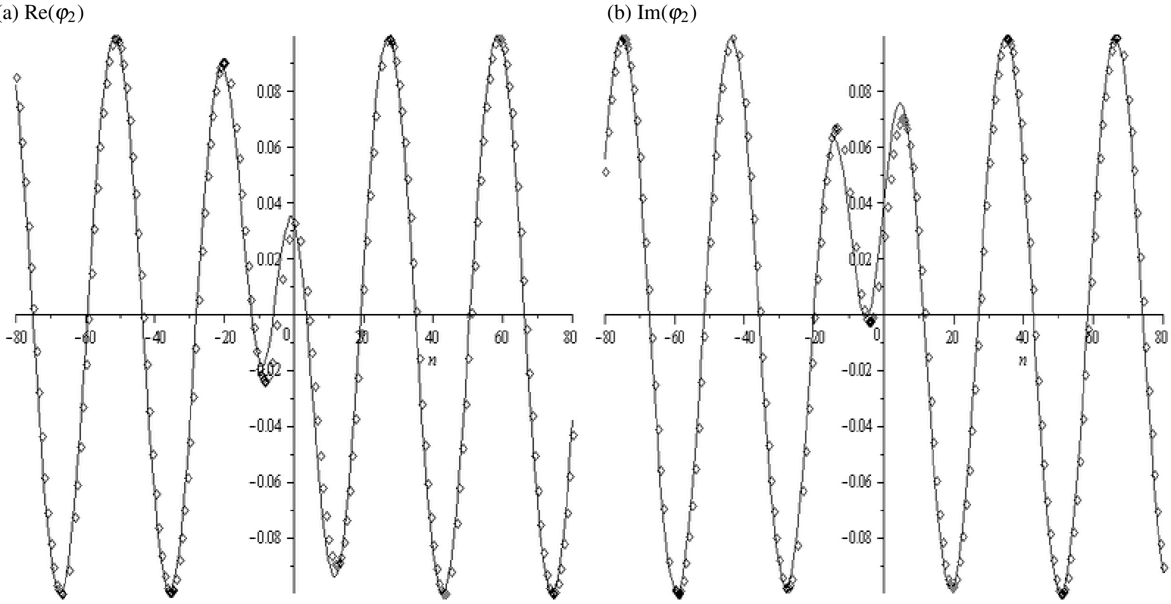

Fig. 3. Results for $t = 15$, obtained form the exact solution, real and imaginary parts (solid line), and the HPM solution $\mathrm{Re}(\varphi_2)$ shown in (a), and $\mathrm{Im}(\varphi_2)$ shown in (b), for equation (20) when $k = 0.1$ and $p = 0.2$.

4. Conclusions

The homotopy perturbation method is extended and utilized to find exact and approximate solutions for NDDEs, including the discretized mKdV lattice equation and the discretized nonlinear Schrödinger equation. A clear conclusion can be drawn from the results that the HPM is an effective, simple, and quite accurate tool for handling and solving nonlinear differential-difference equations in a unified manner. It is predicted

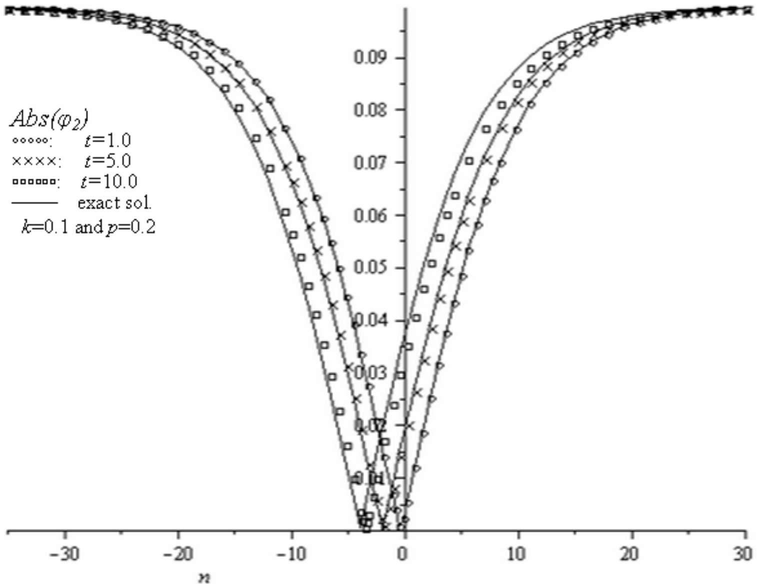

Fig. 4. Results for $t = 1$, 5, and 10 obtained form the exact solution, absolute value (solid line), and HPM solution abs(φ_2) for equation (20) when $k = 0.1$ and $p = 0.2$.

that the HPM can be found widely applicable in science and engineering. The disadvantage of truncated series solution-based methods is that the series solution becomes not exactly coincide with the exact one for a along time.

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