Path Integral Treatment of a Dirac Particle in a Weak Gravitational Plane Wave

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The Green functions for Klein-Gordon and Dirac particles in a weak gravitational field are determined exactly by the path integral formalism. By using simple changes, it is shown that the classical trajectories play an important role in determining these Green functions.

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1. Introduction

We know that the Dirac equation is one of the fundamental equations of physics which allows not only to determine the value $\frac{1}{2}$ of the electron spin, but predict the existence of an antiparticle. However, this equation is not free of difficulties related to states of negative energy, and in order to explain certain phenomena, the use of field theory becomes more than necessary. A good approximation for the treatment of certain problems, when the interaction is not strong enough, is the direct use of the Dirac equation, which is sufficient in this case. For example, for a particle of spin $\frac{1}{2}$, the correction due to the polarization of the vacuum can be obtained, following Pauli's results, in a simple modification of the Dirac equation, where the movement of the particle is corrected by an additional term describing the anomaly of the magnetic moment of the particle.

In addition, we know that the operators of quantum mechanics play a central role in the quantization; later, another means of quantization has been proposed using the path integral. If, in non-relativistic case, the deep link with classical mechanics has been established with the path integral formulation, in the relativistic case there are various formulations. For the Dirac equation, for example, which is a matrix equation in which the spin is described by the matrices γ^μ (which do not commutate), the formalism of path integrals according to Fradkin and Gitman [1, 2] – object of this paper – uses two types of variables: bosonic for the external motion and fermionic (Grassmann) for the interior motion of the particle. Two types of representations of path integrals relative to the equation of Dirac exist at present: one, the so-called global one, is based on the "square" of the Dirac equation, i. e. the Klein-Gordon equation with an additional spin-field term; the other one, the so-called local one, which uses a matrix γ^5 [3] for reasons of homogenization, gives the physical states directly and not artificially, in contrary to the global case.

The purpose of this paper is to examine the possibilities of solving simple problems of particles interacting with an external classical field, using these two representations, equivalent by construction to the Dirac equation. We consider in this paper the Klein-Gordon and the Dirac particles moving in a gravitational field which is relatively weak. Our object is to show by a simple modification, that we can easily obtain from the classical trajectories the Green functions solution of the Klein-Gordon and the Dirac equations.

In the absence of gravitation we have an Euclidian metric $\eta_{\mu\nu}$. When the gravitation is weak, the metric is not flat but can be chosen such that the Green functions have analytical expressions. In first approximation the metric is [4]

$$g_{\mu\nu}(x) = \eta_{\mu\nu} + h_{\mu\nu}(kx) \quad \text{or} \\ g^{\mu\nu}(x) = \eta^{\mu\nu} - h^{\mu\nu}(kx), \tag{1}$$

where the matrix h is considered as a perturbation and is chosen in a simple form. It only depends on the product kx, where k and x are the 4-vectors related to the wave number and the position of the particle.

It is further supposed that the form of h [4] is

$$h^{\mu\nu}(kx) = a^{\mu\nu} F(kx), \tag{2}$$

where the wave number 4-vectors k and the elements of the (4×4) matrix a statisfy the condition

$$k^2 = 0, \quad k^{\mu} a_{\mu\nu} = a_{\mu\nu} k^{\nu} = 0, \quad \mathrm{Tr}\, a = 0. \tag{3}$$

We begin with the simple case of the Klein-Gordon particles (without spin) to determine the Green function, and then we pass to the Green function for the Dirac equation following successively the two representations, without using the γ^5 matrix for the global approach and the local approach, where the matrix γ^5 [1] is necessary. Let us note that this problem has been the object of some other works [4, 5], calculating the Green functions for particles of spin 0 and spin $\frac{1}{2}$ by solving the Klein-Gordon and Dirac equations via the solutions of the respective Heisenberg equations. The same problem has recently been considered with a stochastic approach [6] and with the formalism of path integrals [7], using only the global representation.

Let us begin with the simple case of the Klein-Gordon particles.

2. Green Function for the Klein-Gordon Particle

In this section, we propose to determine the Green function for a particle in a weak gravitational field, which solves the Klein-Gordon equation

$$\left(\hat{p}_b^2 - \hat{p}_b h_b \hat{p}_b - m^2\right) G^c(x_b, x_a) = \delta^4(x_b - x_a). \tag{4}$$

Symbolically we have

$$\hat{G}^c = \frac{I}{\hat{p}^2 - \hat{p} h \hat{p} - m^2},$$

and with the help of a parameter λ, we obtain for $\hat{G}^c$ the exponential form

$$G^c(x_b, x_a) = \mathrm{i} \int_0^{\infty} \mathrm{d}\lambda \, \langle x_b \mid \exp\left[-\mathrm{i}\lambda \hat{H}(\hat{p}, \hat{x})\right] \mid x_a \rangle,$$

where

$$\hat{H}(\hat{p}, \hat{x}) = m^2 - \hat{p}^2 + \hat{p} a \hat{p} F(kx),$$

following [4].

Let us pass now to the path integral formulation. To construct the Green function, first let us eliminate the operators. We use the usual procedure: subdivide the interval $[x_a, x_b]$ into N equal intervals of length $\triangle\tau = \frac{\lambda}{N}$, then insert the closed relations

$$\int \mathrm{d}^4 x \mid x\rangle\langle x \mid = 1, \quad \int \mathrm{d}^4 p \mid p\rangle\langle p \mid = 1,$$

with the scalar product

$$\langle x \mid p \rangle = \frac{1}{(2\pi)^4} \mathrm{e}^{\mathrm{i}px},$$

which allows to pass from one base to another one.

Next we eliminate the operators by using

$$\hat{x}^{\mu} \mid x\rangle = x^{\mu} \mid x\rangle, \quad \hat{p}^{\mu} \mid p\rangle = p^{\mu} \mid p\rangle.$$

Now we choose the Weyl order or the mid-point prescription, and the expression of the Green function then is

$$\begin{aligned} G^c(x_b, x_a) &= \mathrm{i} \int_0^{\infty} \mathrm{d}\lambda \int \mathrm{D}x \int \mathrm{D}p \\ &\cdot \exp\left\{ \mathrm{i} \int_0^{\lambda} \left[p\dot{x} + p^2 - m^2 - papF(kx)\right] \mathrm{d}\tau \right\}. \end{aligned} \tag{5}$$

Let us now proceed to its evaluation: As the action depends on kx, let us define $\varphi = kx$ as a new variable. We introduce the identity [8 – 10]

$$\begin{aligned} &\int \mathrm{d}\varphi_b \int \mathrm{d}\varphi_a \delta(\varphi_a - kx_a) \int \mathrm{D}\varphi \int \mathrm{D}p_{\varphi} \\ &\cdot \exp\left\{ \mathrm{i} \int_0^{\lambda} p_{\varphi}(\dot{\varphi} - k\dot{x}) \mathrm{d}\tau \right\} = 1. \end{aligned} \tag{6}$$

Then (5) becomes

$$\begin{aligned} G^c(x_b, x_a) &= \mathrm{i} \int_0^{\infty} \mathrm{d}\lambda \int \mathrm{D}x \int \mathrm{D}p \int \mathrm{d}\varphi_b \\ &\cdot \int \mathrm{d}\varphi_a \delta(\varphi_a - kx_a) \int \mathrm{D}\varphi \int \mathrm{D}p_{\varphi} \\ &\cdot \exp\left\{ \mathrm{i} \int_0^{\lambda} \left[(p - p_{\varphi}k)\dot{x} + p_{\varphi}\dot{\varphi} + p^2 \right.\right. \\ &\qquad \left.\left. - paF(\varphi)p - m^2\right] \mathrm{d}\tau \right\}. \end{aligned} \tag{7}$$

Introducing $p = P + p_{\varphi}k$, the measure remains unchanged. We then have

$$\begin{aligned} G^c(x_b, x_a) &= \mathrm{i} \int_0^{\infty} \mathrm{d}\lambda \int \mathrm{D}x \int \mathrm{D}p \int \mathrm{d}\varphi_b \\ &\cdot \int \mathrm{d}\varphi_a \delta(\varphi_a - kx_a) \int \mathrm{D}\varphi \int \mathrm{D}p_{\varphi} \\ &\cdot \exp\left\{ \mathrm{i} \int_0^{\lambda} \left[p\dot{x} + p^2 + (\dot{\varphi} + 2pk)\, p_{\varphi} \right.\right. \\ &\qquad \left.\left. - F(\varphi)pap - m^2\right] \mathrm{d}\tau \right\}. \end{aligned} \tag{8}$$

Let us integrate the first term of the action by part,

$$\int_0^\lambda p\dot{x}\mathrm{d}\tau = (p_b x_b - p_a x_a) - \int_0^\lambda \dot{p}x\mathrm{d}\tau,$$

and then integrate over the paths $x(\tau)$,

$$G^c(x_b,x_a) = \mathrm{i}\int_0^\infty \mathrm{d}\lambda \int \mathrm{D}p\,\delta(\dot{p}) \int \mathrm{d}\varphi_b \cdot \int \mathrm{d}\varphi_a \delta(\varphi_a - kx_a) \int \mathrm{D}\varphi \int \mathrm{D}p_\varphi \cdot \exp\left\{\mathrm{i}\left[(x_b p_b - x_a p_a) + \int_0^\lambda \left[p^2 + (2pk + \dot{\varphi})p_\varphi - F(\varphi)pap - m^2\right]\right]\right\}\mathrm{d}\tau.$$

The Dirac function $\delta(\dot{p})$ expresses that the momentum of the particle remains constant during the motion,

$$p_1 = p_2 = \ldots = p_n = p = \text{const.} \tag{9}$$

Integrating successively over all the p_n, the integral $\int \mathrm{D}p$ is finally reduced to a simple integral $\int \frac{\mathrm{d}p}{2\pi}$.

$$G^c(x_b,x_a) = \mathrm{i}\int_0^\infty \mathrm{d}\lambda \int \frac{\mathrm{d}p}{2\pi} \int \mathrm{d}\varphi_b \int \mathrm{d}\varphi_a \delta(\varphi_a - kx_a) \cdot \int \mathrm{D}\varphi \int \mathrm{D}p_\varphi \exp\{\mathrm{i}(x_b p_b - x_a p_a)\}\,\delta(\dot{p}) \cdot \exp\left\{\mathrm{i}\int_0^\lambda (p^2 + (2pk + \dot{\varphi})p_\varphi - F(\varphi)pap - m^2)\mathrm{d}\tau\right\}. \tag{10}$$

Integrating over p_φ also leads to a Dirac function $\delta(2pk + \dot{\varphi})$. The integration over the variables φ shows that the essential contribution to the Green function comes from the trajectory

$$\frac{\mathrm{d}\varphi}{\mathrm{d}\tau} = -2pk, \tag{11}$$

i. e. from the path described by

$$\varphi(\tau) = -2pk\tau + c^{te}, \tag{12}$$

which is obviously that of a line.

Thus G^c becomes

$$G^c(x_b,x_a) = \mathrm{i}\int_0^\infty \mathrm{d}\lambda \int \frac{\mathrm{d}p}{2\pi} \int \mathrm{d}\varphi_b \int \mathrm{d}\varphi_a \delta(\varphi_a - kx_a) \cdot \delta(\varphi_b - kx_a - (2pk)\lambda) \exp\left\{\mathrm{i}\left[p(x_b - x_a) + \int_0^\lambda (p^2 - m^2)\mathrm{d}\tau\right]\right\} \int \mathrm{D}\varphi \exp\left\{\mathrm{i}\frac{pap}{2pk}\int_{kx_a}^{\varphi_b} F(\varphi)\mathrm{d}\varphi\right\}. \tag{13}$$

This Green function $G^c(x_b,x_a)$ has a form which is not symmetrical with respect to the positions x_a and x_b. In order to symmetrize this expression, let us introduce the integral representation of the δ function

$$\delta(\varphi_b - kx_a - (2pk)\lambda) = \frac{1}{2\pi}\int \mathrm{d}p_{\varphi_b} \exp\left\{\mathrm{i}p_{\varphi_b}[\varphi_b - kx_a - (2pk)\lambda]\right\}.$$

Let us change then p into $p - kp_{\varphi_b}$, so we have

$$G^c(x_b,x_a) = \mathrm{i}\int_0^\infty \mathrm{d}\lambda \int \frac{\mathrm{d}p}{2\pi} \exp\left\{\mathrm{i}\left[p(x_b - x_a) + \int_0^\lambda (p^2 - m^2)\mathrm{d}\tau\right]\right\} \exp\left\{\frac{\mathrm{i}pap}{2pk}\int_{kx_a}^{kx_b} F(\varphi)\mathrm{d}\varphi\right\}. \tag{14}$$

After transforming p into $-p$, we integrate over λ. The result is

$$G^c(x_b,x_a) = \frac{1}{(2\pi)^4}\int \frac{\mathrm{d}^4 p}{p^2 - m^2} \cdot \exp\left\{\mathrm{i}\left[p(x_b - x_a) + \frac{(pap)}{2pk}\int_{kx_a}^{kx_b} F(\varphi)\mathrm{d}\varphi\right]\right\}. \tag{15}$$

A simple integration over p^0 leads to the following form:

$$G^c(x_b,x_a) = \frac{-\mathrm{i}}{2}\sum_{\varepsilon=\pm} \theta[\varepsilon(t_b - t_a)] \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \cdot \exp\left\{-\mathrm{i}\varepsilon\omega(t_b - t_a) + \mathrm{i}\vec{p}(\vec{x_b} - \vec{x_a})\right\} \cdot \exp\left\{\frac{\mathrm{i}}{\varepsilon\omega k^0 - 2pk}(pap)\int_{kx_a}^{kx_b} F(\varphi)\mathrm{d}\varphi\right\}, \tag{16}$$

which allows us to extract the wave functions related to the Klein-Gordon particle in a weak gravitational field:

$$\Psi_p^\pm(x) = \left[\frac{1}{(2\pi)^{\frac{3}{2}}}\frac{1}{(2p^0)^{\frac{1}{2}}}\exp\left\{\mp\mathrm{i}\left(px + \frac{(pap)}{2pk}\int^{kx} F(\varphi)\mathrm{d}\varphi\right)\right\}\right]_{p^0=\left(\vec{p}^2+m^2\right)^{1/2}}. \tag{17}$$

3. Green Function for a Dirac Particle: Global and Local Approaches

Let us pass now to the Dirac particle: We consider the motion of a Dirac particle in a weak gravitational field using the path integral approach. The Green function S which we propose is the solution of the following equation:

$$\left[-\gamma\hat{p}_b + \frac{1}{2}F(kx)\gamma a\hat{p}_b - m\right]S(x_b,x_a) = \delta^4(x_b,x_a), \tag{18}$$

where γ^{μ} are the usual Dirac matrices obeying the relations

$$\gamma^{\mu}\gamma^{\nu}+\gamma^{\nu}\gamma^{\mu}=2g^{\mu\nu},\quad \mu,\nu=0,1,2,3.$$

We know that in the global approach of the path integral formulation of the Green function it is not necessary to introduce the matrix γ^5 [3] to the global approach contrary to the local approach.

Thus, in the global approach we can obtain the Green function according to [2, 3] by

$$S(x_b,x_a)=\left[-\gamma\hat{p}_b+\frac{1}{2}F(kx_b)\gamma a\hat{p}_b+m\right]G(x_b,x_a),\quad (19)$$

where

$$G(x_b,x_a)=\mathrm{i}\exp\left\{\mathrm{i}\gamma^{\mu}\frac{\delta_L}{\delta\theta^{\mu}}\right\}\int_0^{\infty}\mathrm{d}\lambda\int\mathrm{D}x\int\mathrm{D}p \int_E \mathrm{D}\Psi\exp\left\{\mathrm{i}\int_0^{\lambda}\mathrm{d}\tau\left[p\dot{x}-\mathrm{i}\Psi_{\mu}\dot{\Psi}^{\mu}+p^2-m^2 -papF-\frac{1}{2}(pa\Psi)\xi F'\right]+\Psi_{\mu}(\lambda)\Psi^{\mu}(0)\right\}_{|\theta=0}. \quad (20)$$

In the local approach we find

$$\tilde{S}(x_b,x_a)=\exp\left\{\mathrm{i}\gamma^{n}\frac{\delta_L}{\delta\theta^{n}}\right\}\int_0^{\infty}\mathrm{d}\lambda\int\mathrm{d}\chi\int\mathrm{D}x \cdot\int\mathrm{D}p\int_E\mathrm{D}\Psi\exp\left\{\mathrm{i}\int_0^{\lambda}\mathrm{d}\tau\left[p\dot{x}-\mathrm{i}\Psi_n\dot{\Psi}^n+p^2 -m^2-papF-\frac{1}{2}(pa\Psi)(k\Psi)F'+\left(-(p\Psi) +\frac{1}{2}F(ap\Psi)-m\Psi^5\right)\chi\right]+\Psi_n(\lambda)\Psi^n(0)\right\}\Bigg|_{\theta=0}. \quad (21)$$

We propose to evaluate the Green function with the method already used in [9] which mainly consists of introducing two identities: The first one depends on the variable which characterizes the gravitation, the second one describing the spin in a form similar to the first. In order to obtain the expressions for the Green functions let us return to the definition of $S(x_b,x_a)$ which is the matrix element

$$S(x_b,x_a)=\langle x_b\mid\hat{S}\mid x_a\rangle$$

of an operator S. Symbolically, we have

$$\left(-\gamma\hat{p}+\frac{1}{2}F(kx)\gamma a\hat{p}-m\right)\hat{S}=I, \quad (22)$$

or

$$\hat{S}=\frac{I}{-\gamma\hat{p}+\frac{1}{2}F(kx)\gamma a\hat{p}-m} =\frac{I}{-\tilde{\gamma}\hat{p}+\frac{1}{2}F\tilde{\gamma}a\hat{p}-m\gamma^5}\gamma^5=\tilde{S}\gamma^5. \quad (23)$$

Let us remind that the matrix γ^5 is defined by

$$\gamma^5=\gamma^0\gamma^1\gamma^2\gamma^3,\quad \left(\gamma^5\right)^2=-1.$$

Then, we have for the global case the following equalities:

$$\frac{I}{\left(-\gamma\hat{p}+\frac{1}{2}F(kx)\gamma a\hat{p}-m\right)} =\left[-\gamma\hat{p}+\frac{1}{2}F(kx)\left(\gamma a\hat{p}\right]+m\right)G, \quad (24)$$

with the relation between $\hat{S}$ and $\hat{G}$ as

$$S(x_b,x_a)=\left[-\gamma\hat{p}_b+\frac{1}{2}F(kx_b)\gamma a\hat{p}_b+m\right]G(x_b,x_a),\quad (25)$$

where

$$G=\mathrm{i}\left[\int_0^{\infty}\mathrm{d}\lambda\exp\left\{\mathrm{i}\lambda\left(-\gamma\hat{p} +\frac{1}{2}F(kx)\gamma a\hat{p}-m\right)\left(-\gamma\hat{p}+\frac{1}{2}F(kx)\gamma a\hat{p}+m\right)\right\}\right]. \quad (26)$$

For the local case the Green function to be evaluated is the following:

$$\tilde{S}=\int_0^{\infty}\mathrm{d}\lambda\int\mathrm{d}\chi\exp\left\{\mathrm{i}\left[\hat{p}^2-m^2-\hat{p}a\hat{p}F +\frac{\mathrm{i}}{2}(\hat{p}a\sigma k)F'+\chi\left(-(\tilde{\gamma}\hat{p})+\frac{1}{2}F(\tilde{\gamma}a\hat{p})-m\gamma^5\right)\right]\right\}, \quad (27)$$

where the parameters λ and χ are the bosonic and fermionic variables, respectively.

The kernels are in the global cases

$$G=\mathrm{i}\int_0^{\infty}\mathrm{d}\lambda\exp\left(-\mathrm{i}\lambda\hat{H}_g(\hat{p},\hat{x})\right), \quad (28)$$

with

$$H_g=p^2-m^2-\hat{p}a\hat{p}F+\frac{\mathrm{i}}{2}\hat{p}a\sigma kF', \quad (29)$$

and in the local case

$$\tilde{S}=\int_0^{\infty}\mathrm{d}\lambda\int\exp\left(-\mathrm{i}\lambda\hat{H}_l\left(\chi,\hat{p},\hat{x}\right)\right)\mathrm{d}\chi, \quad (30)$$

with

$$\hat{H}_l = p^2 - m^2 - \hat{p}a\hat{p}F + \frac{\mathrm{i}}{2}\hat{p}a\sigma kF' + \chi\left(-(\gamma\hat{p}) + \frac{1}{2}F(\gamma a\hat{p}) - m\gamma^5\right). \tag{31}$$

Let us then pass to the path integral formulation. We first have, for the global case,

$$G(x_b,x_a) = \mathrm{i}\int_0^\infty \mathrm{d}\lambda\,\langle x_b|\exp\{-\mathrm{i}\lambda H_g(\hat{p},\hat{x})\}|x_a\rangle, \tag{32}$$

and for the local case

$$\tilde{S}(x_b,x_a) = \int_0^\infty \mathrm{d}\lambda\int\langle x_b|\exp\left(-\mathrm{i}\lambda\hat{H}_l(\chi,\hat{p},\hat{x})\right)|x_a\rangle\mathrm{d}\chi. \tag{33}$$

Following the usual procedure (Trotter formula, insertion of projectors, etc....) we obtain the Green functions in the two cases,

i) global

$$G(x_b,x_a) = \mathrm{i}T\int_0^\infty \mathrm{d}\lambda\int\mathrm{D}x\int\mathrm{D}p\exp\left\{\mathrm{i}\int_0^\lambda\mathrm{d}\tau \cdot\left[p\dot{x} + p^2 - m^2 - papF + \frac{\mathrm{i}}{2}pa\sigma kF'\right]\right\}, \tag{34}$$

ii) local

$$\tilde{S}(x_b,x_a) = T\int_0^\infty \mathrm{d}\lambda\int\mathrm{d}\chi\int\mathrm{D}x\int\mathrm{D}p \cdot\exp\left\{\mathrm{i}\int_0^\lambda\mathrm{d}\tau\left[p\dot{x} + p^2 - m^2 - papF + \frac{\mathrm{i}}{2}(pa\sigma k)F' + \left(-(\gamma p) + \frac{1}{2}F(\gamma ap) - m\gamma^5\right)\chi\right]\right\}. \tag{35}$$

Here the time-ordered product T is introduced because of the presence of the matrices γ.

In order to eliminate the T-product, in a first step we shift the T-symbol by using the following identity:

$$T\exp\{F(\gamma^n(\tau))\} = \exp\left\{F\left(\frac{\delta_L}{\delta\rho_n}\right)\right\}T\exp\left\{\int_0^\lambda\rho_n\gamma^n\mathrm{d}\tau\right\}\Bigg|_{\rho=0}.$$

Then, in a second step, the T-product is replaced by a path integral

$$T\exp\left\{\int_0^\lambda\rho_n\gamma^n\mathrm{d}\tau\right\} = \exp\left\{\mathrm{i}\gamma^n\frac{\delta_L}{\delta\theta^n}\right\}\int\mathrm{D}\Psi\,|_{\Psi_0+\Psi_1=\theta} \cdot\exp\left\{\int_0^\lambda(\Psi_n\dot{\Psi}^n - 2\mathrm{i}\rho_n\Psi^n)\mathrm{d}\tau + \Psi_n(\lambda)\Psi^n(0)\right\}. \tag{36}$$

Let us note in passing that the terms $\sigma^{\mu\nu}$ will be replaced by

$$\sigma^{\mu\nu} \to \mathrm{i}\Psi^\mu\Psi^\nu, \tag{37}$$

and that the term

$$\frac{\mathrm{i}}{2}(pa\sigma k)F' \to -\frac{1}{2}(pa\Psi)(k\Psi)F'$$

is a product of factors. Thus, we have in the global case

$$G(x_b,x_a) = \mathrm{i}\exp\left\{\mathrm{i}\gamma^n\frac{\delta_L}{\delta\theta^n}\right\}\int_0^\infty\mathrm{d}\lambda\int\mathrm{D}x\int\mathrm{D}p \cdot\int_E\mathrm{D}\Psi\exp\left\{\mathrm{i}\int_0^\lambda\mathrm{d}\tau\left[p\dot{x} + p^2 - m^2 - (pap)F - \frac{1}{2}(pa\Psi)(k\Psi)F' - \mathrm{i}\Psi_n\dot{\Psi}^n\right] + \Psi_n(\lambda)\Psi^n(0)\,|_{\theta=0}\right\}, \tag{38}$$

and in the local case

$$\tilde{S}(x_b,x_a) = \exp\left\{\mathrm{i}\gamma^n\frac{\delta_L}{\delta\theta^n}\right\}\int_0^\infty\mathrm{d}\lambda\int\mathrm{d}\chi\int\mathrm{D}x\int\mathrm{D}p \cdot\int_E\mathrm{D}\Psi\exp\left\{\mathrm{i}\int_0^\lambda\mathrm{d}\tau\left[p\dot{x} + p^2 - m^2 - (pap)F - \frac{1}{2}(pa\Psi)(k\Psi)F' + \left(-(p\Psi) + \frac{1}{2}(\Psi ap)F - m\Psi^5\right)\chi - \mathrm{i}\Psi_n\dot{\Psi}^n\right] + \Psi_n(\lambda)\Psi^n(0)\,|_{\theta=0}\right\}. \tag{39}$$

Having formulated the Green function in the path integral approach, let us proceed now to their calculation. For that, let us use the properties of their dependence on the gravitation [8]. Therefore, we insert the identity

$$\int\mathrm{d}\varphi_a\int\mathrm{d}\varphi_b\delta(\varphi_a - kx_a)\int\mathrm{D}p_\varphi\int\mathrm{D}\varphi \cdot\exp\left\{\mathrm{i}\int_0^\lambda p_\varphi(\dot{\varphi} - k\dot{x})\mathrm{d}\tau\right\} = 1. \tag{40}$$

This leads for the global case to

$$G(x_b,x_a) = \mathrm{i}\exp\left\{\mathrm{i}\gamma^n\frac{\delta_L}{\delta\theta^n}\right\}\int_0^\infty\mathrm{d}\lambda\int\mathrm{D}x\int\mathrm{D}p \cdot\int\mathrm{d}\varphi_a\int\mathrm{d}\varphi_b\delta(\varphi_a - kx_a)\int\mathrm{D}p_\varphi\int\mathrm{D}\varphi\int_E\mathrm{D}\Psi \cdot\exp\left\{\mathrm{i}\int_0^\lambda\mathrm{d}\tau\left[\left(p^2 - m^2 - papF - \frac{1}{2}(pa\Psi)\xi F'\right) + (p - kp_\varphi)\dot{x} + p_\varphi\dot{\varphi} - \mathrm{i}\Psi_n\dot{\Psi}^n\right] + \Psi_n(\lambda)\Psi^n(0)\,|_{\theta=0}\right\}, \tag{41}$$

and for the local case to

$$\tilde{S}(x_b,x_a)=\exp\left\{\mathrm{i}\gamma^n\frac{\delta_L}{\delta\theta^n}\right\}\int_0^\infty\mathrm{d}\lambda\int\mathrm{d}\chi\int\mathrm{D}x \cdot\int\mathrm{D}p\int\mathrm{d}\varphi_a\int\mathrm{d}\varphi_b\delta(\varphi_a-kx_a)\int\mathrm{D}p_\varphi\int\mathrm{D}\varphi \cdot\int_E\mathrm{D}\Psi\exp\left\{\mathrm{i}\int_0^\lambda\mathrm{d}\tau\left[p^2-m^2-papF-\frac{1}{2}(pa\Psi)\xi F' +\left(-(p\Psi)+\frac{1}{2}F(ap\Psi)-m\Psi^5\right)\chi+(p-kp_\varphi)\dot{x} +p_\varphi\dot{\varphi}-\mathrm{i}\Psi_n\dot{\Psi}^n\right]+\Psi_n(\lambda)\Psi^n(0)\,|_{\theta=0}\right\}. \quad (42)$$

Let us make the following transformation:

$$p=P+kp_\varphi. \quad (43)$$

Then, we obtain

$$G(x_b,x_a)=\mathrm{i}\exp\left\{\mathrm{i}\gamma^n\frac{\delta_L}{\delta\theta^n}\right\}\int_0^\infty\mathrm{d}\lambda\int\mathrm{D}x\int\mathrm{D}p \cdot\int\mathrm{d}\varphi_a\int\mathrm{d}\varphi_b\delta(\varphi_a-kx_a)\int\mathrm{D}p_\varphi\int\mathrm{D}\varphi\int_E\mathrm{D}\Psi \cdot\exp\left\{\mathrm{i}\int_0^\lambda\mathrm{d}\tau\left[p\dot{x}+\left(p^2-m^2-papF -\frac{1}{2}(pa\Psi)(k\Psi)F'\right)+p_\varphi(\dot{\varphi}+2pk)-\mathrm{i}\Psi_n\dot{\Psi}^n\right] +\Psi_n(\lambda)\Psi^n(0)\,|_{\theta=0}\right\} \quad (44)$$

and

$$\tilde{S}(x_b,x_a)=\exp\left\{\mathrm{i}\gamma^n\frac{\delta_L}{\delta\theta^n}\right\}\int_0^\infty\mathrm{d}\lambda\int\mathrm{d}\chi\int\mathrm{D}x\int\mathrm{D}p \cdot\int\mathrm{d}\varphi_a\int\mathrm{d}\varphi_b\delta(\varphi_a-kx_a)\int\mathrm{D}p_\varphi\int\mathrm{D}\varphi\int_E\mathrm{D}\Psi \cdot\exp\left\{\mathrm{i}\int_0^\lambda\mathrm{d}\tau\left[p\dot{x}+\left(p^2-m^2-papF-\frac{1}{2}(pa\Psi)\xi F'\right) +\left(-p\Psi-kp_\varphi\Psi+\frac{1}{2}F(\Psi ap)-m\Psi^5\right)\chi +p_\varphi(\dot{\varphi}+2kp)-\mathrm{i}\Psi_n\dot{\Psi}^n\right]+\Psi_n(\lambda)\Psi^n(0)\,|_{\theta=0}\right\}. \quad (45)$$

Let us integrate the first term of the action by parts then integrate over the paths $x(\tau)$. This leads in the two cases to

i) global

$$G(x_b,x_a)=\mathrm{i}\exp\left\{\mathrm{i}\gamma^n\frac{\delta_L}{\delta\theta^n}\right\}\int_0^\infty\mathrm{d}\lambda\int\mathrm{D}p\int\mathrm{d}\varphi_a\int\mathrm{d}\varphi_b \cdot\delta(\varphi_a-kx_a)\int\mathrm{D}p_\varphi\int\mathrm{D}\varphi\int_E\mathrm{D}\Psi\exp\{\mathrm{i}(x_bp_b-x_ap_a)\} \cdot\delta(\dot{p})\exp\left\{\mathrm{i}\int_0^\lambda\mathrm{d}\tau\left[\left(p^2-m^2-(pap)F -\frac{1}{2}(pa\Psi)(k\Psi)F'\right)+p_\varphi(\dot{\varphi}+2pk)-\mathrm{i}\Psi_n\dot{\Psi}^n\right] +\Psi_n(\lambda)\Psi^n(0)\,|_{\theta=0}\right\}, \quad (46)$$

ii) local

$$\tilde{S}(x_b,x_a)=\exp\left\{\mathrm{i}\gamma^n\frac{\delta_L}{\delta\theta^n}\right\}\int_0^\infty\mathrm{d}\lambda\int\mathrm{d}\chi\int\mathrm{D}p \cdot\int\mathrm{d}\varphi_a\int\mathrm{d}\varphi_b\delta(\varphi_a-kx_a)\int\mathrm{D}p_\varphi\int\mathrm{D}\varphi\int_E\mathrm{D}\Psi \cdot\exp\{\mathrm{i}(x_bp_b-x_ap_a)\}\,\delta(\dot{p})\exp\left\{\mathrm{i}\int_0^\lambda\mathrm{d}\tau\left[\left(p^2-m^2 -papF-\frac{1}{2}(pa\Psi)(k\Psi)F'\right)-\left(p\Psi-\frac{1}{2}F(\Psi ap) +\xi p_\varphi+m\Psi^5\right)\chi+p_\varphi(\dot{\varphi}+2kp)-\mathrm{i}\Psi_n\dot{\Psi}^n\right] +\Psi_n(\lambda)\Psi^n(0)\,|_{\theta=0}\right\}. \quad (47)$$

The Dirac function $\delta(\dot{p})$ expresses that the momentum is constant during the motion, i. e.,

$$p_1=p_2=\ldots=p_n=p.$$

Let us now integrate on p, implying in the

i) global case

$$G(x_b,x_a)=\mathrm{i}\exp\left\{\mathrm{i}\gamma^n\frac{\delta_L}{\delta\theta^n}\right\}\int_0^\infty\mathrm{d}\lambda\int\frac{\mathrm{d}^4p}{(2\pi)^4} \cdot\exp\{\mathrm{i}p(x_b-x_a)+\mathrm{i}\lambda(p^2-m^2)\}\int\mathrm{d}\varphi_a \cdot\int\mathrm{d}\varphi_b\delta(\varphi_a-kx_a)\int\mathrm{D}p_\varphi\int\mathrm{D}\varphi\int_E\mathrm{D}\Psi \cdot\exp\left\{\mathrm{i}\int_0^\lambda\mathrm{d}\tau\left[\left(-papF-\frac{1}{2}(pa\Psi)(k\Psi)F'\right) +p_\varphi(\dot{\varphi}+2pk)-\mathrm{i}\Psi_n\dot{\Psi}^n\right]+\Psi_n(\lambda)\Psi^n(0)\,|_{\theta=0}\right\}, \quad (48)$$

ii) local case

$$\tilde{S}(x_b,x_a)=\exp\left\{\mathrm{i}\gamma^n\frac{\delta_L}{\delta\theta^n}\right\}\int_0^\infty\mathrm{d}\lambda\int\mathrm{d}\chi\int\frac{\mathrm{d}^4p}{(2\pi)^4} \cdot\exp\{\mathrm{i}p(x_b-x_a)+\mathrm{i}\lambda(p^2-m^2)\}\int\mathrm{d}\varphi_a \cdot\int\mathrm{d}\varphi_b\delta(\varphi_a-kx_a)\int\mathrm{D}p_\varphi\int\mathrm{D}\varphi\int_E\mathrm{D}\Psi \cdot\exp\left\{\mathrm{i}\int_0^\lambda\mathrm{d}\tau\left[\left(-papF-\frac{1}{2}(pa\Psi)(k\Psi)F'\right)\right.\right.$$

$$-\left(p\Psi-\frac{1}{2}F(\Psi ap)+\xi p_\varphi+m\Psi^5\right)\chi$$
$$+p_\varphi(\dot{\varphi}+2kp)-\mathrm{i}\Psi_n\dot{\Psi}^n\Big]+\Psi_n(\lambda)\Psi^n(0)\,|_{\theta=0}\Big\}. \quad (49)$$

Now we use the factorization property of the spin-field interaction [8] by introducing the identity

$$\int \mathrm{d}\xi_a\mathrm{d}\xi_b\delta(\xi_a-k\Psi_a)\int \mathrm{D}p_\xi\int \mathrm{D}\xi \cdot\exp\left\{\mathrm{i}\int_0^\lambda p_\xi(\dot{\xi}-k\dot{\Psi})\mathrm{d}\tau\right\}=1, \quad (50)$$

where ξ and p_ξ are odd Grassmann variables.

Then G and S become in the

i) global case

$$G(x_b,x_a)=\mathrm{i}\exp\left\{\mathrm{i}\gamma^n\frac{\delta_L}{\delta\theta^n}\right\}\int_0^\infty\mathrm{d}\lambda\int\frac{\mathrm{d}^4p}{(2\pi)^4}\int\mathrm{d}\varphi_a$$
$$\cdot\int\mathrm{d}\varphi_b\delta(\varphi_a-kx_a)\int\mathrm{d}\xi_a\mathrm{d}\xi_b\delta(\xi_a-k\Psi_a)\int\mathrm{D}p_\varphi$$
$$\cdot\int\mathrm{D}\varphi\int\mathrm{D}p_\xi\int\mathrm{D}\xi\int_E\mathrm{D}\Psi\exp\{\mathrm{i}p(x_b-x_a)$$
$$+\mathrm{i}\lambda\left(p^2-m^2\right))\}\exp\left\{\mathrm{i}\int_0^\lambda\mathrm{d}\tau\Big[-(pap)F(\varphi)\right.$$
$$-\frac{1}{2}(pa\Psi)\xi F'+p_\varphi(\dot{\varphi}+2kp)$$
$$\left.+p_\xi(\dot{\xi}-k\dot{\Psi})-\mathrm{i}\Psi_\mu\dot{\Psi}^\mu\Big]+\Psi_\mu(\lambda)\Psi^\mu(0)\right\}, \quad (51)$$

ii) local case

$$\tilde{S}(x_b,x_a)=\exp\left\{\mathrm{i}\gamma^n\frac{\delta_L}{\delta\theta^n}\right\}\int_0^\infty\mathrm{d}\lambda\int\mathrm{d}\chi\int\frac{\mathrm{d}^4p}{(2\pi)^4}$$
$$\cdot\exp\{\mathrm{i}p(x_b-x_a)+\mathrm{i}\lambda\left(p^2-m^2\right)\}\int\mathrm{d}\varphi_a$$
$$\cdot\int\mathrm{d}\varphi_b\delta(\varphi_a-kx_a)\int\mathrm{d}\xi_a\mathrm{d}\xi_b\delta(\xi_a-k\Psi_a)\int\mathrm{D}p_\varphi$$
$$\cdot\int\mathrm{D}\varphi\int\mathrm{D}p_\xi\int\mathrm{D}\xi\int_E\mathrm{D}\Psi\exp\left\{\mathrm{i}\int_0^\lambda\mathrm{d}\tau\Big[\Big(-papF\right.$$
$$-\frac{1}{2}(pa\Psi)\xi F'\Big)-\left(p\Psi-\frac{1}{2}F(\Psi ap)+\xi p_\varphi+m\Psi^5\right)\chi$$
$$+p_\xi(\dot{\xi}-k\dot{\Psi})+p_\varphi(\dot{\varphi}+2kp)-\mathrm{i}\Psi_n\dot{\Psi}^n\Big]$$
$$\left.+\Psi_n(\lambda)\Psi^n(0)\,|_{\theta=0}\right\}. \quad (52)$$

At this state, it is preferable to use the velocities ω by making change of variables $\Psi^n\to\omega^n$, defined by

$$\Psi^n(\tau)=\frac{1}{2}\int_0^\lambda\varepsilon(\tau-\tau')\omega^n(\tau')\mathrm{d}\tau'+\frac{\theta^n}{2}. \quad (53)$$

We also adopt the condensed notation of [2] and obtain

$$G(x_b,x_a)=\mathrm{i}\exp\left\{\mathrm{i}\gamma^n\frac{\delta_L}{\delta\theta^n}\right\}\int_0^\infty\mathrm{d}\lambda\int\frac{\mathrm{d}^4p}{(2\pi)^4}$$
$$\cdot\exp\{\mathrm{i}p(x_b-x_a)+\mathrm{i}\lambda(p^2-m^2)\}\int\mathrm{d}\varphi_a$$
$$\cdot\int\mathrm{d}\varphi_b\delta(\varphi_a-kx_a)\int\mathrm{d}\xi_a\int\mathrm{d}\xi_b\int\mathrm{D}\varphi\int\mathrm{D}p_\varphi$$
$$\cdot\int\mathrm{D}\xi\int\mathrm{D}p_\xi\int_E D\omega\delta(\xi_a+\frac{k}{2}(\omega-\theta) \quad (54)$$
$$\cdot\exp\left\{\mathrm{i}\int_0^\lambda\mathrm{d}\tau\Big[\lambda\Big(-(pap)F(\varphi)+\frac{\mathrm{i}}{4}(pa\theta)\xi F'\right.$$
$$+\frac{\mathrm{i}}{4}pa(\varepsilon*\omega)\xi F'\Big)+p_\xi(\dot{\xi}-k\omega)$$
$$\left.+p_\varphi(\dot{\varphi}+2kp)\Big]-\tfrac{1}{2}\omega*\varepsilon*\omega\right\}\Big|_{\theta=0}$$

and

$$\tilde{S}(x_b,x_a)=\exp\left\{\mathrm{i}\gamma^n\frac{\delta_L}{\delta\theta^n}\right\}\int_0^\infty\mathrm{d}\lambda\int\mathrm{d}\chi\int\frac{\mathrm{d}^4p}{(2\pi)^4}$$
$$\cdot\exp\{\mathrm{i}p(x_b-x_a)+\mathrm{i}\lambda(p^2-m^2)\}\int\mathrm{d}\varphi_a$$
$$\cdot\int\mathrm{d}\varphi_b\delta(\varphi_a-kx_a)\int\mathrm{d}\xi_a\mathrm{d}\xi_b\int\mathrm{D}p_\varphi\int\mathrm{D}\varphi\int\mathrm{D}p_\xi$$
$$\cdot\int\mathrm{D}\xi\int_E D\omega\delta(\xi_a+\frac{k}{2}(\omega-\theta)$$
$$\cdot\exp\left\{\mathrm{i}\int_0^\lambda\mathrm{d}\tau\Big[\lambda\Big(-papF(\varphi)+\frac{\mathrm{i}}{4}(pa(\varepsilon*\omega\right. \quad (55)$$
$$+\theta))\xi F'\Big)-\frac{1}{2}\Big((\varepsilon*\omega+\theta)p-\frac{1}{2}F(\varepsilon*\omega+\theta)ap$$
$$+2\xi p_\varphi+m(\varepsilon*\omega^5+\theta^5)\Big)\chi+p_\xi(\dot{\xi}-k\omega)$$
$$\left.+p_\varphi(\dot{\varphi}+2kp)\Big]-\frac{1}{2}\omega*\varepsilon*\omega\right\}.$$

Let us now introduce the transformation

$$\bar{\omega}^\mu(\tau)\to\omega^\mu(\tau)+\mathrm{i}k^\mu(\varepsilon^{-1}*p_\xi),$$

leading to

$$k\bar{\omega}(\tau)=k\omega(\tau).$$

Since

$$\bar{\omega} * \varepsilon * \bar{\omega} = \omega * \varepsilon * \omega + 2\mathrm{i}k\omega p_\xi,$$
$$\varepsilon * \bar{\omega} = \varepsilon * \omega + \mathrm{i}kp_\xi,$$

we obtain for the two cases

i) global

$$G(x_b,x_a) = \mathrm{i}\exp\left\{\mathrm{i}\gamma^n \frac{\delta_L}{\delta\theta^n}\right\} \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \int_0^\infty \mathrm{d}\lambda \int \mathrm{d}\varphi_a \cdot \int \mathrm{d}\varphi_b \delta(\varphi_a - kx_a) \int \mathrm{D}\varphi \int \mathrm{D}p_\varphi \int \mathrm{d}\xi_a \int \mathrm{d}\xi_b \int \mathrm{D}\xi \cdot \int \mathrm{D}p_\xi \int_E D\omega \delta\left(\xi_a + \frac{k}{2}(\omega - \theta)\right) \exp\{\mathrm{i}p(x_b - x_a) + \mathrm{i}\lambda(p^2 - m^2)\} \exp\left\{\mathrm{i}\int_0^\lambda \mathrm{d}\tau\left[\left(-(pap)F(\varphi) + \frac{\mathrm{i}}{4}(pa(\varepsilon * \omega + \theta))\xi F'\right) + p_\xi(\dot{\xi} - k\omega) + p_\varphi(\dot{\varphi} + 2kp)\right] - \frac{1}{2}(\omega * \varepsilon * \omega - 2\mathrm{i}k\omega p_\xi)\right\}, \tag{56}$$

ii) local

$$\tilde{S}(x_b,x_a) = \exp\left\{\mathrm{i}\gamma^n \frac{\delta_L}{\delta\theta^n}\right\} \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \exp\{\mathrm{i}p(x_b - x_a) + \mathrm{i}\lambda(p^2 - m^2)\} \int_0^\infty \mathrm{d}\lambda \int \mathrm{d}\chi \int \mathrm{d}\varphi_a \int \mathrm{d}\varphi_b \delta(\varphi_a - kx_a) \cdot \int \mathrm{D}\varphi \int \mathrm{D}p_\varphi \int \mathrm{d}\xi_a \int \mathrm{d}\xi_b \int \mathrm{D}\xi \int \mathrm{D}p_\xi \int_E D\omega \delta(\xi_a + \frac{k}{2}(\omega - \theta)) \exp\left\{\mathrm{i}\int_0^\lambda \mathrm{d}\tau(p_\xi(\dot{\xi} - k\omega) + p_\varphi(\dot{\varphi} + 2kp))\right\} \exp\left\{\mathrm{i}\int_0^\lambda \mathrm{d}\tau\left[\left(-papF(\varphi) + \frac{\mathrm{i}}{4}(pa(\varepsilon * \omega - \mathrm{i}kp_\xi + \theta))\xi F'\right) - \frac{1}{2}\left((\varepsilon * \omega - \mathrm{i}kp_\xi + \theta)p_\mu - \frac{1}{2}F(\varepsilon * \omega - \mathrm{i}kp_\xi + \theta)ap + 2\xi p_\varphi + m(\varepsilon * \omega^5 + \theta^5)\right)\chi\right] - \frac{1}{2}(\omega * \varepsilon * \omega - 2\mathrm{i}k\omega p_\xi)\right\}. \tag{57}$$

One can replace the Dirac function by its integral representation

$$\delta\left(\xi_a + \frac{k}{2}(\omega - \theta)\right) = \int \mathrm{d}p_{\zeta_a} \exp\left\{\mathrm{i}p_{\zeta_a}\left(\xi_a + \frac{k}{2}(\omega - \theta)\right)\right\}, \tag{58}$$

where p_{ζ_a} is a Grassmann variable.

We have for the two cases

i) global

$$G(x_b,x_a) = \mathrm{i}\exp\left\{\mathrm{i}\gamma^n \frac{\delta_L}{\delta\theta^n}\right\} \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \int_0^\infty \mathrm{d}\lambda \int \mathrm{d}\varphi_a \cdot \int \mathrm{d}\varphi_b \delta(\varphi_a - kx_a) \int \mathrm{D}\varphi \int \mathrm{D}p_\varphi \int \mathrm{d}\xi_a \int \mathrm{d}\xi_b \int \mathrm{D}\xi \cdot \int \mathrm{D}p_\xi \int_E D\omega \exp\{\mathrm{i}p(x_b - x_a) + \mathrm{i}\lambda\left(p^2 - m^2\right)\} \cdot \exp\left\{\mathrm{i}\int_0^\lambda \mathrm{d}\tau\left[\left(-papF(\varphi) + \frac{\mathrm{i}}{4}\left[pa(\varepsilon * \omega - \mathrm{i}kp_\xi + \theta)\right]\xi F'\right) + p_\varphi(\dot{\varphi} + 2kp) + p_\xi(\dot{\xi} - k\omega)\right] - \frac{1}{2}(\omega * \varepsilon * \omega - 2\mathrm{i}k\omega p_\xi) + p_{\zeta_a}\left(\xi_a + \frac{k}{2}(\omega - \theta)\right)\right\}\Bigg|_{\theta=0}, \tag{59}$$

ii) local

$$\tilde{S}(x_b,x_a) = \exp\left\{\mathrm{i}\gamma^n \frac{\delta_L}{\delta\theta^n}\right\} \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \exp\{\mathrm{i}p(x_b - x_a) + \mathrm{i}\lambda(p^2 - m^2)\} \int_0^\infty \mathrm{d}\lambda \int \mathrm{d}\chi \int \mathrm{d}\varphi_a \int \mathrm{d}\varphi_b \delta(\varphi_a - kx_a) \cdot \int \mathrm{D}\varphi \int \mathrm{D}p_\varphi \int \mathrm{d}\xi_a \int \mathrm{d}\xi_b \int \mathrm{D}\xi \int \mathrm{D}p_\xi \int_E D\omega \cdot \exp\left\{\mathrm{i}\int_0^\lambda \mathrm{d}\tau\left[p_\xi(\dot{\xi} - k\omega) + p_\varphi(\dot{\varphi} + 2kp)\right]\right\} \cdot \exp\left\{\mathrm{i}\int_0^\lambda \mathrm{d}\tau\left[\left(-papF(\varphi) + \frac{\mathrm{i}}{4}(pa(\varepsilon * \omega - \mathrm{i}kp_\xi + \theta))\xi F'\right) - \frac{1}{2}\left((\varepsilon * \omega - \mathrm{i}kp_\xi + \theta)p - \frac{1}{2}F(\varepsilon * \omega - \mathrm{i}kp_\xi + \theta)ap + 2\xi p_\varphi + m(\varepsilon * \omega^5 + \theta^5)\right)\chi\right] - \frac{1}{2}(\omega * \varepsilon * \omega - 2\mathrm{i}k\omega p_\xi) + p_{\zeta_a}\left(\xi_a + \frac{k}{2}(\omega - \theta)\right)\right\}\Bigg|_{\theta=0}. \tag{60}$$

The integration on the velocity ω^5 is simple:

$$\int D\omega^5 \exp\left\{-\frac{1}{2}\left(\omega_5 * \varepsilon * \omega^5 + \mathrm{i}m\varepsilon * \omega^5\chi\right)\right\} = 1. \tag{61}$$

The integrals over ω^μ $(\mu = 0,1,2,3)$ are

$$\int D\omega^\mu \exp\left\{\int_0^\lambda \left(-\frac{1}{2}\omega_\mu \varepsilon \omega^\mu + J_\mu \omega^\mu\right) \mathrm{d}\tau\right\},$$

where the sources $J_\mu(\tau)$ are in the cases

i) global

$$J_\mu(\tau)=-\frac{1}{4}\int_0^\lambda \mathrm{d}s\,(pa)_\mu\,\xi\,(s)\,F'\,(s)\,\varepsilon(\tau-s)+\frac{\mathrm{i}}{2}k_\mu p_{\xi_a}.$$

ii) local

$$J_\mu(\tau)=-\frac{1}{4}\int_0^\lambda \mathrm{d}s\,(pa)_\mu\,\xi\,(s)\,F'\,(s)\,\varepsilon(\tau-s)$$
$$-\frac{1}{2}\chi\int_0^\lambda (p+(ap))_\mu\,F\,(s)\,\varepsilon(\tau-s)\mathrm{d}s+\frac{\mathrm{i}}{2}p_{\xi_a}k_\mu,$$

By using the properties $ka=0$ and $k^2=0$, which allow to simplify the Green functions, G and $\tilde{S}$ have the following forms:

i) global case

$$G(x_b,x_a)=\mathrm{i}\exp\left\{\mathrm{i}\gamma^n\frac{\delta_L}{\delta\theta^n}\right\}\int\frac{\mathrm{d}^4p}{(2\pi)^4}\int_0^\infty \mathrm{d}\lambda\int \mathrm{d}\varphi_a\int \mathrm{d}\varphi_b\,\delta(\varphi_a-kx_a)$$
$$\cdot\int \mathrm{D}\varphi\int \mathrm{D}p_\varphi\int \mathrm{d}\xi_a\int \mathrm{d}\xi_b\int \mathrm{D}\xi\int \mathrm{D}p_\xi\exp\left\{\mathrm{i}p(x_b-x_a)+\mathrm{i}\lambda(p^2-m^2)\right\}$$
$$\cdot\exp\left\{\begin{array}{l}\mathrm{i}\int_0^\lambda \mathrm{d}\tau\left[-(pap)F(\varphi)+p_\varphi(\dot{\varphi}+2kp)+p_\xi\dot{\xi}+p_{\zeta_a}\left(\xi_a-\frac{k}{2}\theta\right)+\frac{\mathrm{i}}{4}(pa\theta)\xi F'\right]\\ +\frac{1}{32}\left(pa^2p\right)\int_0^\lambda\int_0^\lambda \mathrm{d}s\mathrm{d}s'\varepsilon(s-s')\xi(s)\xi(s')F'(s)F'(s')\end{array}\right\}\Bigg|_{\theta=0}, \tag{62}$$

ii) local case

$$\tilde{S}(x_b,x_a)=\mathrm{i}\exp\left\{\mathrm{i}\gamma^n\frac{\delta_L}{\delta\theta^n}\right\}\int\frac{\mathrm{d}^4p}{(2\pi)^4}\int_0^\infty \mathrm{d}\lambda\int \mathrm{d}\varphi_a\int \mathrm{d}\varphi_b\,\delta(\varphi_a-kx_a)$$
$$\cdot\int \mathrm{D}\varphi\int \mathrm{D}p_\varphi\int \mathrm{d}\xi_a\int \mathrm{d}\xi_b\int \mathrm{D}\xi\int \mathrm{D}p_\xi\exp\left\{\mathrm{i}p(x_b-x_a)+\mathrm{i}\lambda(p^2-m^2)\right\}$$
$$\cdot\exp\left\{\begin{array}{l}\mathrm{i}\int_0^\lambda \mathrm{d}\tau\left[\begin{array}{c}-papF(\varphi)+p_\varphi(\dot{\varphi}+2kp)+p_\xi\left(\dot{\xi}+\frac{1}{2}pk\chi\right)+p_{\zeta_a}\left(\xi_a-\frac{k}{2}\theta\right)\\ +\frac{\mathrm{i}}{4}(pa\theta)\xi F'-\frac{1}{2}\left(\theta p+2\xi p_\varphi+m\theta-\frac{1}{2}F(\theta ap)\right)\chi\end{array}\right]\\ +\frac{1}{32}(pa^2p)\int_0^\lambda\int_0^\lambda \mathrm{d}s\mathrm{d}s'\varepsilon(s-s')\xi(s)\xi(s')F'(s)F'(s')\\ +\frac{1}{8}((pap)+pa^2p)\int_0^\lambda\int_0^\lambda \mathrm{d}s\mathrm{d}s'\varepsilon(s-s')\\ F'(s)F(s')\xi(s)\chi+\frac{\mathrm{i}}{4}(pk)p_{\xi_a}\chi\int_0^\lambda \mathrm{d}s'F(s')\end{array}\right\}\Bigg|_{\theta=0}. \tag{63}$$

Let us now integrate on the p_ξ. The two expressions (62) and (63) are then reduced to

i) global case

$$G(x_b,x_a)=\mathrm{i}\exp\left\{\mathrm{i}\gamma^n\frac{\delta_L}{\delta\theta^n}\right\}\int\frac{\mathrm{d}^4p}{(2\pi)^4}\int_0^\infty \mathrm{d}\lambda\int \mathrm{d}\varphi_a$$
$$\cdot\int \mathrm{d}\varphi_b\delta(\varphi_a-kx_a)\int \mathrm{D}\varphi\int \mathrm{D}p_\varphi\int \mathrm{d}\xi_a\exp\{\mathrm{i}p(x_b-x_a)$$
$$+\mathrm{i}\lambda(p^2-m^2)\}\exp\left\{\mathrm{i}\int_0^\lambda \mathrm{d}\tau\left[-(pap)F(\varphi)\right.\right.$$
$$+p_\varphi(\dot{\varphi}+2kp)+p_{\zeta_a}\left(\xi_a-\frac{k}{2}\theta\right)$$
$$\left.\left.+\frac{\mathrm{i}}{4}(pa\theta)\xi_aF'\right]\Bigg|_{\theta=0}\right\}, \tag{64}$$

ii) local case

$$\tilde{S}(x_b,x_a) = \exp\left\{\mathrm{i}\gamma^n \frac{\delta_L}{\delta\theta^n}\right\}\int \frac{\mathrm{d}^4 p}{(2\pi)^4}\int_0^\infty \mathrm{d}\lambda \int \mathrm{d}\chi \cdot \exp\left\{\mathrm{i}p(x_b-x_a)+\mathrm{i}\lambda(p^2-m^2)\right\}\int \mathrm{d}\varphi_a \int \mathrm{d}\varphi_b \cdot \delta(\varphi_a - kx_a)\int \mathrm{D}\varphi \int \mathrm{D}p_\varphi \int \mathrm{d}\xi_a \int \mathrm{d}p_{\zeta_a}\int \mathrm{d}\zeta_b \cdot \delta\left(\zeta_b - \xi_a + \frac{1}{2}pk\chi\right)\exp\left\{\mathrm{i}\int_0^\lambda \mathrm{d}\tau\left[-(pap)F(\varphi) + p_\varphi(\dot\varphi + 2kp - \mathrm{i}\zeta_a\chi) + p_{\zeta_a}\left(\xi_a - \frac{k}{2}\theta - \frac{1}{4}kp\chi\right) + \frac{\mathrm{i}}{4}(pa\theta)F'\xi_a - \frac{1}{2}\left(\theta p + m\theta^5 - \frac{1}{2}(\theta ap)F\right)\chi\right] + \frac{1}{8}\int_0^\lambda\int_0^\lambda \mathrm{d}s\mathrm{d}s'\varepsilon(s-s')(pap+pa^2p)F'(s)F(s') \cdot \left(\zeta_a - \frac{1}{2}pk\chi s\right)\chi + \frac{1}{32}\int_0^\lambda\int_0^\lambda \mathrm{d}s\mathrm{d}s'\varepsilon(s-s')pa^2p \cdot \left(\zeta_a - \frac{pk}{2}\chi s\right)\left(\zeta_a - \frac{pk}{2}\chi s'\right)F'(s)F'(s')\right\}\Bigg|_{\theta=0}. \tag{65}$$

I. e., we have, respectively, in the

i) global case

$$\xi = \xi_a = \xi_b = c^{te}, \tag{66}$$

ii) and local case

$$\xi(\tau) = \xi_a - \frac{pk}{2}\chi\tau. \tag{67}$$

Integrating over the p_{ζ_a}, we obtain

i) global case

$$\xi_a = \xi_b = \frac{k}{2}\theta, \tag{68}$$

ii) local case

$$\xi_a = \frac{k}{2}\theta + \frac{1}{4}kp\chi. \tag{69}$$

We notice that we have in both cases

$$\xi_a + \xi_b = k\theta,$$

Thus, the Green functions reduce to:

i) global case

$$G(x_b,x_a) = \mathrm{i}\exp\left\{\mathrm{i}\gamma^n\frac{\delta_L}{\delta\theta^n}\right\}\int\frac{\mathrm{d}^4p}{(2\pi)^4}\int_0^\infty \mathrm{d}\lambda\int\mathrm{d}\varphi_a \cdot \int \mathrm{d}\varphi_b\delta(\varphi_a - kx_a)\int\mathrm{D}\varphi\int\mathrm{D}p_\varphi\exp\{\mathrm{i}p(x_b-x_a) + \mathrm{i}\lambda(p^2-m^2)\}\exp\left\{\mathrm{i}\int_0^\lambda\mathrm{d}\tau\left[-(pap)F(\varphi) + p_\varphi(\dot\varphi+2kp) + \frac{\mathrm{i}}{2}(pa\theta)(\theta k)F'\right]\Bigg|_{\theta=0}\right\}, \tag{70}$$

ii) local case

$$\tilde{S}(x_b,x_a) = \exp\left\{\mathrm{i}\gamma^n\frac{\delta_L}{\delta\theta^n}\right\}\int\frac{\mathrm{d}^4p}{(2\pi)^4}\int_0^\infty\mathrm{d}\lambda\int\mathrm{d}\chi \cdot \exp\{\mathrm{i}p(x_b-x_a)+\mathrm{i}\lambda(p^2-m^2)\}\int\mathrm{d}\varphi_a\int\mathrm{d}\varphi_b \cdot\delta(\varphi_a-kx_a)\int\mathrm{D}\varphi\int\mathrm{D}p_\varphi\exp\left\{\mathrm{i}\int_0^\lambda\mathrm{d}\tau\left[-papF(\varphi) + p_\varphi\left(\dot\varphi+2kp-\mathrm{i}\frac{k}{2}\theta\chi\right) - \frac{1}{2}(\theta p+m\theta^5)\chi + \frac{1}{4}(\theta ap)F\chi + \frac{\mathrm{i}}{4}(pa\theta)\left(k\theta+\frac{1}{2}kp\chi\right)F'\right] + \frac{1}{16}\int_0^\lambda\int_0^\lambda\mathrm{d}s\mathrm{d}s'\varepsilon(s-s')[(pap)+(pa^2p)]F'(s)F(s') \cdot(k\theta)\chi + \frac{1}{32}\int_0^\lambda\int_0^\lambda\mathrm{d}s\mathrm{d}s'\varepsilon(s-s')(pa^2p) + \frac{1}{4}(k\theta-pk\chi s)(k\theta-pk\chi s')F'(s)F'(s')\right\}\Bigg|_{\theta=0}. \tag{71}$$

An integration on p_φ allows to obtain the relation existing between τ and φ:

i) global case

$$\mathrm{d}\tau = -\frac{\mathrm{d}\varphi}{2kp}, \tag{72}$$

ii) local case

$$\frac{\mathrm{d}\tau}{\mathrm{d}\varphi} = -\frac{1}{2kp}\left(1 - \frac{\mathrm{i}k\theta}{2kp}\chi\right). \tag{73}$$

Then (70) and (71) reduce to

$$G(x_b,x_a) = \mathrm{i}\exp\left\{\mathrm{i}\gamma^n\frac{\delta_L}{\delta\theta^n}\right\}\int\frac{\mathrm{d}^4p}{(2\pi)^4}\int_0^\infty\mathrm{d}\lambda\int\mathrm{d}\varphi_a \cdot\int\mathrm{d}\varphi_b\delta(\varphi_a-kx_a-(2kp)\lambda)\exp\{\mathrm{i}p(x_b-x_a) + \mathrm{i}\lambda(p^2-m^2)\}\int\mathrm{D}\varphi\exp\left\{-\mathrm{i}\int_{\varphi_a}^{\varphi_b}\left[(pap)F(\varphi) + \frac{1}{2}(pa\theta)(\theta k)F'\right]\frac{\mathrm{d}\varphi}{2kp}\right\}\Bigg|_{\theta=0}, \tag{74}$$

and

$$\tilde{S}(x_b,x_a) = \exp\left\{\mathrm{i}\gamma^n\frac{\delta_L}{\delta\theta^n}\right\}\int\frac{\mathrm{d}^4p}{(2\pi)^4}\int_0^\infty\mathrm{d}\lambda\int\mathrm{d}\chi \cdot\exp\{\mathrm{i}p(x_b-x_a)+\mathrm{i}\lambda(p^2-m^2)\} \cdot\exp\left\{-\mathrm{i}\left[\frac{(pap)}{2kp}\int_{\varphi_a}^{\varphi_b}F(\varphi)\mathrm{d}\varphi\right.\right.$$

$$+\frac{(pa\theta)(\theta k)}{4kp}\int_{kx_a}^{kx_b}\frac{\mathrm{d}F}{\mathrm{d}\varphi}\mathrm{d}\varphi+\frac{1}{4}(\theta ap)\chi\int_{\varphi_a}^{\varphi_b}F\mathrm{d}\varphi\Bigg] -\frac{1}{2}(\theta p+m\theta^5)\chi+\frac{1}{8}\int_0^\lambda\int_0^\lambda \mathrm{d}s\mathrm{d}s'\varepsilon(s-s')(pap) \cdot F'(s)F(s')\frac{k}{2}\theta\chi\Bigg\}\Bigg|_{\theta=0}. \quad (75)$$

Now, let us integrate on the variable χ [11] for only the local case. Then

$$\tilde{S}(x_b,x_a)=\exp\left\{\mathrm{i}\gamma^n\frac{\delta_L}{\delta\theta^n}\right\}\left(\frac{\mathrm{i}}{4}(\theta ap)\int_{\varphi_a}^{\varphi_b}F\mathrm{d}\varphi +\frac{1}{2}(\theta p+m\theta^5)+\frac{1}{4}(pap)(F'*\varepsilon*F)\frac{k}{2}\theta\right) \cdot\int\frac{\mathrm{d}^4p}{(2\pi)^4}\int_0^\infty \mathrm{d}\lambda\exp\left\{\mathrm{i}p(x_b-x_a)+\mathrm{i}\lambda(p^2-m^2)\right\} \cdot\exp\left\{-\mathrm{i}\frac{1}{2kp}\left[(pap)\int_{\varphi_a}^{\varphi_b}F(\varphi)\mathrm{d}\varphi +\frac{1}{2}(pa\theta)(\theta k)\int_{kx_a}^{kx_b}\frac{\mathrm{d}F}{\mathrm{d}\varphi}\mathrm{d}\varphi\right]\Bigg|_{\theta=0}\right\}. \quad (76)$$

Let us reintroduce now the matrices γ^μ by performing the derivation with the help of the following formula

$$\exp\left(\mathrm{i}\gamma^n\frac{\delta_L}{\delta\theta^n}\right)f(\theta)\,|_{\theta=0}=f\left(\frac{\partial_L}{\partial\theta}\right)\exp(\mathrm{i}\gamma^n\theta_n)\,|_{\theta=0},$$

where

$$f(\theta)=\exp\left[\mathrm{i}\frac{(pa\theta)(\theta k)}{4k(x_b-x_a)}\int_{kx_a}^{kx_b}\frac{\mathrm{d}F}{\mathrm{d}\varphi}\mathrm{d}\varphi\right] =1+\frac{\mathrm{i}(pa\theta)(\theta k)}{4k(x_b-x_a)}\int_{kx_a}^{kx_b}\frac{\mathrm{d}F}{\mathrm{d}\varphi}\mathrm{d}\varphi,$$

in the global case and

$$f(\theta)=\left(\frac{\mathrm{i}}{4}(\theta ap)\int_{\varphi_a}^{\varphi_b}F\mathrm{d}\varphi+\frac{1}{2}(\theta p+m\theta^5) +\frac{1}{4}(pap)(F'*\varepsilon*F)\frac{k}{2}\theta\right) \cdot\exp\left\{-\mathrm{i}\frac{(pa\theta)(\theta k)}{4k(x_b-x_a)}\int_{kx_a}^{kx_b}\frac{\mathrm{d}F}{\mathrm{d}\varphi}\mathrm{d}\varphi\right\} \quad (77)$$

in the local case.

With

$$\exp(\mathrm{i}\gamma^\mu\theta_\mu)=1+\mathrm{i}\gamma_\mu\theta^\mu-\frac{1}{2}\theta_\mu\theta_\nu\gamma^\mu\gamma^\nu +\frac{\mathrm{i}}{6}\theta_\mu\theta_\nu\theta_\sigma\gamma^\mu\gamma^\nu\gamma^\sigma+\theta_0\theta_1\theta_2\theta_3\gamma^5$$

and the derivative

$$\frac{\partial_L^2}{\partial\theta^\alpha\partial\theta^\beta}\theta_\mu\theta_\nu\gamma^\mu\gamma^\nu=-2\mathrm{i}\sigma_{\alpha\beta},$$

it is

$$\exp\left\{\mathrm{i}\gamma^n\frac{\delta_L}{\delta\theta^n}\right\}\left(\frac{\mathrm{i}(p\theta)(\theta ak)}{4k(x_b-x_a)}\int_{kx_a}^{kx_b}\frac{\mathrm{d}F}{\mathrm{d}\varphi}\mathrm{d}\varphi\right)\Bigg|_{\theta=0} =\frac{(p\sigma ak)}{4k(x_b-x_a)}\int_{kx_a}^{kx_b}\frac{\mathrm{d}F}{\mathrm{d}\varphi}\mathrm{d}\varphi. \quad (78)$$

The kernel are in the

i) global case

$$G(x_b,x_a)=\mathrm{i}\int\frac{\mathrm{d}^4p}{(2\pi)^4}\exp\{\mathrm{i}p(x_b-x_a)\}\int_0^\infty\mathrm{d}\lambda \cdot\exp\{\mathrm{i}\lambda(p^2-m^2)\}\exp\left\{-\mathrm{i}\left[\frac{(pap)}{2kp}\int_{kx_a}^{kx_b}F(\varphi)\mathrm{d}\varphi -\frac{(pa\sigma k)}{4kp}(F(\varphi_b)-F(\varphi_a))\right]\right\}, \quad (79)$$

ii) local case

$$\tilde{S}(x_b,x_a)=\mathrm{i}\int\frac{\mathrm{d}^4p}{(2\pi)^4}\int_0^\infty\mathrm{d}\lambda\exp\{\mathrm{i}p(x_b-x_a) +\mathrm{i}\lambda(p^2-m^2)\}(-\gamma^\mu p_\mu+m\gamma^5) \cdot\exp\left\{-\mathrm{i}\left[\frac{(pap)}{2kp}\int_{\varphi_a}^{\varphi_b}F(\varphi)\mathrm{d}\varphi -\frac{(pa\sigma k)}{4kp}(F(\varphi_b)-F(\varphi_a))\right]\right\}. \quad (80)$$

Using the action of the operator

$$\left(-\gamma\hat{p}_b+\frac{1}{2}F(kx_b)\gamma a\hat{p}_b+m\right)$$

on the Green function, obtained in the global case, we find for the same Green function (without γ^5) on the Dirac equation:

$$S(x_b,x_a)=\int\frac{\mathrm{d}^4p}{(2\pi)^4}\int_0^\infty\mathrm{d}\lambda\exp\{\mathrm{i}\lambda(p^2-m^2)\} \cdot(-\gamma p+m)\exp\left\{\mathrm{i}\left[p(x_b-x_a)-\frac{(pap)}{2kp} \cdot\int_{kx_a}^{kx_b}F(\varphi)\mathrm{d}\varphi-\frac{(pa\sigma k)}{4kp}(F(kx_b)-F(kx_a))\right]\right\}. \quad (81)$$

We can integrate again on λ and obtain

$$S(x_b,x_a)=\int\frac{\mathrm{d}^4p}{(2\pi)^4}\frac{(-\gamma p+m)}{p^2-m^2+\mathrm{i}\varepsilon}\exp\left\{\mathrm{i}\left[p(x_b-x_a) -\frac{(pap)}{2kp}\int_{kx_a}^{kx_b}F(\varphi)\mathrm{d}\varphi-\frac{(pa\sigma k)}{4kp}(F(kx_b)-F(kx_a))\right]\right\}. \quad (82)$$

This can be made symmetrical with respect to the positions x_a and x_b thanks to the properties

$$ka = 0, \quad k^2 = 0.$$

Finally, the Green function reads

$$S(x_b,x_a) = \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \frac{1}{p^2 - m^2 + \mathrm{i}\varepsilon} \left(1 + \frac{(pa\sigma k)}{4kp} F(kx_b)\right) \cdot (-\gamma p + m) \left(1 - \frac{(pa\sigma k)}{4kp} F(kx_a)\right) \cdot \exp\left\{\mathrm{i}\left[p(x_b - x_a) - \frac{(pap)}{2kp}\int_{kx_a}^{kx_b} F(\varphi)\mathrm{d}\varphi\right]\right\}. \tag{83}$$

This Green function describes Dirac particles moving in a weak gravitational field.

From the two poles

$$p_\pm^0 = \pm\omega - \mathrm{i}0 = \pm(\mathbf{p}^2 + m^2)^{\frac{1}{2}} - \mathrm{i}0 \tag{84}$$

and using the residues from these two poles, we get the following expression:

$$\begin{aligned} S(x_b,x_a) &= -\mathrm{i}\theta(t_b - t_a)\int \frac{\mathrm{d}^3 p}{(2\pi)^3}\left(\frac{m}{p^0}\right) \cdot \left(1 + \frac{(pa\sigma k)}{4kp}F(kx_b)\right)\frac{(-\gamma p + m)}{2m} \\ &\cdot \left(1 - \frac{(pa\sigma k)}{4kp}F(kx_a)\right)\exp\left\{\mathrm{i}\left(p(x_b - x_a) - \frac{(pap)}{2kp}\int_{kx_a}^{kx_b} F(\varphi)\mathrm{d}\varphi\right)\right\} = -\mathrm{i}\theta(t_a - t_b) \\ &\cdot \int \frac{\mathrm{d}^3 p}{(2\pi)^3}\left(\frac{m}{p^0}\right)\left(1 - \frac{(pa\sigma k)}{4kp}F(kx_a)\right) \cdot \frac{(\gamma p + m)}{2m}\left(1 + \frac{(pa\sigma k)}{4kp}F(kx_b)\right) \\ &\cdot \exp\left\{\mathrm{i}\left(p(x_b - x_a) - \frac{(pap)}{2kp}\int_{kx_a}^{kx_b} F(\varphi)\mathrm{d}\varphi\right)\right\}. \end{aligned} \tag{85}$$

Now using the decomposition on states of positive and negative energy [12]

$$\begin{aligned} \Lambda_+ &= \sum_{\pm s} u(p,s)\bar{u}(p,s) = \frac{\gamma p + m}{2m}, \\ \Lambda_- &= \sum_{\pm s} v(p,s)\bar{v}(p,s) = \frac{-\gamma p + m}{2m}, \end{aligned} \tag{86}$$

and identifying the spectral decomposition

$$\begin{aligned} S(x_b,x_a) = &-\mathrm{i}\theta(t_b - t_a)\int \mathrm{d}^3 p \sum_{\pm s} \Psi_{s,p}^+(x_b)\bar{\Psi}_{s,p}^+(x_a) \\ &+ \mathrm{i}\theta(t_b - t_a)\int \mathrm{d}^3 p \sum_{\pm s} \Psi_{s,p}^-(x_b)\bar{\Psi}_{s,p}^-(x_a), \end{aligned} \tag{87}$$

we can extract the respective wave functions related to the particle and the antiparticle in a weak gravitational field:

$$\begin{aligned} \Psi_{s,p}^+(x) &= \frac{1}{(2\pi)^{\frac{3}{2}}}\left(\frac{m}{p^0}\right)^{\frac{1}{2}}\left(1 + \frac{(pa\sigma k)}{4(kp)}F(kx)\right) \\ &\cdot u(p,s)\exp\left\{\mathrm{i}px - \mathrm{i}\frac{(pap)}{2(kp)}\int^{kx} F(\varphi)\mathrm{d}\varphi\right\} \end{aligned} \tag{88}$$

and

$$\begin{aligned} \Psi_{s,p}^-(x) &= \frac{1}{(2\pi)^{\frac{3}{2}}}\left(\frac{m}{p^0}\right)^{\frac{1}{2}}\left(1 - \frac{(pa\sigma k)}{4(kp)}F(kx_a)\right) \\ &\cdot v(p,s)\exp\left[-\mathrm{i}\left(px + \frac{(pap)}{2(kp)}\int^{kx} F(\varphi)\mathrm{d}\varphi\right)\right]. \end{aligned} \tag{89}$$

The quantities $u(p,s)$ and $v(p,s)$ are spinors, i. e., solutions of the free Dirac equation and such that

$$\bar{u}(p,s)u(p,s) = 1, \quad \bar{v}(p,s)v(p,s) = -1.$$

4. Conclusion

In this paper, we determined the Green functions related to Klein-Gordon and Dirac particles by using only some transformations in order to make the δ functions and the calculation of the integrals easier. The arguments of the δ functions are the equations for the trajectories of the respective classical particles.

Indeed, for the Klein-Gordon particle, the equations of motion are the following:

$$\dot{p}^\mu = -(pap)k^\mu \frac{\mathrm{d}F(kx)}{\mathrm{d}(kx)},$$

$$\dot{x}^\mu = -2p^\mu + ((ap)^\mu + (pa)^\mu)F(kx).$$

After multiplying k_μ, $a_{\rho\mu}$, and $a_{\mu\rho}$, we have

$$k\dot{p} = 0 \to kp = \text{const.},$$

$$a\dot{p} = \dot{p}a = 0 \to pap = \text{const.}, \quad k\dot{x} = -2pk,$$

$$\frac{\mathrm{d}p^\mu}{\mathrm{d}\tau} = \frac{(pap)}{2pk}k^\mu \frac{\mathrm{d}F(kx)}{\mathrm{d}\tau} \to$$

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left(p^\mu - k^\mu \frac{(pap)}{2pk}F(kx)\right) = 0.$$

This leads to the two equations

$$P^\mu = p^\mu - k^\mu \frac{(pap)}{2pk} F(kx) \to \frac{\mathrm{d}P^\mu}{\mathrm{d}\tau} = 0 \to P^\mu = \text{const.}, \tag{90}$$

and

$$\dot{\varphi} = k\dot{x} = -2Pk = -2pk. \tag{91}$$

For the Dirac particle, we have the corresponding equations of motion in the

i) global case

$$\dot{p}^\mu = -k^\mu \left[(pap)\frac{\mathrm{d}F}{\mathrm{d}(kx)} + \frac{1}{2}(pa\Psi)(k\Psi)\frac{\mathrm{d}^2F}{\mathrm{d}(kx)^2} \right],$$

$$\dot{x}^\mu = -2p^\mu + (ap+pa)^\mu F + \frac{1}{2}(a\Psi)^\mu (k\Psi)\frac{\mathrm{d}F}{\mathrm{d}(kx)},$$

$$\dot{\Psi}^\mu = \frac{\mathrm{i}}{4}[(pa)^\mu (k\Psi) - k^\mu (pa\Psi)]F.$$

Together with

$$pk = (pap), \quad k\dot{x} = -2pk,$$

and by using

$$\frac{\mathrm{d}(pa\Psi)}{\mathrm{d}\tau} = \dot{p}a\Psi + pa\dot{\Psi} = -\frac{1}{4i}[(pa^2p)(k\Psi)]\frac{\mathrm{d}F}{\mathrm{d}(kx)},$$

$$(k\Psi)^2 = 0,$$

we obtain the equations

$$\frac{\mathrm{d}P^\mu}{\mathrm{d}\tau} = \frac{\mathrm{d}}{\mathrm{d}\tau}\left[p^\mu - k^\mu \left(\frac{(pap)}{2pk}F - \frac{1}{8(pk)^2}(pa\Psi)(k\Psi)\frac{\mathrm{d}F}{\mathrm{d}\tau} \right) \right] = 0$$

$$P^\mu = \text{const}, \quad k\dot{x} = -2Pk = -2pk,$$

$$k\Psi(\tau) = k\Psi(\lambda) = k\Psi(0) = \frac{k\theta}{2}.$$

ii) In the local case we have

$$\dot{p}^\mu = -k^\mu \left[(pap)k^\mu \frac{\mathrm{d}F}{\mathrm{d}(kx)} + \frac{1}{2}(pa\Psi)(k\Psi)\frac{\mathrm{d}^2F}{\mathrm{d}(kx)^2} - \frac{1}{2}(\Psi ap)\chi \frac{\mathrm{d}F}{\mathrm{d}(kx)} \right],$$

$$\dot{x}^\mu = -2p^\mu + ((pa)^\mu + (ap)^\mu)F + \left(\frac{1}{2}(a\Psi)^\mu (k\Psi)\frac{\mathrm{d}F}{\mathrm{d}(kx)} + \left(\Psi^\mu - \frac{1}{2}(\Psi a)^\mu F \right)\chi \right),$$

$$\dot{\Psi}^\mu = \frac{\mathrm{i}}{4}[(pa)^\mu (k\Psi) - k^\mu (pa\Psi)]F' - \frac{1}{2}\left(p^\mu - \frac{1}{2}(ap)^\mu F \right)\chi,$$

$$\dot{\Psi}^5 = \mathrm{i}\frac{m}{2}\chi.$$

By using

$$k\dot{\Psi} = -\frac{kp}{2}\chi,$$

$$k\Psi(\tau) = -\frac{(kp)}{2}\left(\tau - \frac{\lambda}{2} \right)\chi + \frac{k\theta}{2},$$

$$\Psi^5 = \left(\mathrm{i}\frac{m\tau}{2} + \frac{(kp)\lambda}{4} \right)\chi + \frac{k\theta}{2},$$

we finally obtain the following equations:

$$\begin{aligned} k\dot{x} &= -2(pk) - 2\mathrm{i}(k\Psi)\chi \\ &= -2(pk) + \left[\mathrm{i}\frac{(kp)\tau}{2}\chi + \frac{k\theta}{2} - \mathrm{i}\frac{(kp)\lambda}{4}\chi \right]\chi \\ &= -2(pk) - \mathrm{i}k\theta\chi, \end{aligned}$$

$$\frac{\mathrm{d}\varphi}{\mathrm{d}\tau} = -2(pk) - \mathrm{i}k\theta\chi = -2(pk)\left[1 + \mathrm{i}\frac{k\theta}{2pk}\chi \right],$$

or we find

$$\frac{\mathrm{d}\tau}{\mathrm{d}\varphi} = -\frac{1}{2(pk)}\left[1 - \mathrm{i}\frac{k\theta}{2pk}\chi \right]$$

and

$$k\dot{\Psi} = -\frac{kp}{2}\chi,$$

$$\dot{\Psi}^5 = \mathrm{i}\frac{m}{2}\chi.$$

Thus, it was shown that the classical trajectories play essential roles in the calculation of the Green functions, and consequently a semi classical calculation should give the same result, what will be published separately.

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