Periodic Solutions of Asymptotically Linear Hamiltonian Systems without Twist Conditions

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In dynamical system theory, especially in many fields of applications from mechanics, Hamiltonian systems play an important role, since many related equations in mechanics can be written in an Hamiltonian form. In this paper, we study the existence of periodic solutions for a class of Hamiltonian systems. By applying the Galerkin approximation method together with a result of critical point theory, we establish the existence of periodic solutions of asymptotically linear Hamiltonian systems without twist conditions. Twist conditions play crucial roles in the study of periodic solutions for asymptotically linear Hamiltonian systems. The lack of twist conditions brings some difficulty to the study. To the authors' knowledge, very little is known about the case, where twist conditions do not hold.

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1. Introduction and Main Result

A Hamiltonian system is a system of $2N$ ordinary differential equations of the form

$$q_i' = \frac{\partial H}{\partial p_i}, \quad p_i' = \frac{\partial H}{\partial q_i} \quad i = 1,2,\dots,N, \tag{1}$$

where $H = H(t, p_1, \dots, p_N, q_1, \dots, q_N)$ is a real-valued smooth function, which is called the Hamiltonian or the energy function. In fact, when H is independent of t, H is a first integral of Hamilton's equation (1) as

$$L_t H = \sum_{i=1}^{N} \left(\frac{\partial H}{\partial p_i} p_i' + \frac{\partial H}{\partial q_i} q_i' \right) = 0,$$

and (1) is a conservative mechanic system. The two N-vectors $q = (q_1, q_2, \dots, q_N)$ and $p = (p_1, p_2, \dots, p_N)$ are called the position and momentum vectors, respectively. The variable t is called time. The integer N is called the degree of freedom of the system (1). If we introduce the $2N$-vector $x = (q, p)$ and the $2N \times 2N$ skew-symmetric matrix J defined by

$$J = \begin{pmatrix} 0 & -I_N \\ I_N & 0 \end{pmatrix},$$

where I_N is the identity matrix in $\mathbb{R}^N$, then the system (1) can be written as

$$\dot{x}(t) = JH'(t,x), \tag{2}$$

where $H'(t,x)$ denotes the gradient of the Hamiltonian function H with respect to x.

Since Newton's second law gives rise to differential equations of second order in $\mathbb{R}^N$ and therefore to equations with first order in $\mathbb{R}^{2N}$, Hamiltonian mechanics as a large field can be dated to the 18th century celestial mechanics. Most of the equations of motion in mechanics follow from the fact that the equations of motion can be written as Hamiltonian systems. In fact, the natural mathematical structure to develop the theory of conservative systems is the Hamiltonian formalism.

Complex dynamical behaviour of Hamiltonian systems has attracted mathematicians and physicists to study it by using a large variety of analytical and geometric tools. Now as an important part in those tools, the variational method has been applied by many authors to study the existence of periodic solutions of (2) satisfying the properties

$$\begin{aligned} &H'(t,x) = B_0(t)x + o(|x|) \\ &\text{as } |x| \to 0 \text{ uniformly in } t, \end{aligned} \tag{3}$$

$$H'(t,x) = B_\infty(t)x + o(|x|) \quad \text{as } |x| \to \infty \text{ uniformly in } t, \tag{4}$$

where $H(t,x)$ is T-periodic with respect to t, $B_0(t)$ and $B_\infty(t)$ are both T-periodic and continuous symmetric matrices. (2) is called asymptotically linear both at zero and at infinity because of (3) and (4). Precisely speaking, periodic solutions of (2) are exactly critical points of the functional

$$\Phi(x) = \int_0^T (-J\dot{x},x)\mathrm{d}t - \int_0^T H(t,x(t))\mathrm{d}t$$

defined on some suitable spaces of T-periodic functions, where $(\cdot,\cdot)$ denotes the inner product in $\mathbb{R}^{2N}$. However, since Φ is strong indefinite, i. e., it is unbounded from below and from above, the classical methods of the calculus of variations did not apply for a long time until the pioneering work done by Rabinowitz [1] from 1978 was done. In the celebrated paper [1], Rabinowitz obtained periodic solutions of (2) with H independent of t for the first time by using variational methods. Since then, more and more variational principles for indefinite functional have been established and applied to Hamiltonian systems.

Now let us recall some earlier work on asymptotically linear Hamiltonian systems like (2) with twist conditions by variational methods. Twist conditions play crucial roles in studying the existence and multiplicity of periodic solutions and can be measured by the gap between the Maslov index (or Conley-Zehnder index) at zero and at infinity. For the reader's convenience, here we give a brief introduction to the Maslov index. The Maslov index was introduced by Gel'fand and Lidskiĭ in [2], and developed by Conley and Zehnder, and Long and Zehnder in [3,4] for the study of Hamiltonian systems in relation to the Morse theory and the critical point theory. Let $W = Sp(N,\mathbb{R}) = \{M : M^\top JM = J\}$ be the set of all $2N \times 2N$ symplectic matrices. Write

$$W^* = \{\gamma \in C([0,T],W) : \gamma(0) = I_{2N}, \; 1 \text{ is not an eigenvalue of } \gamma(T)\}.$$

According to Conley and Zehnder [3] and Long and Zehnder [4], there is a map $j : W^* \to \mathbb{Z}$ satisfying that $j(\gamma_1) = j(\gamma_2)$ if and only if γ_1 and γ_2 lie in the same component of W^*. For a non-degenerate symmetric matrix $B(t)$, i. e., 1 is not a Floquet multiplier of the linear system

$$\dot{y}(t) = JB(t)y, \tag{5}$$

we can define the Maslov index of B by $i(B) = k$ if $j(M) = k$, where M is the fundamental solution matrix of the linear system (5). For $B(t)$ is degenerate, i. e., 1 is a Floquet multiplier of (5), Long [5] generalized the definition and the Maslov index of $B(t)$ is a pair of integers denoted by $(i(B),n(B))$, where $i(B)$ and $n(B)$ are defined by

$$i(B) = \lim_{C\to B} i(C), \text{ where } C \text{ is non-degenerate,}$$
$$n(B) = \dim\ker(M(T) - I_{2N}).$$

Obviously, B is non-degenerate if and only if $n(B) = 0$. That is, the linear system (5) has only the trivial T-periodic solution 0. Now, we denote by $(i_\infty(B_\infty, n_\infty(B_\infty))$ and $(i_0(B_0, n_0(B_0))$ the Maslov indices of B_∞ and B_0, respectively. (2) is called non-resonant at infinity if $n_\infty(B_\infty) = 0$ and non-resonant at zero if $n_0(B_0) = 0$. We refer to the books of Abbondandolo [6] and Long [7] for more details about the Maslov index.

The above problem of the existence and multiplicity of periodic solutions of (2) was first studied by Amman and Zehnder in [8,9], where B_∞ and B_0 were assumed to be constant matrices with B_∞ being non-degenerate and $i_\infty(B_\infty) \notin [i_0(B_0), i_0(b_0) + n_0(b_0)]$. Later, Conley and Zehnder [3], and Long and Zehnder [4] studied the case where B_∞ and B_0 are non-degenerate and depend on t and $i_\infty(B_\infty) \neq i_0(B_0)$. Many other authors followed the study in the case, where $B_\infty(t)$ is non-degenerate, see Long [5], Chang et al. [10], Li and Liu [11], Ding and Liu [12], etc. As to degenerate, but constant B_∞, Szulkin [13] studied the Landesman-Lazer type resonace condition, Su [14] studied the problem by Morse theory and Galerkin methods. If $B_\infty(t)$ depends actually on t, Chang et al. [10], Szulkin and Zou [15], Fei [16], Piasni [17] considered the strongly resonant case (the nonlinearities are bounded globally). Fei and Qiu [18] and Su [19] studied the problem under the assumption that $B_\infty(t)$ and $B_0(t)$ are finitely degenerate which is a strong condition on the nonlinearities.

In all those papers, twist conditions have been assumed in different forms. We say that twist conditions are actually various relations between the Maslov indices of $B_\infty(t)$ and $B_0(t)$. With these twist conditions, one can get the existence of a non-trivial periodic solutions of (2) since $B_\infty(t)$ can not be continuously deformed to $B_0(t)$. It is clear that if $B_\infty(t) = B_0(t)$, i. e., (2) is asymptotically linear at zero and at infinity with the same coefficient matrix, then $i_\infty(B_\infty) = i_0(B_0)$. This implies that the twist conditions do not hold anymore, which brings a difficulty to the arguments of the problem. To the authors' knowledge, very little is known about the case where $B_\infty(t) = B_0(t)$.

Motivated by this situation, we study (2) in this paper with $B_\infty(t) = B_0(t) = h$ being constant matrices. More precisely, we study the following Hamiltonian systems:

$$\dot{x}(t) = JH'(x), \tag{6}$$

satisfying

$$H'(x) = hx + o(|x|) \text{ as } |x| \to 0 \text{ or } |x| \to \infty, \tag{7}$$

where $h = \mathrm{diag}(\alpha, \alpha, \ldots, \alpha)$, $\alpha \in \mathbb{R}$. We shall establish the existence of periodic solutions of (6) with period 2π, i.e., solutions $x(t)$ of (6) satisfying $x(t+2\pi) = x(t)$. Moreover, these periodic solution also satisfy $x(t+\pi) = -x(t)$.

Our main result states as follows.

Theorem 1. Suppose that $H \in C^1(\mathbb{R}^{2N}, \mathbb{R})$ is even and satisfies (7), $\alpha \in 2\mathbb{Z} - 1$ and

(H1) $H(x) - \frac{1}{2}(hx, x) > 0$ as $|x|$ being small;

(H2) $|H'(x) - hx|$ is bounded and $H(x) - \frac{1}{2}(hx, x) \to -\infty$ as $|x| \to \infty$.

Then (6) possesses at least one non-trivial periodic solution.

Remark 2. We will use Theorem 6.2 of [20] and the Galerkin approximation methods to prove our main result. We give the proof of Theorem 1 in Section 2.

2. Proof of the Main Result

Let E be a real Hilbert space on which $S^1 = \mathbb{R}/2\pi\mathbb{Z}$ acts. Let $\mathrm{Fix}(S^1)$ be the set of fixed points of the action.

Definition 3. Φ satisfies $(PS)_c$ condition for any $c \in \mathbb{R}$ on E, if every sequence $\{x_n\} \subset E$ with $\Phi(x_n) \to c$ and $\Phi'(x_n) \to 0$ possesses a convergent subsequence.

The following theorem (Theorem 6.2 of [20]) on the existence of critical orbits of Φ will be useful in our discussion.

Theorem 4. Let $\Phi \in C^1(E, \mathbb{R})$ be an S^1-invariant functional satisfying $(PS)_c$ conditions. Let X and Y be closed invariant subspaces of E with $\mathrm{codim}\, X$ and $\dim Y$ finite and $\mathrm{codim}\, X < \dim Y$. Assume that the following conditions are satisfied:

(i) $\mathrm{Fix}(S^1) \subset X$, $Y \cap \mathrm{Fix}(S^1) = \{0\}$;

(ii) $\inf_X \Phi > -\infty$;

(iii) there exist $r > 0$ and $c < 0$ such that $\Phi(x) \le c$ whenever $x \in Y$ and $\|x\| = r$;

(iv) if $x \in \mathrm{Fix}(S^1)$ and $\Phi'(x) = 0$, then $\Phi(x) \ge 0$.

Then Φ possesses at least $\frac{1}{2}(\dim Y - \mathrm{codim}\, X)$ distinct critical orbits.

Let $E = W^{\frac{1}{2},2}(S^1, \mathbb{R})$ be the Sobolev space of 2π-periodic real functions

$$x(t) = a_0 + \sum_{k=1}^{+\infty} (a_k \cos kt + b_k \sin kt),$$

where $a_0, a_k, b_k \in \mathbb{R}^{2N}$ such that

$$\sum_{k=1}^{+\infty} k(|a_k|^2 + |k_k|^2) < +\infty.$$

Then E is a Hilbert space with a norm $\|\cdot\|$ induced by the inner product $\langle \cdot, \cdot \rangle$ defined by

$$\langle x, y \rangle = 2\pi(a_0, a_0') + \pi \sum_{k=1}^{\infty} k((a_k, a_k') + (b_k, b_k')),$$

where $y = a_0' + \sum_{k=1}^{+\infty} (a_k' \cos kt + b_k' \sin kt)$.

For each $x \in E$, define a functional $\Phi : E \to \mathbb{R}$ by

$$\Phi(x) = \int_0^{2\pi} (-Jx', y)\mathrm{d}t - \int_0^{2\pi} H(x(t))\mathrm{d}t. \tag{8}$$

It is well known that critical points of Φ are solutions of (6). As a Hilbert space, E is isometric to its dual space E^* by the Riesz representation theorem. Thus we can define an operator $A : E \to E$ by extending the bilinear form

$$\langle Ax, y \rangle = \int_0^{2\pi} (-Jx', y)\mathrm{d}t. \tag{9}$$

It is not difficult to see that A is linear bounded and self-adjoint. For each $x(t) = a_0 + \sum_{k=1}^{+\infty} (a_k \cos kt + b_k \sin kt) \in E$,

$$Ax = \sum_{k=1}^{+\infty} (-Jb_k \cos kt + Ja_k \sin kt). \tag{10}$$

Write

$$F(x) = \int_0^{2\pi} H(x(t))\mathrm{d}t.$$

According to [11], $F' : E \to E$ is compact. Moreover, for any $y \in E$,

$$\langle F'(x), y \rangle = \int_0^{2\pi} (H'(x), y)\mathrm{d}t.$$

Then Φ can be rewritten as

$$\Phi(x) = \frac{1}{2}\langle Ax, x\rangle - F(x). \tag{11}$$

In order to obtain the solution $x(t)$ of (6) satisfying $x(t+\pi) = -x(t)$, we introduce an action Γ on E by

$$\Gamma x(t) = h_{2N}x(t-s), \quad \forall x \in E, \tag{12}$$

where $s = \frac{\pi}{2N}$ and

$$h_2N = \begin{pmatrix} 0 & 0 & \cdots & 0 & -1 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

A direct computation shows that $\Gamma^{2N}x(t) = -x(t-\pi)$ and $\Gamma^{4N}x(t) = x(t)$. Therefore, if $\Gamma x(t) = x(t)$, then $x(t) = \Gamma^{2N}x(t) = -x(t-\pi)$. Now we define a subset SE of E as

$$SE = \{x \in E : \Gamma x(t) = x(t)\}.$$

Lemma 5. As a closed subspace of E, SE has the following structure:

$$SE = \left\{x(t) : x(t) = \sum_{k=1}^{\infty}(a_k \cos(2k-1)t + b_k \sin(2k-1)t)\right\},$$

where $a_k = pu_k - qv_k$, $b_k = pv_k + qu_k$, $p, q \in \mathbb{R}$ and

$$u_k = \big(1, \cos\tau_k, \cos(2\tau_k), \cdots, \cos((2N-1)\tau_k)\big),$$
$$v_k = \big(0, \sin\tau_k, \sin(2\tau_k), \cdots, \sin((2N-1)\tau_k)\big),$$

for $\tau_k = (2k-1)s$.

Proof. It is easy to see that SE is a closed subspace of E. For any $x(t) = a_0 + \sum_{k=1}^{+\infty}(a_k \cos kt + b_k \sin kt) \in E$, we have $x(t-\pi) = -x(t)$, that is

$$a_0 = -a_0, a_k = (-1)^{k+1}a_k, b_k = (-1)^{k+1}bk,$$

i.e.,

$$a_k = b_k = 0 \ \text{ for even } \ k.$$

Thus, for any $x(t) \in SE$,

$$x(t) = \sum_{k=1}^{+\infty}(a_k \cos(2k-1)t + b_k \sin(2k-1)t).$$

Note that $\Gamma x(t) = x(t)$ yields $h_{2N}x(t-s) = x(t)$. For $x \in SE$, we have

$$\begin{aligned}&\sum_{k=1}^{+\infty}(a_k \cos(2k-1)t + b_k \sin(2k-1)t) \\ &= \sum_{k=1}^{+\infty}\big[h_{2N}a_k \cos(2k-1)(t-s) \\ &\qquad + h_{2N}b_k \sin(2k-1)(t-s)\big].\end{aligned}$$

This means that for $k \geq 1$ and $\tau_k = (2k-1)s$,

$$\begin{aligned} a_k &= h_{2N}a_k \cos\tau_k - h_{2N}b_k \sin\tau_k, \\ b_k &= h_{2N}a_k \sin\tau_k + h_{2N}b_k \cos\tau_k. \end{aligned} \tag{13}$$

Let $e_k = a_k + \mathrm{i}b_k$. Then (13) can be written as

$$e_k = \mathrm{e}^{\mathrm{i}\tau_k}h_{2N}e_k. \tag{14}$$

Let λ be an eigenvalue of h_{2N}. By a direct computation of determinant, we have

$$\lambda = \mathrm{e}^{\pm\frac{(2j+1)\pi}{2N}\mathrm{i}}, \quad j = 0, 1, 2, \cdots, N-1.$$

(15) shows that

$$e_k = (1, \mathrm{e}^{\mathrm{i}\tau_k}, \mathrm{e}^{\mathrm{i}2\tau_k}, \cdots, \mathrm{e}^{\mathrm{i}(2N-1)\tau_k}). \tag{15}$$

Set $u_k = \mathrm{Re}(e_k)$, $v_k = \mathrm{Im}(e_k)$. Then the conclusion follows from a direct check. □

Remark 6. It is easy to see that critical points of Φ in SE are non-trivial periodic solutions of (6) satisfying the symmetric property.

For each symmetric matrix h, we define an operator $B : E \to E$ by

$$\langle Bx, y\rangle = \int_0^{2\pi}(hx, y)\mathrm{d}t. \tag{16}$$

It is easy to see that B is linear bounded and self-adjoint. Moreover, for any $x \in E$,

$$Bx(t) = ha_0 + \sum_{k=1}^{+\infty}\frac{1}{k}(ha_k \cos kt + hb_k \sin kt). \tag{17}$$

Let

$$E_n = \left\{x(t) : x(t) = a_0 + \sum_{k=1}^{n}(a_k \cos kt + b_k \sin kt)\right\}.$$

Let $P_n : E \to E_n$ be the orthogonal projection. Then p_n satisfies the following properties:

(1) The image of P_n has finite dimension;
(2) $P_n x \to x$ as $n \to \infty$ for any $x \in E$;
(3) P_n commutes with $A - B$.

Therefore $\Omega = \{P_n : n = 1, 2, \cdots\}$ is a Galerkin approximation method with respect to operator $A - B$. Set

$$SE(k) = \{x \in SE : x(t) = a_k \cos(2k-1)t + b_k \sin(2k-1)t\}$$

and

$$SE_n = SE(1) \oplus SE(2) \oplus \cdots \oplus SE(n).$$

Definition 7. For each $\alpha \in \mathbb{R}$, define

$$i(\alpha) = \sum_{k=1}^{\infty} \{\sigma^-[(-1)^{k+1}(2k-1) - \alpha] - \sigma^-((-1)^{k+1})\},$$
$$j(\alpha) = \sum_{k=1}^{\infty} \sigma^0[(-1)^{k+1}(2k-1) - \alpha],$$

where $\sigma^-(t) = 1$ if $t < 0$ and $\sigma^-(t) = 0$ if $t \geq 0$, $\sigma^0(t) = 1$ if $t = 0$, $\sigma^0(t) = 0$, otherwise.

It is easy to see that $i(\alpha)$ and $j(\alpha)$ are well defined. Denote by $M^+(A-B)$, $M^-(A-B)$, and $M^0(A-B)$ the positive definite, the negative definite, and the null subspaces of the self-adjoint operator $A - B$, respectively. We have the following lemma.

Lemma 8. For $k \geq 1$, let $\lambda_k = (-1)^{k+1}(2k-1) - \alpha$. Then the following conclusions hold.

$$M^+(A-B) = \bigoplus_{k=1, \lambda_k > 0}^{\infty} SE(k),$$
$$M^-(A-B) = \bigoplus_{k=1, \lambda_k < 0}^{\infty} SE(k),$$
$$M^0(A-B) = \bigoplus_{k=1, \lambda_k = 0}^{\infty} SE(k).$$

Moreover, for k large enough,

$$\dim M^-(P_k(A-B)P_k) = 2i(\alpha) + 2\sum_{j=1}^{k} \sigma^-((-1)^{j+1}),$$
$$\dim M^0(P_k(A-B)P_k) = 2j(\alpha),$$

where P_k is restricted over SE.

Proof. For $x_k = a_k \cos(2k-1)t + b_k \sin(2k-1)t \in SE$, consider the following eigenvalue problem:

$$(A-B)x_k = \lambda x_k.$$

From (10) and (17), one has

$$-Jb_k - \frac{\alpha}{2k-1} a_k = \lambda a_k, Ja_k - \frac{\alpha}{2k-1} b_k = \lambda b_k. \quad (18)$$

By a direct computation, for $e_k = u_k + \mathrm{i} v_k$,

$$Je_k = \mathrm{e}^{\mathrm{i} N \tau_k} e_k = (-1)^{k+1} \mathrm{i} e_k.$$

This yields $Ju_k = (-1)^k v_k$, $jv_k = (-1)^{k+1} u_k$. By Lemma 5, $a_k = pu_k - qv_k$, $b_k = pv_k + qu_k$. Then (18) turns to

$$(-1)^{k+1} a_k - \frac{\alpha}{2k-1} a_k = \lambda a_k,$$
$$(-1)^{k+1} b_k - \frac{\alpha}{2k-1} b_k = \lambda b_k.$$

The above two equalities show that $A - B$ is positive definite, negative definite, and null on $SE(k)$ if and only if $\lambda_k = (-1)^{k+1}(2k-1) - \alpha$ is positive, negative, and zero, respectively.

For k large enough, Definition 7 implies

$$\begin{aligned} &\dim M^-(P_k(A-B)P_k) \\ &= 2\sum_{j=1}^{k} \sigma^-[(-1)^{k+1}(2k-1) - \alpha] \\ &= 2\sum_{j=1}^{k} \sigma^-[(-1)^{k+1}(2k-1) - \alpha] - 2\sum_{j=1}^{k} \sigma^-((-1)^{j+1}) \\ &\quad + 2\sum_{j=1}^{k} \sigma^-((-1)^{j+1}) \\ &= 2i(\alpha) + 2\sum_{j=1}^{k} \sigma^-((-1)^{j+1}). \end{aligned}$$

The last equality of Lemma 8 is obvious. The proof is complete. □

Let Φ_{SE} and Φ_{SE_n} be the restrictions of Φ over SE and SE_n, respectively.

Lemma 9. Under the assumptions of Theorem 1, the following conclusions hold:
(i) Φ_{SE} satisfies $(PS)_c$ condition on SE,
(ii) Φ_{SE_n} satisfies $(PS)_c$ condition on SE_n,
(iii) Φ_{SE} satisfies $(PS)^*$ condition on SE, i. e., every sequence $\{x_n\}$ with $x_n \in \Phi_{SE_n}$, $\Phi_{SE_n}(x_n)$ being bounded and $\Phi'_{SE_n}(x_n) \to 0$, possesses a convergent subsequence.

Proof. Case (i). We first show that $\{x_n\} \subset SE$ is bounded. Assume $\{x_n\}$ is not bounded. By passing to a subsequence we may assume $\|x_n\| \to \infty$ as $n \to \infty$.

For any $x \in SE$, define

$$\Psi_{SE}(x) = F(x) - \frac{1}{2}\langle Bx, x\rangle. \tag{19}$$

Then $\Phi_{SE}(x)$ can be written as

$$\Phi_{SE}(x) = \frac{1}{2}\langle (A-B)x, x\rangle - \Psi_{SE}(x).$$

Since $|H'(x) - hx|$ is bounded, there exists a constant $C > 0$ such that

$$\|\Psi'_{SE}(x)\| < C \quad \forall x \in SE.$$

Observing that $(A-B)x = \Phi'_{SE}(x) + \Psi'_{SE}(x)$ and $\Phi'_{SE}(x_n) \to 0$ as $n \to 0$, we have

$$\|(A-B)x_n\| \le \|\Phi'_{SE}(x_n)\| + \|\Psi'_{SE}(x_n)\| \le C+1 \quad \text{as } n \to \infty.$$

Since $\|x_n\| \to \infty$, $H(x_n) - \frac{1}{2}(hx_n, x_n)$ converges to $-\infty$ as $n \to \infty$,

$$\begin{aligned}\Psi_{SE}(x_n) &= -\Phi_{SE}(x_n) + \frac{1}{2}\langle (A-B)x_n, x_n\rangle \\ &> -\Phi_{SE}(x_n) - \frac{C+1}{2}\|x_n\|.\end{aligned} \tag{20}$$

Let $n \to \infty$, (20) implies $-\infty > -\infty$. This is a contradiction. Therefore $\{x_n\}$ is bounded.

Now, we show $\{x_n\}$ has a convergent subsequence. Notice that $P_0 : SE \to \ker A = \mathbb{R}^{2N}$ is of finite rank and $F' : SE \to SE$ is compact. Therefore we may suppose that

$$F'(x_n) \to y \text{ and } P_0 x_n \to z \text{ in } SE \text{ as } n \to \infty.$$

Since $A + P_0$ has a continuous inverse $(A+P_0)^{-1}$, it follows from

$$(A - P_0)x_n = \Phi'_{SE}(x_n) + F'(x_n) + P_0 x_n$$

that

$$\begin{aligned} x_n &= (A+P_0)^{-1}(\Phi'_{SE}(x_n) + F'(x_n) + P_0 x_n) \\ &\to (A+P_0)^{-1}(y+z) \text{ as } n \to \infty. \end{aligned}$$

Henceforth, $\{x_n\}$ has a convergent subsequence.

With a similar argument to Case (i), we can prove the other two cases. The proof is complete. □

For $\theta \in S^1$ and $x \in SE$, define an action T on SE by

$$T(\theta)x(t) = x(t+\theta).$$

A direct computation shows that

$$\begin{aligned}\mathrm{Fix}(S^1) &= \{x \in SE : T(\theta)x(t) = x(t+\theta), \quad \forall \theta \in S^1\} \\ &= \{0\}.\end{aligned} \tag{21}$$

Now we are ready to prove our main result.

Proof of Theorem 1. Since $x(t) \in SE$ is periodic and $F(x(t))$ is independent of t, Φ_{SE_n} is S^1-invariant according to the action T on SE.

Take the subspace X and Y as

$$X = M^+(A-B) \oplus M^0(A-B) \cap SE_n,$$

$$Y = M^-(A-B) \oplus M^0(A-B) \cap SE_n,$$

It is easy to see that $\operatorname{codim} X < \dim Y$. By (21), we can verify easily that the assumptions (i) and (iv) of Theorem 4 hold.

For $x \in M^+(A-B)$, there is a suitable constant $c_1 > 0$ such that

$$\langle (A-B)x, x\rangle \ge c_1\|x\|^2. \tag{22}$$

Let Ψ_n be the restriction of Ψ_{SE} on SE_n. By the condition (H2) of Theorem 1, $\Psi'_n(x)$ is bounded, i. e., $\exists c_2 > 0$ such that

$$\|\Psi'_n(x)\| \le c_2, \quad \forall x \in SE.$$

Moreover, $\Psi_n(x_0) \to -\infty$ as $x_0 \in M^0(A-B) \cap SE_n$ with $\|x_0\| \to +\infty$.

Write $x = x_+ + x_0 \in X$, where $x_+ \in M^+(A-B) \cap SE_n$. We have

$$\begin{aligned}\Phi_{SE_n}(x) &= \frac{1}{2}\langle (A-B)x_+, x_+\rangle - \Psi_n(x_+ + x_0) \\ &> \frac{1}{2}c_1\|x_+\|^2 - c_2\|x\| - \Psi_n(x_0).\end{aligned}$$

Thus the assumption (ii) of Theorem 4 holds. If the assumption (iii) of Theorem 4 holds, then by Definition 7, Lemma 8, Lemma 9, and Theorem 4, Φ_{SE_n} possesses at least

$$\frac{1}{2}(\dim Y - \operatorname{codim} X) = i(\alpha) + j(\alpha) - i(\alpha) = j(\alpha)$$

distinct orbits. From Definition 7 and $\alpha \in 2\mathbb{Z} - 1$, $j(\alpha) = 1$. Therefore, Φ_{SE_n} has at least one non-trivial critical point x_n. By Lemma 9, Φ_{SE_n} satisfies the $(PS)^*$ condition. So $\{x_n\}$ has a subsequence convergent to a point x with $\Phi'_{SE}(x) = 0$, i. e., x is a critical point of Φ_{SE}. Therefore, (6) has at least one non-trivial periodic solution. Thus the proof of Theorem 1 will be finished by the following lemma.

Lemma 10. Under the conditions of Theorem 1, the assumptions (iii) of Theorem 4 holds.

The main idea of the proof of Lemma 10 comes from [11].

Proof of Lemma 10. Let

$$G(x) = -\left(H(x) - \frac{1}{2}(hx,x)\right). \quad (23)$$

By (H2), there exists a constant $\rho > 0$ being small such that $G(x) < 0$ for any $x \in B_\rho = \{x : \|x\| \le \rho\}$. Define a function $g : M^0(A-B) \cap SE_0 \cap B_r \to \mathbb{R}$ by

$$g(x_0) = \sup\{\Phi_{SE}(x_- + x_0) : x_- \in M^-(A-B) \cap SE_n \cap B_r\}.$$

Since Φ_{SE} and Φ'_{SE} are both bounded in $M^-(A-B) \cap M^0(A-B) \cap SE_n$, g is well defined and continuous. First, we claim that $\Phi_{SE}(x_- + x_0)$ attains its maximum in the domain $M^-(A-B) \cap B_r \cap SE_n$ at an interior point for a given point $x_0 \in M^0(A-B) \cap B_r \cap SE_n$.

By (7), for any $\kappa > 0$ there exists $\delta > 0$ such that

$$|G(x)| \le \kappa|x|^2 + \delta|x|^4$$

which implies

$$\int_0^{2\pi} |G(x)|\mathrm{d}t \le \int_0^{2\pi} (\kappa|x(t)|^2 + \delta|x(t)|^4)\mathrm{d}t \le (\kappa|x(t)|^2_{L^2} + \delta|x(t)|^4_{L^4}) \le \beta\kappa\|x\|^2, \quad (24)$$

as $x \in B_r$ and r small enough.

Given $x_- \in M^-(A-B) \cap B_r \cap SE_n$. If $x \in \partial(M^-(A-B) \cap B_r \cap SE_n)$ and letting κ be small enough, then we have $\|x_-\| = r$ and $\|x_- + x_0\| \le 2r$ and

$$\Phi_{SE}(x_- + x_0) = \frac{1}{2}\langle (A-B)x_-, x_-\rangle + \int_0^{2\pi} G(x_- + x_0)\mathrm{d}t \quad (25)$$

$$\le -c_3\|x_-\|^2 + \beta\kappa\|x_- + x_0\|^2 < -\frac{c_3}{2}r^2. \quad (26)$$

If $x_- = 0$, then

$$\Phi_{SE}(x_- + x_0) = \Phi_{SE}(x_0) = \int_0^{2\pi} G(x_0)\mathrm{d}t > \beta\kappa\|x_0\|^2 > \beta\kappa r^2. \quad (27)$$

Thus we get the claim. Let $y = y(x_0) \in \mathrm{int}M^-(A-B) \cap B_r \cap SE_n$ be the maximum point of $\Phi_{SE}(x_- + x_0)$. Then y satisfies

$$\langle (A-B)y, z\rangle + \int_0^{2\pi} G'(y + x_0)\mathrm{d}t = 0, \quad \forall z \in M^-(A-B) \cap SE_n. \quad (28)$$

Let

$$u(t) = hy + G'(y + x_0),$$

then $u(t) \in L^2(S^1, \mathbb{R}^{2N})$ and

$$\|u(t)\|_{L^2} < h|y|_{L^2} + G'(y + x_0)|_{L^2} < h|y|_{L^2} + c_4|y + x_0|_{L^2} < c_5 r, \quad (29)$$

since $|G'(x)| = |H'(x) - hx|$ is bounded.

It follows from (28) that

$$y' = Ju.$$

From the regularity theory, (29) yields that

$$\|y\|_C \le c_6(|u|_{L^2} + |y|_{L^2}) \le c_7 r,$$

where $\|\cdot\|_C$ denotes the maximum norm in $C(S^1, \mathbb{R}^{2N})$. Note that $M^0(A-B) \cap SE_n$ is a space with finite dimension, all norms are equivalent to each other. Hence, one has $\|x_0\|_C < c_8 r$ and

$$\|y(x_0) + x_0\|_C < c_9 r,$$

for any $x_0 \in M^0(A-B) \cap B_r$, where c_9 is a constant. This enables us to choose r small enough such that $c_9 r < \rho$. Then by (23), $G(y(t) + x(t)) < 0$ for any t such that $(y(x_0) + x_0)(t) \ne 0$. Then we have

$$\Phi_{SE}(x_- + x_0) \le g(x_0) = \Phi_{SE}(y + x_0) = \frac{1}{2}\langle (A-B)y, y\rangle + \int_0^{2\pi} G(y + x_0)\mathrm{d}t \le 0 \quad (30)$$

for any $x_- + x_0 \in M^-(A-B) \cap M^0(A-B) \cap SE_n \cap B_r$.

Notice that $y + x_0 \ne 0$ if $x_0 \ne 0$. Thus $\int_0^{2\pi} G(y + x_0)\mathrm{d}t < 0$ and $g(x) < 0$. So if we restrict g to $M^0(A-B) \cap SE_n \cap \partial(B_r)$, then g is strict negative. This implies that there exists a constant $\eta > 0$ such that

$$g(x_0) \le -\eta < 0,$$

for any $x_0 \in M^0(A-B) \cap SE_n \cap B_r$. Then the assumption (iii) of Theorem 4 holds with $c =$

$\min\{-\frac{1}{2}c_3r^2, -\eta\}$. The proof of Theorem 1 is complete. □

Now let us take an example as an application of Theorem 1. For any $x \in \mathbb{R}^{2N}$, let $\rho(|x|)$ be a smooth function satisfying

$$\rho(|x|) = \begin{cases} 1, \text{ as } & |x| < r, \\ 0, \text{ as } & |x| > R, \end{cases}$$

with $0 \le \rho(|x|) \le 1$ and $|\rho'(|x|)| < 2$, where r and R are given positive constants with $r < R$. Write

$$H_0(x) = \rho(|x|)|x|^\mu \sin|x| + (\rho(|x|) - 1)|x|\left(1 - \frac{1}{\ln(\mathrm{e} + |x|)}\right),$$

where $\mu \ge 2$. Let $H(x) = \frac{1}{2}(hx, x) + H_0(x)$. Then $H \in C^1(\mathbb{R}^{2N}, \mathbb{R})$ is even and defines the following Hamiltonian system

$$\dot{x} = JH'(x). \qquad (31)$$

By a direct computation, $H(x)$ satisfies (7), (H1) and (H2). From Theorem 1, (31) has at least one nontrivial periodic solution which can not be obtained by previous results, since there is no twist condition for (31).

3. Conclusion

Now, we have studied a class of asymptotically linear Hamiltonian systems and proved Theorem 1. Applying Theorem 1, we can obtain non-trivial periodic solutions of asymptotically linear Hamiltonian systems which have no twist conditions. Therefore, our result is a good supplement to previous results on the existence of periodic solutions of asymptotically linear Hamiltonian systems.

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[1] P.H. Rabinowitz, Commun. Pure Appl. Math. **31**, 157C184(1978).
[2] I.M. Gel'fand and V.B. Lidskiĭ, Amer. Math. Soc. Transl. Ser. 2 **8**, 143 (1958).
[3] C. Conley and E. Zehnder, Commun. Pure Appl. Math. **37**, 207 (1984).
[4] Y. Long and E. Zehnder, Morse theory for forced oscillations of asymptotically linear Hamiltonian systems, in: Stochastic Processes in Physics and Geometry, World Scientific, Singapore 1990, pp. 528.
[5] Y. Long, Sci. (Series A) **33**, 1409 (1990).
[6] A. Abbondandolo, Morse Theory for Hamiltonian Systems, Chapman, Hall, London 2001.
[7] Y. Long, Index Theory for Symplectic Paths with Applications, Prog. Math. Vol. **207**, Birkhäuser, Basel 2002.
[8] H. Amann and E. Zehnder, Ann. Scuola Norm. Sup. Pisa (4) **7**, 39 (1980).
[9] H. Amann and E. Zehnder, Manuscripta Math. **32**, 149 (1980).
[10] K.C. Chang, J. Liu and M. Liu, Ann. Inst. H. Poincare **14**, 103 (1997).
[11] S. Li and J. Liu, J. Diff. Eq. **78**, 53 (1989).
[12] Y. Ding and J. Liu, J. System Sci. Math. Sci. **9**, 30 (1990).
[13] A. Szulkin, Math. Z. **209**, 375 (1992).
[14] J. Su, J. Math. Anal. Appl. **233**, 1 (1999).
[15] A. Szulkin and W. Zou, J. Diff. Eq. **174**, 369 (2001).
[16] G. Fei, J. Diff. Eq. **121**, 121 (1995).
[17] L. Pisani, Nonlinear Diff. Eq. Appl. **4**, 61 (1997).
[18] G. Fei and Q. Qiu, Chinese Ann. Math. Ser. B **18**, 359 (1997).
[19] J. Su, J. Diff. Eq. **145**, 252 (1998).
[20] M. Willem, Critical Point Theory and Hamiltonian Systems, Springer-Verlag, Berlin 1989.