Fractional Diffusion Equation and External Forces: Solutions in a Confined Region

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We investigate the $\mathcal{N}$-dimensional fractional diffusion equation with radial symmetry by taking time dependent boundary conditions and external forces into account in a confined region. A spatial and time dependence on the diffusion coefficient is also considered. The results obtained show an anomalous dispersion of the solutions and non-usual behaviour for the survival probability.

Key words: Diffusion Equation; Confined Geometry; Anomalous Diffusion.

1. Introduction

Typical diffusive phenomena found in usual diffusion equation an appropriate description which has as main characteristic the square mean displacement with a linear dependence on time, i. e., $\langle (r-\langle r\rangle)^2\rangle \propto t$ [1]. This feature is deeply related to the central limit theorem and also to the Markovian nature of this stochastic process. However, a large number of experimental observations show that more complex processes, in which the mean-square displacement is not proportional to t, may also occur in nature. These situations can be found, for instance, in atom deposition into a porous substrate [2], diffusion of high molecular weight polyisopropylacrylamide in nanopores [3], highly confined hard disk fluid mixtures [4], fluctuating particle fluxes [5], conservative motion in a $d = 2$ periodic potential [6], transport of fluid in porous media, diffusion on fractals [7], and many other interesting physical systems.

Several formalisms [8 – 11] have been employed to investigate these phenomena which present a non-conventional diffusion process. One of them is based on fractional diffusion equations [12 – 15], which have been widely applied to different physical scenarios. The advantage of these equations is the simple way to deal with boundary values problems and incorporate forced fields which in other approaches can lead to cumbersome situations. In addition, they may be related to continuous time random walk (CTRW) [16], Langevin equations [17], and master equations [18]. These features associated to fractional diffusion equations have motivated the study of their solutions in presence of, for instance, external forces [19 – 21], spatial dependent diffusion coefficients [22], reaction terms [23 – 26], cylindrical symmetry [27 – 29], and in confined regions [30 – 32]. Towards this direction, we address this work to the investigation of the $\mathcal{N}$-dimensional fractional diffusion equation with radial symmetry

$$\frac{\partial^\gamma}{\partial t^\gamma}\rho(r,t) = \int_0^t \mathrm{d}t'\,\nabla\cdot\left(\mathcal{D}(r,t-t')\nabla\rho(r,t')\right) - \int_0^t \mathrm{d}'t\,\nabla\cdot\left(\overline{F}(r,t-t')\rho(r,t')\right), \tag{1}$$

where $\mathcal{D}(r,t)$ represents the diffusion coefficient, $\overline{F}(r,t)$ is an external force acting on the system, and $0 < \gamma \le 1$ (subdiffusive case). The fractional time derivative representation considered here is the Caputo one [33] and the boundary conditions are $\rho(a,t) = \phi_a(t)$ and $\rho(b,t) = \phi_b(t)$, where $\phi_a(t)$ and $\phi_b(t)$ are arbitrary time dependent functions. Consequently, the system described by (1) is restricted to the region $a \leq r \leq b$ and subjected to time dependent boundary conditions.

In order to investigate the solutions of (1) subjected to time dependent boundary conditions, we use the Green function approach [34]. The solutions obtained within this formalism will be useful to investigate the dynamic properties related to the system and, in particular, clarify the role of the surface on the diffusive process. It is worth mentioning the importance of the solutions of (1) as they may be used to investigate, for example, adsorption-desorption processes with memory [35], situations characterized by reactive boundaries [36], and first time distribution in confined regions [37]. We start the analysis by considering (1) with $\mathcal{D}(r,t) = \overline{\mathcal{D}}(t) r^{-\theta}$, where $\overline{\mathcal{D}}(t) = t^{\eta-1}/\Gamma(\eta-1)$ with $\eta \geq 0$, in absence of external forces, i. e., $\overline{F}(r,t) = 0$. The solution obtained for this case as a particular case results found in [30–32] and may recover for $\alpha = 1$ results found in [38] by taking the limit $a \to 0$ with $\rho(b,t) = 0$. In this context, we also analyze the survival probability which for this case exhibits an anomalous behaviour. Afterwards, we incorporate the external force $\overline{F}(r,t) = -(kr - \mathcal{K}/r^{\varepsilon}) F'(t)\hat{r}$, where $F'(t) = t^{\eta-1}/\Gamma(\eta-1)$ and $\varepsilon = 1+\theta$, in our analysis which is firstly addressed by considering $k = 0$ and after this for the case $k \neq 0$. These developments are reported in Section 2, while conclusions are presented in Section 3.

2. Fractional Diffusion Equation

Let us consider (1) without external forces subjected to the time dependent boundary conditions $\rho(a,t) = \phi_a(t)$ and $\rho(b,t) = \phi_b(t)$, with the initial condition given by an arbitrary function $\rho(r,0) = \tilde{\rho}(r)$ which is initially normalized, i. e., $\int_0^\infty \mathrm{d}r r^{\mathcal{N}-1} \rho(r,t) = 1$. By also considering a spatial dependence on the diffusion coefficient, i. e., it is given by $\mathcal{D}(r,t) = \overline{\mathcal{D}}(t) r^{-\theta}$, (1) may be written as

$$\frac{\partial^\gamma}{\partial t^\gamma} \rho(r,t) = \frac{1}{r^{\mathcal{N}-1}} \cdot \int_0^t \mathrm{d}t' \frac{\partial}{\partial r} \left(\overline{\mathcal{D}}(t-t') r^{\mathcal{N}-1-\theta} \frac{\partial}{\partial r} \rho(r,t') \right) \tag{2}$$

with $0 < \gamma \leq 1$ and $\overline{\mathcal{D}}(t) = t^{\eta-1}/\Gamma(\eta-1)$. This equation extends the usual diffusion equation by incorporating fractional time derivatives and a spatial and time dependence on the diffusion coefficient. The spatial dependence adopted for the diffusion coefficient has been used to investigate several physical contexts such as diffusion on fractals [7], turbulence [39], and fast electrons in hot plasma in the presence of an electric field [40]. A direct consequence of the extensions incorporated in (2) accomplishing time dependent boundary conditions $\phi_a(t)$ and $\phi_b(t)$ is the anomalous spreading of the distribution and the quantities related to it. The solution anomalous behaviour is noticed by the behaviour of the mean-square displacement $\sigma^2 = \langle (r - \langle r \rangle)^2 \rangle$ when compared to the usual one. In addition, it is interesting to note that for a suitable time dependence on the diffusion coefficient the solutions may exhibit different diffusive regimes and the choice of boundary conditions represents a key role regarding the presence of stationary solutions which, in particular, are equal to the usual one. This last feature shows that the fractional time derivative produces an anomalous relaxation of the solution to the stationary state, in contrast to the spatial fractional derivatives which lead to Lévy like distributions.

In order to obtain the solution for (2) and investigate the surface effect on the solution, i. e., how the boundary conditions may change the relaxation of the system, we use the Green function approach and the Laplace transform. Applying the Laplace transform to (2) and performing some calculations, it is possible to show that the solution is given by

$$\begin{aligned} \rho(r,s) = & -s^{\gamma-1} \int_a^b \mathrm{d}r'\, r'^{\mathcal{N}-1} G(r,r';s) \overline{\rho}(r) \\ & + \overline{\mathcal{D}}(s) \left[b^{\mathcal{N}-\theta-1} \phi_b(s) \frac{\partial}{\partial r'} G(r,r';s) \bigg|_{r'=b} \right. \\ & \left. - a^{\mathcal{N}-\theta-1} \phi_a(s) \frac{\partial}{\partial r'} G(r,r';s) \bigg|_{r'=a} \right], \end{aligned} \tag{3}$$

where the last term represents the surface effect and determines the existence of an stationary solution. The Green function $G(r,r';s)$ present in above equation is obtained by solving the equation

$$\begin{aligned} & \frac{1}{r^{\mathcal{N}-1}} \frac{\mathrm{d}}{\mathrm{d}r} \left(\overline{\mathcal{D}}(s) r^{\mathcal{N}-\theta-1} \frac{\mathrm{d}}{\mathrm{d}r} G(r,r';s) \right) \\ & - s^\gamma G(r,r';s) = \frac{1}{r^{\mathcal{N}-1}} \delta(r-r') \end{aligned} \tag{4}$$

subjected to the boundary conditions $G(r,r';s)|_{r=b} = 0$ and $G(r,r';s)|_{r=a} = 0$. Equation (4) may be solved by applying different procedures. We use the eigenfunctions of Sturm-Liouville related to the spatial operator of (4), i. e., the eigenfunctions which emerge from the equation $\partial_r \left(r^{\mathcal{N}-1-\theta}\partial_r \psi_n(k_n,r)\right) = -k_n^2 r^{\mathcal{N}-1}\psi_n(k_n,r)$ with $\psi_n(k_n,r)|_{r=a} = \psi_n(k_n,r)|_{r=b} = 0$. By employing this procedure it can be shown that the Green function for this case is given by

$$G(r,r';s) = -\frac{1}{2+\theta}\sum_{n=1}^{\infty}\frac{\mathcal{N}_n}{s^\gamma+\overline{\mathcal{D}}(s)k_n^2}\psi_n(k_n,r')\psi_n(k_n,r) \quad (5)$$

with

$$\psi_n(k_n,r) = r^{\frac{1}{2}(2+\theta-\mathcal{N})} \cdot\left[\mathrm{J}_{|\alpha|}\left(\frac{2k_n}{2+\theta}r^{\frac{1}{2}(2+\theta)}\right)\mathrm{N}_{|\alpha|}\left(\frac{2k_n}{2+\theta}a^{\frac{1}{2}(2+\theta)}\right) -\mathrm{J}_{|\alpha|}\left(\frac{2k_n}{2+\theta}a^{\frac{1}{2}(2+\theta)}\right)\mathrm{N}_{|\alpha|}\left(\frac{2k_n}{2+\theta}r^{\frac{1}{2}(2+\theta)}\right)\right], \quad (6)$$

where $\alpha = \mathcal{N}/(2+\theta)-1$, $\mathrm{J}_\alpha(x)$ and $\mathrm{N}_\alpha(x)$ are Bessel functions of first and second species. The eigenvalues k_n are determined by using the equation

$$\mathrm{J}_{|\alpha|}\left(\frac{2k_n}{2+\theta}b^{\frac{1}{2}(2+\theta)}\right)\mathrm{N}_{|\alpha|}\left(\frac{2k_n}{2+\theta}a^{\frac{1}{2}(2+\theta)}\right) -\mathrm{J}_{|\alpha|}\left(\frac{2k_n}{2+\theta}a^{\frac{1}{2}(2+\theta)}\right)\mathrm{N}_{|\alpha|}\left(\frac{2k_n}{2+\theta}b^{\frac{1}{2}(2+\theta)}\right) = 0 \quad (7)$$

and

$$\mathcal{N}_n = \frac{\pi^2 k_n^2}{\mathrm{J}^2_{|\alpha|}\left(\frac{2k_n}{2+\theta}a^{\frac{1}{2}(2+\theta)}\right)\Big/\mathrm{J}^2_{|\alpha|}\left(\frac{2k_n}{2+\theta}b^{\frac{1}{2}(2+\theta)}\right)-1}. \quad (8)$$

Applying the inverse Laplace transform (3) and (5) with $\overline{\mathcal{D}}(s) = \mathcal{D}/s^\eta$ $(\overline{\mathcal{D}}(t) = \mathcal{D}t^{\eta-1}/\Gamma(\eta-1))$, we obtain that

$$\begin{aligned}\rho(r,t) = &-\frac{1}{\Gamma(1-\gamma)}\int_0^t \mathrm{d}t'\frac{1}{(t-t')^{\gamma-1}} \\ &\cdot\int_a^b \mathrm{d}r' r'^{\mathcal{N}-1}G(r,r';s)\overline{\rho}(r) + \int_0^t \mathrm{d}\bar{t}\,\overline{\mathcal{D}}(t-\bar{t}) \\ &\cdot\left\{b^{\mathcal{N}-\theta-1}\int_0^{\bar{t}}\mathrm{d}t'\phi_b(\bar{t}-t')\frac{\partial}{\partial r'}G(r,r';t')\bigg|_{r'=b}\right. \\ &\left.-a^{\mathcal{N}-\theta-1}\int_0^{\bar{t}}\mathrm{d}t'\phi_a(\bar{t}-t')\frac{\partial}{\partial r'}G(r,r';t')\bigg|_{r'=a}\right\}\end{aligned} \quad (9)$$

with

$$G(r,r';t) = -\frac{t^{\gamma-1}}{2+\theta}\sum_{n=1}^{\infty}\mathcal{N}_n\psi_n(k_n,r')\psi_n(k_n,r) \cdot\mathrm{E}_{\gamma+\eta,\gamma}\left(-Dk_n^2t^{\gamma+\eta}\right), \quad (10)$$

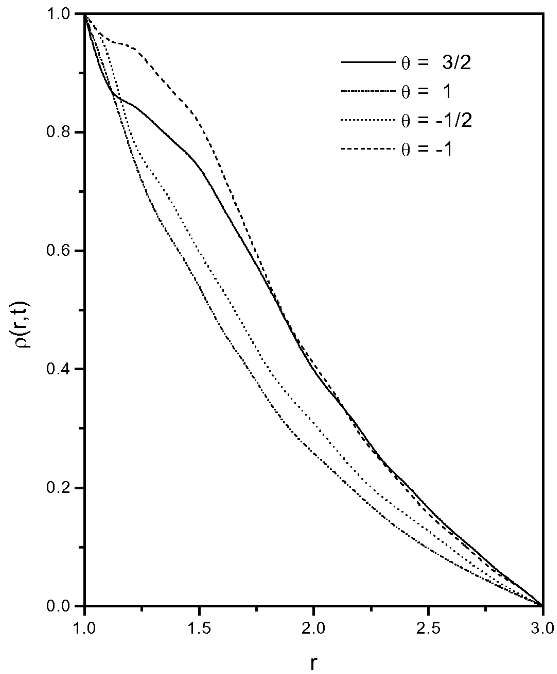

Fig. 1. Behaviour of $\rho(r,t)$ versus r obtained from equation (9) for typical values of θ, accomplishing $\Phi_a(t) = 1$ and $\Phi_b(t) = 0$. We consider, for simplicity, $\mathcal{D} = 1$, $\mathcal{N} = 2$, $a = 1$, $b = 3$, $t = 1$, $\gamma+\eta = 1/2$, and the initial condition $\rho(r,0) = \delta(r-3/2)/r^{\mathcal{N}-1}$. Note that the value of θ modifies the diffusive process of the system and for $\theta < 0$ the diffusion is faster than $\theta > 0$.

where $\mathrm{E}_{\alpha,\beta}(x) = \sum_{n=0}^{\infty}x^n/\Gamma(\beta+\alpha n)$ is the generalized Mittag-Leffler function [33] (see Fig. 1). The presence of this function in above equation indicates an anomalous dispersion of the solution due to the presence of fractional time derivatives in (2). In particular, it is interesting to note that the time dependence in the diffusion coefficient also produces an anomalous dispersion of the solution and, in particular, depending on the choice of the time dependence different diffusive regimes may be obtained as mentioned before. The anomalous spreading may be verified by analyzing the dispersion relation, i. e., $\sigma^2 = \langle(r-\langle r\rangle)^2\rangle$. For the case characterized by the boundary condition given by $\rho(a,t) = \rho(b,t) = 0$, the initial condition $\rho(r,0) = \delta(r-\tilde{r})/t^{\mathcal{N}-1}$ and $\theta = 0$, we have that

$$\sigma^2(t) = \langle r^2\rangle - (2-\mathcal{S}(t))\langle r\rangle^2 \quad (11)$$

(see Fig. 2) with

$$\langle r^2\rangle = \sum_{n=1}^{\infty}\frac{\mathcal{N}_n}{\pi k_n^2}\left(b^{\frac{\mathcal{N}}{2}+1}\frac{\mathrm{J}_{|\alpha|}(k_na)}{\mathrm{J}_{|\alpha|}(k_nb)} - a^{\frac{\mathcal{N}}{2}+1}\right) \cdot\psi_n(\tilde{r},k_n)\mathrm{E}_{\gamma+\eta}\left(-Dk_n^2t^{\gamma+\eta}\right)$$

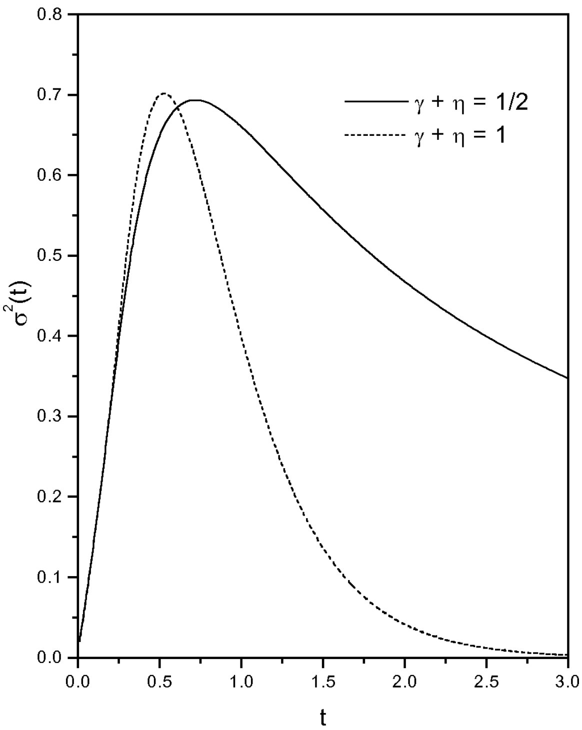

Fig. 2. Behaviour of $\sigma^2(t)$ versus t in (11) for typical values of $\gamma+\eta$ by considering $\mathcal{D}=1$, $\mathcal{N}=1$, $a=1$, $b=3$, and $\rho(r,0)=\delta(r-2)/r^{\mathcal{N}-1}$.

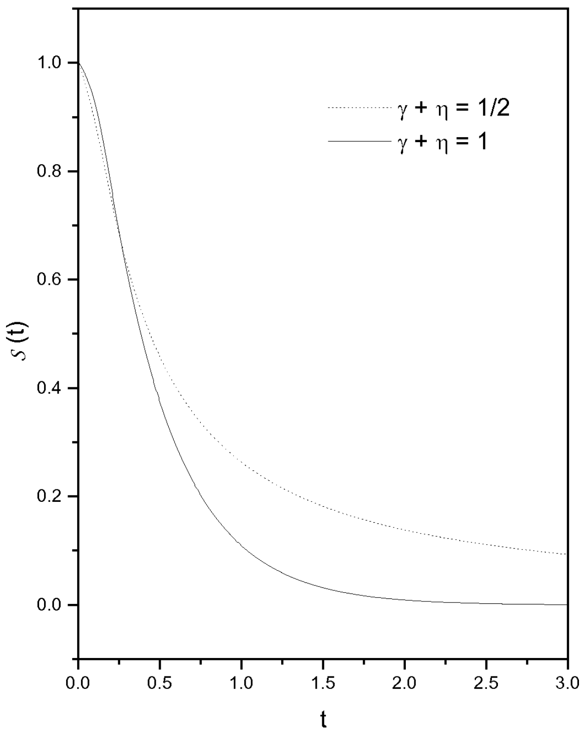

Fig. 3. Behaviour of $\mathcal{S}(t)$ versus t in (14) for typical values of $\gamma+\eta$ by considering $\Phi_a(t)=1$, $\Phi_b(t)=0$, $\mathcal{D}=1$, $\mathcal{N}=1$, $a=1$, $b=3$, and the initial condition $\rho(r,0)=\delta(r-2)/r^{\mathcal{N}-1}$.

$$-\sum_{n=1}^{\infty}\frac{2\mathcal{N}\mathcal{N}_n}{\pi k_n^4}\left(b^{\frac{\mathcal{N}}{2}-1}\frac{\mathrm{J}_{|\alpha|}(k_n a)}{\mathrm{J}_{|\alpha|}(k_n b)}-a^{\frac{\mathcal{N}}{2}-1}\right)\cdot\psi_n(\tilde{r},k_n)\mathrm{E}_{\gamma+\eta}\left(-Dk_n^2t^{\gamma+\eta}\right), \quad (12)$$

$$\begin{aligned}\langle r\rangle=&\sum_{n=1}^{\infty}\frac{\mathcal{N}_n}{\pi k_n^2}\left(b^{\frac{\mathcal{N}}{2}}\frac{\mathrm{J}_{|\alpha|}(k_n a)}{\mathrm{J}_{|\alpha|}(k_n b)}-a^{\frac{\mathcal{N}}{2}}\right)\cdot\psi_n(\tilde{r},k_n)\mathrm{E}_{\gamma+\eta}\left(-Dk_n^2t^{\gamma+\eta}\right)\\&-\sum_{n=1}^{\infty}\frac{\mathcal{N}_n}{\pi k_n^4}\left(b^{\frac{\mathcal{N}}{2}-1}\frac{\mathrm{J}_{|\alpha|}(k_n a)}{\mathrm{J}_{|\alpha|}(k_n b)}-a^{\frac{\mathcal{N}}{2}-1}\right)\cdot\psi_n(\tilde{r},k_n)\mathrm{E}_{\gamma+\eta}\left(-Dk_n^2t^{\gamma+\eta}\right),\end{aligned} \quad (13)$$

and

$$\mathcal{S}(t)=\sum_{n=1}^{\infty}\frac{\mathcal{N}_n}{\pi k_n^2}\left(b^{\frac{\mathcal{N}}{2}-1}\frac{\mathrm{J}_{|\alpha|}(k_n a)}{\mathrm{J}_{|\alpha|}(k_n b)}-a^{\frac{\mathcal{N}}{2}-1}\right)\cdot\psi_n(\tilde{r},k_n)\mathrm{E}_{\gamma+\eta}\left(-Dk_n^2t^{\gamma+\eta}\right) \quad (14)$$

(see Fig. 3). Note that the quantity $\mathcal{S}(t)=\int_a^b \mathrm{d}r r^{\mathcal{N}-1}\cdot\rho(r,t)$ is the survival probability [41], i. e., it is related to the quantity of particles which are present in the region defined by the interval $a\le r\le b$. This manner, the quantity $1-\mathcal{S}(t)$ gives the quantity of particles absorbed by the surfaces, i. e., removed of the system.

Now, let us incorporate in our analysis the external force $F(r,t)=\mathcal{K}/r^{\varepsilon}$ ($\varepsilon=1+\theta$) which can be derived from a power law potential and has the logarithmic potential as particular case and a reaction term represented by $\tilde{\alpha}(r,t)$. In this way, we expect to extend the previous solution to a broader context in order to make possible the investigation of, for example, drug absorption or deliver [42], tumor development [43], and heat production [44]. Employing the previous procedure, the solution for (1) subject to the previous external force and with reaction term is given by

$$\begin{aligned}\rho(r,t)=&\\&-\frac{1}{\Gamma(1-\gamma)}\int_0^t\mathrm{d}t'\frac{1}{(t-t')^{\gamma-1}}\int_a^b\mathrm{d}r'^{\mathcal{N}-1}\widetilde{G}(r,r';t')\overline{\rho}(r)\\&-\int_0^t\mathrm{d}t'\int_a^b\mathrm{d}r'^{\mathcal{N}-1}\widetilde{G}(r,r';t-t')\tilde{\alpha}(r',t')\end{aligned}$$

$$+\int_0^t d\bar{t}\mathcal{D}(t-\bar{t})\left\{b^{\mathcal{N}-\theta-1}\int_0^{\bar{t}} dt'\phi_b(\bar{t}-t')\frac{\partial}{\partial r'}\widetilde{G}(r,r';t')\Big|_{r=b} - a^{\mathcal{N}-\theta-1}\int_0^{\bar{t}} dt'\phi_a(\bar{t}-t')\frac{\partial}{\partial r'}\widetilde{G}(r,r';t')\Big|_{r=a}\right\} \quad (15)$$

with the Green function given by

$$\widetilde{G}(r,r';t) = -\frac{t^{\gamma-1}r^{\frac{\mathcal{K}}{\mathcal{D}}}}{2+\theta}\sum_{n=1}^{\infty}\widetilde{\mathcal{N}}_n\widetilde{\psi}_n(k_n,r')\widetilde{\psi}_n(k_n,r)\mathrm{E}_{\gamma+\eta,\gamma}(-Dk_n^2t^{\gamma}), \quad (16)$$

where

$$\widetilde{\psi}_n(k_n,r) = r^{\frac{1}{2}\left(2+\theta-\mathcal{N}-\frac{\mathcal{K}}{\mathcal{D}}\right)}\left[\mathrm{J}_{|\nu|}\left(\frac{2k_n}{2+\theta}r^{\frac{1}{2}(2+\theta)}\right)\mathrm{N}_{|\nu|}\left(\frac{2k_n}{2+\theta}a^{\frac{1}{2}(2+\theta)}\right) - \mathrm{J}_{|\nu|}\left(\frac{2k_n}{2+\theta}a^{\frac{1}{2}(2+\theta)}\right)-\mathrm{N}_{|\nu|}\left(\frac{2k_n}{2+\theta}r^{\frac{1}{2}(2+\theta)}\right)\right] \quad (17)$$

with $\nu = (\mathcal{N}+\mathcal{K}/\mathcal{D})/(2+\theta)-1$. The k_n are obtained from (7) by replacing the index α of the Bessel functions by the index ν and finally, with the normalization factor given by

$$\widetilde{\mathcal{N}}_n = \frac{\pi^2k_n^2}{\mathrm{J}^2_{|\nu|}\left(\frac{2k_n}{2+\theta}a^{\frac{1}{2}(2+\theta)}\right)/\mathrm{J}^2_{|\nu|}\left(\frac{2k_n}{2+\theta}b^{\frac{1}{2}(2+\theta)}\right)-1}. \quad (18)$$

This result extends the results presented in [30–32] to the $\mathcal{N}$-dimensional case, and recovers the results obtained in [38] when the external force is absent for $\eta = 0$. In addition, the results found in [29] and [45] are also a particular case of the above equations. By applying the limit $b\to\infty$ in (15) and also considering the boundary condition $\rho(\infty,t)=0$, we obtain an interesting result. It is given by

$$\rho(r,t) = -\frac{1}{\Gamma(1-\gamma)}\int_0^t d\tilde{t}\frac{1}{(t-\tilde{t})^{\gamma}}\int_0^{\infty} d\tilde{r}\tilde{r}^{\mathcal{N}-1}\tilde{\rho}(\tilde{r})G(r,\tilde{r};s) - \int_0^t dt'\int_a^{\infty} dr' r'^{\mathcal{N}-1}\widetilde{G}(r,r';t-t')\alpha(r',t') - a^{\mathcal{N}-\theta-1}\int_0^t d\bar{t}\int_0^{\bar{t}} dt'\mathcal{D}(t-\bar{t})\phi_a(\bar{t}-t')\frac{\partial}{\partial r'}\widetilde{\mathcal{G}}(r,r';t')\Big|_{r'=a} \quad (19)$$

with the Green function

$$\mathcal{G}(r,\tilde{r};t) = -\frac{2t^{\gamma-1}}{2+\theta}\int_0^{\infty} dk\frac{k\psi(\tilde{r},k)\psi(r,k)\mathrm{E}_{\gamma+\eta,\gamma}\left(-k^2t^{\gamma+\eta}\right)}{\mathrm{J}^2_{|\nu|}\left(\frac{2k}{2+\theta}a^{\frac{1}{2}(2+\theta)}\right)+\mathrm{N}^2_{|\nu|}\left(\frac{2k}{2+\theta}a^{\frac{1}{2}(2+\theta)}\right)}. \quad (20)$$

Note that this solution is restricted to the interval $a\le r<\infty$ and may be extended in order to cover the interval $0\le r<\infty$. In this direction, we need to take the limit $a\to 0$ and the restriction $2+\theta-\mathcal{N}-\mathcal{K}/\mathcal{D}>0$ in (20) to preserve the boundary conditions $\mathcal{G}(0,\tilde{r},t)=0$ and $\mathcal{G}(\infty,\tilde{r},t)=0$ for the Green function. These conditions lead to the Green function

$$\mathcal{G}(r,\tilde{r};t) = -\frac{2t^{\gamma-1}}{2+\theta}(r\tilde{r})^{\frac{1}{2}\left(2+\theta-\mathcal{N}-\frac{\mathcal{K}}{\mathcal{D}}\right)}\int_0^{\infty} dkk\mathrm{J}_{|\nu|}\left(\frac{2k}{2+\theta}r^{\frac{1}{2}(2+\theta)}\right)\mathrm{J}_{|\nu|}\left(\frac{2k}{2+\theta}\tilde{r}^{\frac{1}{2}(2+\theta)}\right)\mathrm{E}_{\gamma+\eta,\gamma}\left(-k^2t^{\gamma}\right). \quad (21)$$

For the case $\nu = 1/2$ and $\eta = 0$ the above equation may be simplified to

$$\mathcal{G}(r,\tilde{r};t) = -\frac{1}{2t}\sqrt{\frac{t^{\gamma}}{\mathcal{D}}}(r\tilde{r})^{\frac{1}{2}\left(\frac{1}{2}(2+\theta)-\mathcal{N}-\frac{\mathcal{K}}{\mathcal{D}}\right)}\left\{\mathbf{H}^{10}_{11}\left[\frac{2}{(2+\theta)\sqrt{\mathcal{D}t^{\gamma}}}\left|r^{\frac{1}{2}(2+\theta)}-\tilde{r}^{\frac{1}{2}(2+\theta)}\right|\Big|^{(\gamma/2,\gamma/2)}_{(0,1)}\right] - \mathbf{H}^{10}_{11}\left[\frac{2}{(2+\theta)\sqrt{\mathcal{D}t^{\gamma}}}\left|r^{\frac{1}{2}(2+\theta)}+\tilde{r}^{\frac{1}{2}(2+\theta)}\right|\Big|^{(\gamma/2,\gamma/2)}_{(0,1)}\right]\right\}, \quad (22)$$

where $\mathbf{H}^{\mathrm{m,n}}_{\mathrm{p,q}}\left[x\Big|^{(a_p,A_p)}_{(b_q,B_q)}\right]$ is the H Fox function [46]. For $\gamma = 1$ with $\eta = 0$ we have

$$\mathcal{G}(r,\tilde{r};t)=\frac{1}{(2+\theta)\mathcal{D}t}(r\tilde{r})^{\frac{1}{2}\left(2+\theta-\mathcal{N}-\frac{\mathcal{K}}{\mathcal{D}}\right)}\mathrm{e}^{-\frac{1}{(2+\theta)^2\mathcal{D}t}\left(r^{2+\theta}+\tilde{r}^{2+\theta}\right)}\mathrm{I}_{|\nu|}\left(\frac{(r\tilde{r})^{2+\theta}}{(2+\theta)^2\mathcal{D}t}\right), \tag{23}$$

where $\mathrm{I}_\nu(x)$ is a Bessel function of modified argument (see Fig. 4).

The external force worked out above may be extended by incorporating a linear term, i. e., $-kr$, which results in $F(r,t)=-kr+\mathcal{K}/r^{\varepsilon}$ with $\varepsilon=1+\theta$. The solution for this case is formally given by (15) with the Green function

$$\mathcal{G}(r,\overline{r},t)=-r^{\frac{\mathcal{K}}{\mathcal{D}}}\mathrm{e}^{-\frac{kr^{2+\theta}}{(2+\theta)\mathcal{D}}}t^{\gamma-1}\sum_{n=1}^{\infty}\overline{\mathcal{N}}_n\overline{\Psi}_n(r,\lambda_n)\overline{\Psi}_n(\overline{r},\lambda_n)\mathrm{E}_{\gamma+\eta,\gamma}\left(-\mathcal{D}\lambda_n^2t^{\gamma+\eta}\right), \tag{24}$$

where

$$\overline{\Psi}_n(r,\lambda_n)=\Phi\left(\overline{\lambda}_n,\beta;\frac{kr^{2+\theta}}{(2+\theta)\mathcal{D}}\right)\Psi\left(\overline{\lambda}_n,\beta;\frac{ka^{2+\theta}}{(2+\theta)\mathcal{D}}\right)-\Phi\left(\overline{\lambda}_n,\beta;\frac{ka^{2+\theta}}{(2+\theta)\mathcal{D}}\right)\Psi\left(\overline{\lambda}_n,\beta;\frac{kr^{2+\theta}}{(2+\theta)\mathcal{D}}\right) \tag{25}$$

with $\overline{\lambda}_n=-\lambda_n/(2+\theta)k$, $\beta=(\mathcal{K}+\mathcal{N}\mathcal{D})/((2+\theta)\mathcal{D})$. $\Phi(\alpha,\beta;x)$ is the confluent hypergeometric function,

$$\Psi(\alpha,\beta;x)=\frac{\Gamma(1-\beta)}{\Gamma(1+\alpha-\beta)}\Phi(\alpha,\beta;x)+\frac{\Gamma(\beta-1)}{\Gamma(\alpha)}x^{1-\beta}\Phi(1+\alpha-\beta,2-\beta;x).$$

The eigenvalues λ_n are determined by the equation

$$\Phi\left(\overline{\lambda}_n,\beta;\frac{kb^{2+\theta}}{(2+\theta)\mathcal{D}}\right)\Psi\left(\overline{\lambda}_n,\beta;\frac{ka^{2+\theta}}{(2+\theta)\mathcal{D}}\right)-\Phi\left(\overline{\lambda}_n,\beta;\frac{ka^{2+\theta}}{(2+\theta)\mathcal{D}}\right)\Psi\left(\overline{\lambda}_n,\beta;\frac{kb^{2+\theta}}{(2+\theta)\mathcal{D}}\right)=0 \tag{26}$$

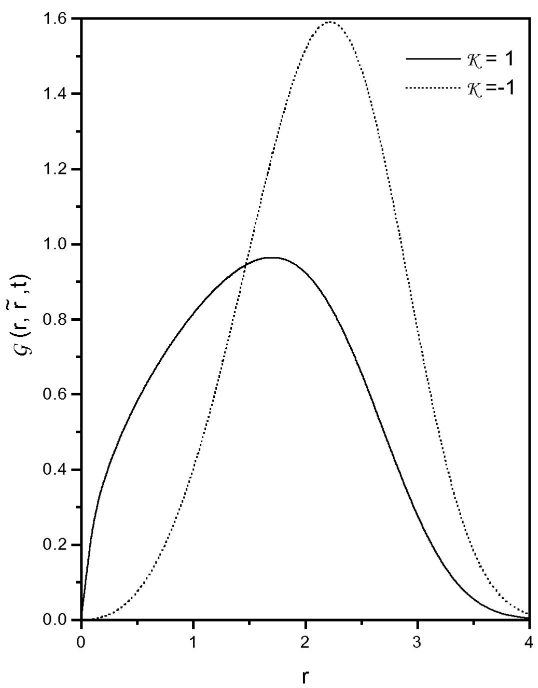

Fig. 4. Behaviour of $\mathcal{G}(r,\tilde{r},t)$ versus r which illustrates equation (23) for typical values of $\mathcal{K}$ by considering, for simplicity, $\mathcal{D}=1$, $\mathcal{N}=2$, $t=1$, $\tilde{r}=1$, and $\theta=1$.

and the normalization condition is given by $\overline{\mathcal{N}}_n=1/\int_a^b \mathrm{d}r r^{\mathcal{N}-1-\frac{\mathcal{K}}{\mathcal{D}}}\mathrm{e}^{-\frac{kr^{2+\theta}}{(2+\theta)\mathcal{D}}}\overline{\Psi}_n^2(r,\lambda_n)$. Notice that (24) may be extended to the interval $0\le r<\infty$ by taking the limit $a\to 0$ and $b\to\infty$ accomplishing the boundary condition $\lim_{b\to\infty}\rho(b,t)=0$. In this direction, after some calculations, it is possible to show that the solution is given by

$$\begin{aligned}\rho(r,t)=&\\ -&\frac{1}{\Gamma(1-\gamma)}\int_0^t \mathrm{d}t\frac{1}{(t-\tilde{t})^{\gamma}}\int_0^{\infty}\mathrm{d}\tilde{r}\tilde{r}^{\mathcal{N}-1}\overline{\rho}(\tilde{r})\mathcal{G}(r,\tilde{r};s)\\ -&a^{\mathcal{N}-\theta-1}\int_0^t \mathrm{d}\overline{t}\int_0^{\overline{t}}\mathrm{d}t'\mathcal{D}(t-\overline{t})\phi_a(\overline{t}-t')\frac{\partial}{\partial r'}\mathcal{G}(r,r';t')\Big|_{r'=a}\end{aligned} \tag{27}$$

with the Green function

$$\begin{aligned}\mathcal{G}(r,r',t)=&-r^{\frac{\mathcal{K}}{\mathcal{D}}}t^{\gamma-1}\mathrm{e}^{-\frac{kr^{2+\theta}}{(2+\theta)\mathcal{D}}}\\ &\cdot\sum_{n=1}^{\infty}\widehat{\mathcal{N}}_n\hat{\psi}_n(r,\lambda_n)\hat{\psi}_n(r',\lambda_n)\mathrm{E}_{\gamma+\eta,\gamma}\left(-\mathcal{D}\lambda_n^2t^{\gamma+\eta}\right)\end{aligned} \tag{28}$$

with the eigenfunction

$$\hat{\psi}_n(r,\lambda_n)=\mathrm{L}_n^{\xi}\left(\frac{kr^{2+\theta}}{(2+\theta)\mathcal{D}}\right) \tag{29}$$

and

$$\hat{N}_n = \left(\frac{k}{(2+\theta)\mathcal{D}}\right)^{\frac{\mathcal{K}+\mathcal{N}\mathcal{D}}{(2+\theta)\mathcal{D}}} \frac{(2+\theta)\Gamma(n+1)}{\Gamma\left(\frac{\mathcal{K}+\mathcal{N}\mathcal{D}}{(2+\theta)\mathcal{D}}+n\right)}, \quad (30)$$

where $\xi = (\mathcal{K}+\mathcal{N}\mathcal{D})/[(2+\theta)\mathcal{D}]-1$, $\mathrm{L}_n^{\xi}(x)$ are associated Laguerre polynomials, and $\lambda_n = (2+\theta)nk$. It is interesting to note that the solution for cases analyzed above may present a stationary solution depending on the choice of the time dependent functions $\phi_a(t)$ and $\phi_b(t)$. This feature indicates that the fractional time derivative and the form of the time dependence of the diffusion coefficient may be useful to describe systems with anomalous relaxation which have as stationary solution the usual one.

3. Conclusion and Discussion

We investigated a $\mathcal{N}$-dimensional fractional diffusion equation with radial symmetry accomplishing external forces and spatial and time dependent diffusion coefficient. We also consider the arbitrary reaction term $\alpha(r,t)$. Equation (1) is firstly analyzed by considering the spatial and time dependent diffusion coefficient $\mathcal{D}(r,t) = \overline{\mathcal{D}}(t)r^{-\theta}$, where $\overline{\mathcal{D}}(t) = \overline{\mathcal{D}}t^{\eta-1}/\Gamma(\eta-1)$, without external forces and reaction term subjected to a time dependent boundary conditions. The solution obtained for this case shows an anomalous behaviour which may be verified by analyzing the distribution or the mean square displacement. The survival probability also manifests a non-usual behaviour, in particular, when the boundary condition is given by $\rho(a,t) = \rho(b,t) = 0$ the asymptotic behaviour is $\mathcal{S}(t) \sim 1/t^{\gamma+\eta}$. Afterwards, we have incorporated external forces in our analysis and have seen that they play an important role on the diffusion process by affecting the spreading of the distribution and consequently of the system. Finally, we expect that the results found here may be useful to investigate situations where the fractional diffusion equations are present.

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