Multi-Soliton Excitations and Chaotic Patterns for the (2+1)-Dimensional Breaking-Soliton System

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Starting from a projective equation and a linear variable separation approach, some solitary wave solutions with arbitrary functions for the (2+1)-dimensional breaking soliton system are derived. Based on the derived solution and by selecting appropriate functions, some novel localized excitations such as multi-solitons and chaotic-solitons are investigated.

Key words: Breaking-Soliton System; Solitary Wave Solutions; Multi-Solitons; Chaotic-Solitons.
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1. Introduction

Modern soliton theory is widely applied in many natural sciences [1 – 4] such as chemistry, biology, mathematics, communication, and particularly in almost all branches of physics like fluid dynamics, plasma physics, field theory, optics, and condensed matter physics, etc. [5 – 9]. Seeking the exact solutions of partial differential equations has long been an interesting and hot topic in nonlinear mathematical physics. In order to find some new exact solutions, a wealth of effective methods have been set up, for instance, the bilinear method, the standard Painlevé truncated expansion, the method of 'coalescence of eigenvalue' or 'wavenumbers', the homogenous balance method, the homotopy perturbation method, the hyperbolic function method, the Jacobian elliptic method, the variable separation method [10 – 18], and the mapping method [19 – 21], etc. The mapping approach is a kind of classic, efficient, and well-developed method to solve nonlinear evolution equations, the remarkable characteristic of which is that we can have many different ansatzs and therefore, a large number of solutions. In the past, we have solved the exact solutions of some nonlinear systems via the Ricati equation ($\phi' = \sigma + \phi^2$) mapping method, such as (1+1)-dimensional related Schrödinger equation, (2+1)-dimensional Generalized Breor-Kaup system, (3+1)-dimensional Burgers system, (3+1)-dimensional Jimbo-Miwa system, (2+1)-dimensional modified dispersive water-wave system, (2+1)-dimensional Boiti-Leon-Pempinelli system, (2+1)-dimensional Korteweg-de Vries system, (2+1)-dimensional asymmetric Nizhnik-Novikov-Veselov system, etc. [22 – 28]. In this paper, with a new projective equation ($\phi' = \sigma\phi + \phi^2$) and a linear variable separation approach, we obtain some solitary wave solutions to the (2+1)-dimensional breaking soliton system

$$u_{xt} - 4u_{xy}u_x - 2u_{xx}u_y - u_{xxxy} = 0. \tag{1}$$

Equation (1) was used to describe the (2+1)-dimensional interaction of Riemann waves propagated along the y-axis with long waves propagated along the x-axis [29].

2. Solitary Wave Solutions to the (2+1)-Dimensional Breaking Soliton System

The basic idea of the equation ($\phi' = \sigma\phi + \phi^2$) mapping approach is as follows. For a given nonlinear partial differential equation (NPDE) with the independent variables $x = (x_0 = t, x_1, x_2, \cdots, x_m)$ and the dependent variable u in the form

$$P(u, u_t, u_{x_i}, u_{x_i x_j}, \cdots) = 0, \tag{2}$$

where P is in general a polynomial function of its arguments and the subscripts denote the partial derivatives, the solution may be assumed to be in the form

$$u = \sum_{i=0}^{n} \{A_i(x)\phi^i[q(x)]\} \tag{3}$$

with

$$\phi' = \sigma\phi + \phi^2, \tag{4}$$

where $A_i(x)$ and $q(x)$ are functions of the indicated argument to be determined, σ is an arbitrary constant, and the prime denotes ϕ differentiation with respect to q. To determine u explicitly, one substitutes (3) and (4) into the given NPDE and collects the coefficients of the polynomials of ϕ, then eliminates each coefficient to derive a set of partial differential equations for A_i and q, and solves the system of partial differential equations to obtain A_i and q. Finally, as (4) is known to possess the general solutions

$$\phi = \begin{cases} -\frac{1}{2}\sigma\left[1+\tanh\left(\frac{1}{2}\sigma q\right)\right] \\ -\frac{1}{2}\sigma\left[1+\coth\left(\frac{1}{2}\sigma q\right)\right] \end{cases}. \tag{5}$$

Substituting A_i, q, and (5) into (3), one obtains the exact solutions to the given NPDE.

Now we apply the mapping approach to (1). By the balancing procedure, the ansatz (3) becomes

$$u(x,y,t) = f(x,y,t) + g(x,y,t)\phi, \tag{6}$$

where f, g, and q are functions of (x,y,t) to be determined. Substituting (6) and (4) into (1) and collecting coefficients of polynomials of ϕ, then setting each coefficient to zero, we have

$$\begin{aligned} f &= \frac{1}{4}\int \frac{1}{q_x q_y}[4q_x^2 q_y q_{xx}\sigma - q_{xx}^2 q_y - +q_x^4 q_y \sigma^2 \\ &\qquad + 2q_x q_y q_{xxx} - q_x^2 q_t]\mathrm{d}x, \\ g &= -2q_x, \end{aligned} \tag{7}$$

with the function q in a special variable separation form

$$q = \chi(x) + \varphi(y - ct), \tag{8}$$

c is an arbitrary constant. Based on the solutions of (4), one thus obtains following solitary wave solutions of (1):

$$\begin{aligned} u_1 = &-\frac{1}{4}\int \frac{1}{\chi_x^2}[4\chi_x^2\chi_{xx}\sigma - \chi_{xx}^2 + \chi_x^4\sigma^2 \\ &\qquad + 2\chi_x\chi_{xxx} + \chi_x^2 c]\mathrm{d}x \\ &+ \chi_x\sigma + \chi_x\sigma\tanh\left[\frac{1}{2}\sigma(\chi+\varphi)\right], \end{aligned} \tag{9}$$

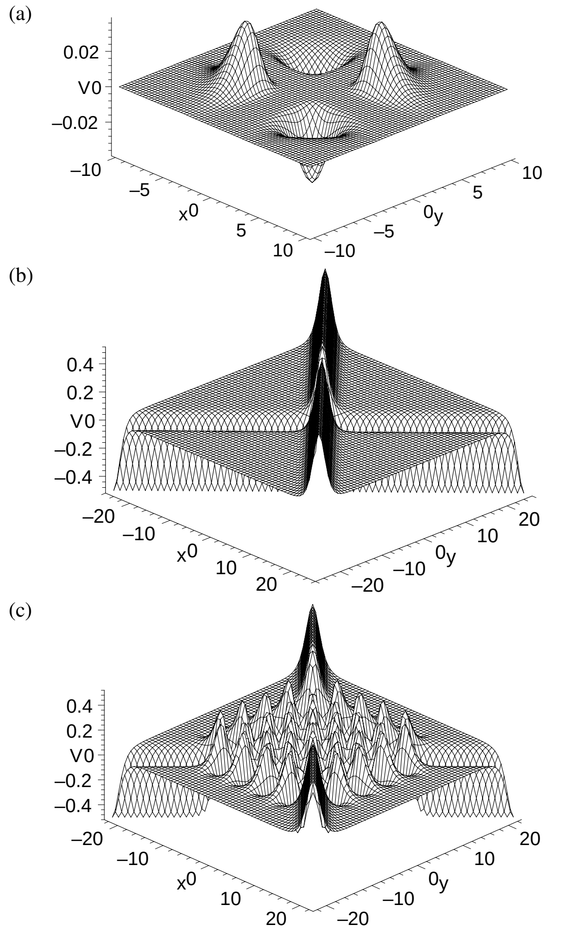


Fig. 1. Three types of multi-soliton localized excitation (a) multi-dromion excitation; (b) multi-solitoff excitation; (c) multi-dromion-solitoff excitation.

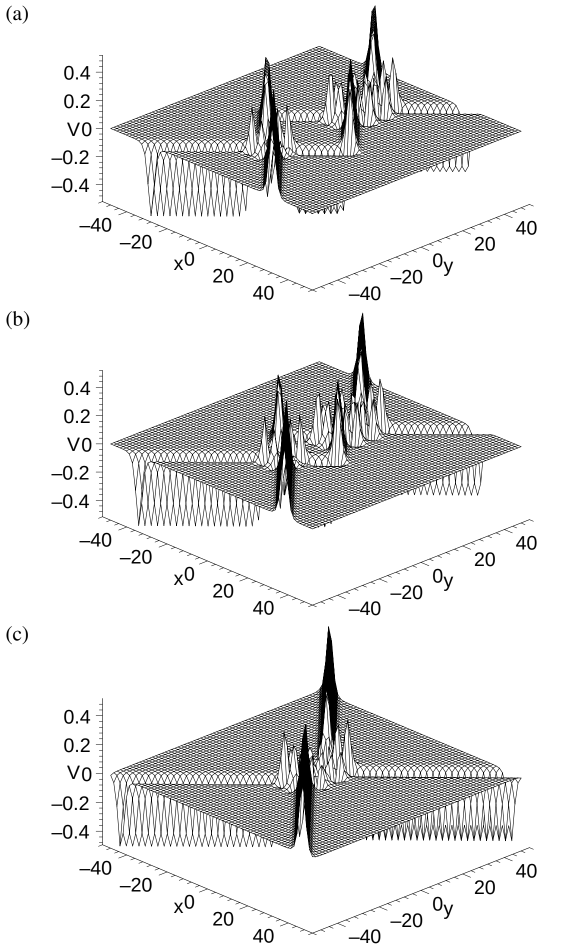
$$\begin{aligned} u_2 = &-\frac{1}{4}\int \frac{1}{\chi_x^2}[4\chi_x^2\chi_{xx}\sigma - \chi_{xx}^2 + \chi_x^4\sigma^2 \\ &\qquad + 2\chi_x\chi_{xxx} + \chi_x^2 c]\mathrm{d}x \\ &+ \chi_x\sigma + \chi_x\sigma\coth\left[\frac{1}{2}\sigma(\chi+\varphi)\right] \end{aligned} \tag{10}$$

with two arbitrary functions being $\chi(x)$ and $\varphi(y-ct)$.

3. Multi-Soliton Localized Excitations

Now we will discuss some new type of multi-soliton localized excitations from the potential of the solitary wave solution determined by (10) and rewrite it in a simple form, namely

$$\begin{aligned} V &= u_{2y} \\ &= -\frac{1}{2}\chi_x\varphi_y\sigma^2\left\{\coth\left[\frac{1}{2}\sigma(\chi+\varphi)^2\right] - 1\right\}. \end{aligned} \tag{11}$$



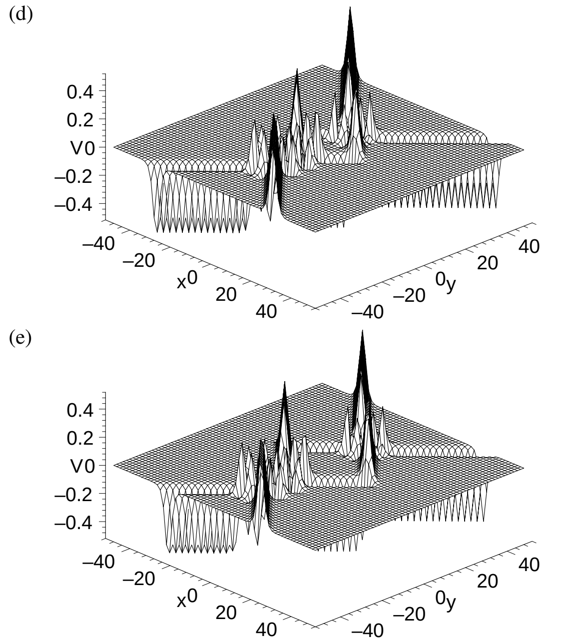


Fig. 2. Evolutional profile of two multi-dromion-solitoffs for the solution V with the condition (15) at different times (a) $t=-30$, (b) $t=-12$, (c) $t=0$, (d) $t=12$, and (e) $t=30$, respectively.

According to the solution V (11), we first discuss its multi-soliton excitations. For instance, if we choose χ and φ as

$$\chi = 1+10\operatorname{sech}(x), \quad \varphi = 1+10\operatorname{sech}(y-ct), \tag{12}$$

we can obtain a multi-dromion excitation for the physical quantity V of (11) presented in Figure 1a with fixed parameters $\sigma=-0.5$, $c=1$, and $t=0$.

If we choose χ and φ as

$$\begin{aligned}\chi &= \sum_{n=-N}^{N} 0.1\tanh(x+4n),\\ \varphi &= \sum_{m=-M}^{M} 0.1\tanh(y+4m-ct),\ M=N=0.1,\end{aligned} \tag{13}$$

we can obtain a multi-solitoff excitation for the physical quantity V under the condition (13) presented in Figure 1b with fixed parameters $\sigma=-0.5$, $c=1$, and $t=0$.

Furthermore, if we choose χ and φ as

$$\begin{aligned}\chi &= \sum_{n=-N}^{N} 0.1\operatorname{sech}(x+5n),\\ \varphi &= \sum_{m=-M}^{M} 0.1\operatorname{sech}(y+5m-ct),\ M=N=2,\end{aligned} \tag{14}$$

we can obtain a multi-dromion-solitoff excitation for the physical quantity V under the condition (14) presented in Figure 1c with fixed parameters $\sigma=-0.5$, $c=1$, and $t=0$.

Now we focus our attention on the intriguing evolution of two multi-dromion-solitoffs for the solution V. When χ and φ are considered to be

$$\begin{aligned}\chi &= \sum_{n=-N}^{N} 0.1\operatorname{sech}(x+5n),\\ \varphi &= \sum_{m=-M}^{M} \{0.1\operatorname{sech}(y+5m-c_1t)\\ &\qquad + 0.5\operatorname{sech}(x+5m-c_2t)\},\\ M &= N = 2,\end{aligned} \tag{15}$$

and $\sigma = 1$, $c_1 = 1$, $c_2 = -1$ in (11), we can obtain the interactions between two multi-dromion-solitoffs. Figure 2 shows an evolutional profile corresponding to the physical quantity V expressed by (11) at different times (a) $t = -30$, (b) $t = -12$, (c) $t = 0$, (d) $t = 12$, (e) $t = 30$. From Figure 2 and through detailed analysis, we find that the shapes, amplitudes, and velocities of the two multi-dromion-solitoffs are completely conserved after their interactions.

4. Localized Excitations with Chaotic Behaviours

Just as solitons, chaos is another important part of nonlinear science. It has been widely applied in many natural sciences. In this section, we mainly discuss some localized coherent excitations with chaotic behaviour in the (2+1)-dimensional breaking soliton system.

Recently, Lü et al. have introduced a new chaotic system (LCC system) of three-dimensional quadratic autonomous ordinary differential equations [30], which can display two 1-scroll chaotic attractors simultaneously with only three equilibria and two 2-scroll chaotic attractors simultaneously with five equilibria [31]:

$$m_\xi = -12m + ln, \quad n_\xi = -5n + ml, \quad l_\xi = 4.5l - mn, \tag{16}$$

where m, n, and l are functions of ξ ($\xi = x$ or $\xi = y - ct$). A novel butterfly-like chaotic attractor for the LCC system (16) is depicted in Figure 3 when

$$m(0) = 0.1, \quad n(0) = 0.1, \quad l(0) = 0.1. \tag{17}$$

If the functions χ and/or φ are assumed to be solutions of a chaotic dynamical system, we can derive some localized excitations with chaotic behaviour. For example, χ is defined to be a solution of the LCC system (16), and take

$$\chi = 1 + 0.01m(x), \quad \varphi = 1 + 0.01\exp(y - ct), \tag{18}$$

where $m(x)$ is a solution of the LCC system (16) with the initial conditions (17). By this choice, the dromion localized in all directions is changed into a chaotic line soliton, which presents chaotic behaviour in the x-direction though still localized in y-direction. Figure 4 shows the corresponding plot of the chaotic line soliton for the field V of (11) with parameter $\sigma = 1$, $c = 1$ at time $t = 0$.

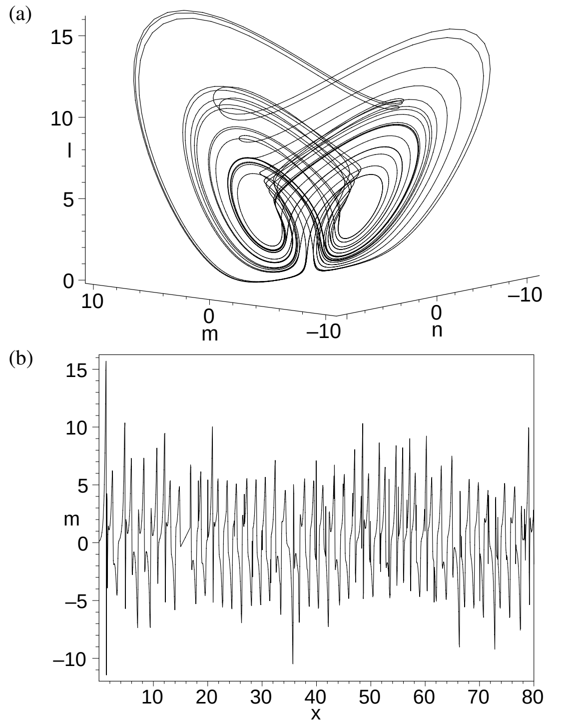


Fig. 3. (a) Novel butterfly-like attractor plot of the chaotic LCC system (16) with the initial condition (17). (b) A typical plot of the chaotic solution m of (16) related to (a).

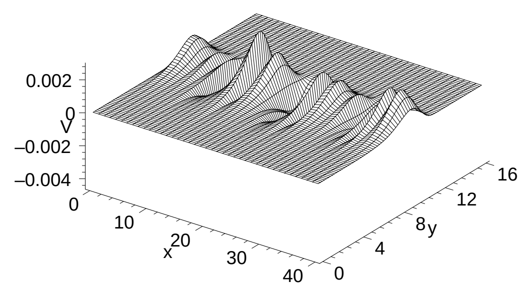


Fig. 4. Plot of the chaotic line soliton for the field V determined by (11) with condition (18) and $c = 1$, $t = 0$.

Furthermore, if χ and φ are all selected as chaotic solutions of the LCC system, the field V of (11) will behave chaotically in all directions and will yield a chaotic pattern. For example, χ and φ may be chosen as

$$\chi(x) = 1 + m(x), \quad \varphi(y) = 1 + m(y - ct), \tag{19}$$

where $m(x)$, and $m(y - ct)$ are the solutions of the LCC system (16) with the initial conditions (17). Figure 5a shows a plot of the special chaotic pattern for the field V expressed by (11) with the condition (19) at

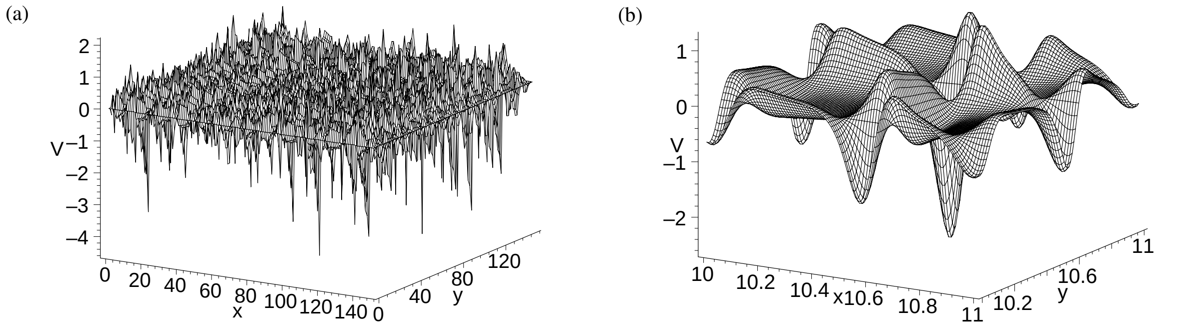


Fig. 5. (a) Plot of the chaotic patten for the field V expressed by (11) with condition (19) and $c = 1$, $t = 0$. (b) An enlargement of the center area ($x \in [10,11], y \in [10,11]$) of (a).

time $t = 0$. In order to show that the chaotic behaviour is due to the peak value of solitons, we enlarge small regions ($x \in [10,11], y \in [10,11]$) of Figure 5a. The result is shown in Figure 5b, which presents a kind of dromion with a chaotic structure clearly.

5. Summary and Discussion

In the past, many authors have solved the exact solutions of some nonlinear systems via the Ricatti equation ($\phi' = \sigma + \phi^2$) mapping method. In summary, via a new projective equation ($\phi' = \sigma\phi + \phi^2$) and a linear variable separation approach, the (2+1)-dimensional breaking soliton system is solved. Based on the derived solitary wave solution, we obtain some special multi-soliton excitations such as multi-dromion, multi-solitoff, and multi-dromion-solitoff, which are different from the ones presented in the previous work. Additionally, using the nuclear spin generator (NSG) chaotic system [32], Fang and Zheng recently have obtained some chaotic solitons of the (2+1)-dimensional Generalized Broer-Kaup system. Along the above line , we use the LCC chaotic system to get some new chaotic solutions. Since the wide applications of the soliton theory to learn more about the localized excitations and their applications in reality is worth to studing further.

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